SHEARING STRESS IN BENDING OF I BEAMS

by

V. BASILEVICH (Beograd)

In the case of bending of a cantilever of a constant cross-section of any shape by a force P applied at the end parallel to one of the principal axes of the cross-section, in addition to normal stresses, proportional in each cross-section to the bending moment, there will act also shearing stresses proportional to the shearing force.

Exact solutions of these problems are known for only a few special cross-sections having certain simple boundaries: rectangle, ellipse, triangle, etc.

This paper presents the solution of the problem in case of I cross-section.

Nomenclature

The following nomenclature is used in the paper:

P = force applied at the end,

J = moment of inertia of the cross-section,

μ = Poisson's ratio,

n = normal to the boundary,

x, y = principal axes of the cross-section.

Solution

The solution of the problem of bending of prismatic cantilever can be reduced to the determination of the stress function Φ (x, y) which satisfies the differential equation in the region of the cross-section

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \qquad (1)$$

and the condition

$$\left(\frac{\partial\Phi}{\partial y}-\frac{P}{2J}x^2+\frac{\mu}{1+\mu}\frac{P}{2J}y^2\right)\cos nx-\frac{\partial\Phi}{\partial x}\cos ny=0 \tag{2}$$

on the boundary.

We denote by Φ_1 the value of the function $\Phi(x,y)$ in the region 1, 2, 3, 6 (Fig. 1) and by Φ_2 its value in the region 4, 4′ 5, 5′. These functions must be harmonic and must satisfy the boundary condition (2) and the condition of continuity on the line 4 – 5.

If we take Φ_1 in the form

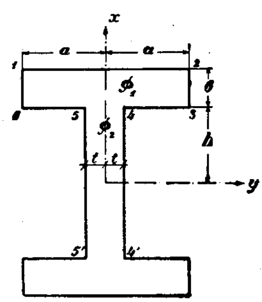

Fig. 1

$$\Phi_1=\frac{P}{2J}\left[(h+b)^2-\frac{\mu}{1+\mu}\frac{a^2}{3}\right]y\ +$$

$$+\sum_{n=1,2}^{\infty}\left(A_n\, ch\,\frac{n\pi(x-h)}{a}+B_n\, sh\,\frac{n\pi(x-h)}{a}\right)\sin\frac{n\pi y}{a}=$$

$$=\sum_{n=1,2}^{\infty}\left\{-\frac{a}{\pi}\frac{P}{J}\left[(h+b)^2-\frac{\mu}{1+\mu}\frac{a^2}{3}\right]\frac{(-1)^n}{n}+\right. \tag{3}$$

$$\left.+A_n\, ch\,\frac{n\pi(x-h)}{a}+B_n\, sh\,\frac{n\pi(x-h)}{a}\right\}\sin\frac{n\pi y}{a},$$

this function satisfies the differential equation (1) and the boundary conditions along the edges 2—3 and 1—6.

Taking Φ_2 in the form

$$\Phi_2=\frac{P}{2J}\left[h^2-\frac{\mu}{1+\mu}\frac{t^2}{3}+(2bh+b^2)\frac{a}{t}\right]y+\sum_{m=1,2}^{\infty}C_m\, ch\,\frac{m\pi x}{t}\sin\frac{m\pi y}{t}=$$

$$=\sum_{m=1,2}^{\infty}\left\{-\frac{1}{\pi}\frac{P}{J}\left[h^2t-\frac{\mu}{1+\mu}\frac{t^3}{3}+(2bh+b^2)a\right]\frac{(-1)^m}{m}+C_m\, ch\,\frac{m\pi x}{t}\right\}\sin\frac{m\pi y}{t}, \tag{4}$$

this function satisfies the differential equation (1) and the boundary conditions along the edges 4 – 4′ and 5 – 5′.

The condition (2) along the edge 1 – 2 becomes

$$\left(\frac{\partial\Phi}{\partial y}\right)_{x=h+b}=\frac{P}{2J}\left[(h+b)^2-\frac{\mu}{1+\mu}y^2\right] \tag{5}$$

which is satisfied, if

$$\left(\frac{\partial \Phi_1}{\partial y}\right)_{x=h+b} = \frac{P}{2J}\left[(h+b)^2 - \frac{\mu}{1+\mu}\frac{a^2}{3}\right] + $$

$$+ \sum_{n=1,2}^{\infty} \frac{n\pi}{a}\left(A_n ch\,\frac{n\pi b}{a} + B_n sh\,\frac{n\pi b}{a}\right)\cos\frac{n\pi y}{a} =$$

$$= \left(\frac{\partial \Phi}{\partial y}\right)_{x=h+b} = \frac{P}{2J}\left[(h+b)^2 - \frac{\mu}{1+\mu}y^2\right] \quad . \tag{6}$$

Substituting in the equation (6) the known development

$$y^2 = \frac{a^2}{3} + \frac{4a^2}{\pi^2}\sum_{n=1,2}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi y}{a} \tag{7}$$

we obtain

$$\sum_{n=1,2}^{\infty}\frac{n\pi}{a}\left(A_n\,ch\,\frac{n\pi b}{a} + B_n\,sh\,\frac{n\pi b}{a}\right)\cos\frac{n\pi y}{a} =$$

$$= -\frac{P}{J}\,\frac{\mu}{1+\mu}\,\frac{2a^2}{\pi^2}\sum_{n=1,2}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi y}{a} \quad . \tag{8}$$

From the equation (8) it follows that the coefficients B_n and A_n must satisfy the condition

$$B_n = -\frac{P}{J}\,\frac{\mu}{1+\mu}\,\frac{2a^3}{\pi^3}\,\frac{(-1)^n}{n^3}\,\frac{1}{sh\,\frac{n\pi b}{a}} - A_n\,\mathrm{cth}\,\frac{n\pi b}{a} \quad . \tag{9}$$

Along the edges 3—4 and 5—6 the function Φ_1 must satisfy the condition (2). Its value along the line 4—5 must coincide with the value of function Φ_2.

The boundary condition along the edges 3—4 and 5-6 from (2)

$$f_1(y) = \begin{cases} \frac{P}{2J}\left[h^2 y - \frac{\mu}{1+\mu}\frac{y^3}{3} - (2bh+b^2)\,a\right] & for\ \ -a < y < -t \\ 0 & for\ \ -t < y < +t \\ \frac{P}{2J}\left[h^2 y - \frac{\mu}{1+\mu}\frac{y^3}{3} + (2bh+b^2)\,a\right] & for\ \ +t < y < +a \end{cases} \tag{10}$$

can be developed in the following Fourier's series

$$f_1(y)=\sum_{n=1,2}^{\infty}\delta_n \sin\frac{n\pi y}{a}=$$

$$=\sum_{n=1,2}^{\infty}\left\{\frac{Ph^2 a}{Jn\pi}\left[\frac{t}{a}\cos\frac{n\pi t}{a}-\frac{1}{n\pi}\sin\frac{n\pi t}{a}-(-1)^n\right]-\right.$$

$$-\frac{\mu}{1+\mu}\frac{P}{3J}\left[\frac{t}{n\pi}\left(t^2-\frac{6a^2}{n^2\pi^2}\right)\cos\frac{n\pi t}{a}-\frac{3a}{n^2\pi^2}\left(t^2-\frac{2a^2}{n^2\pi^2}\right)\sin\frac{n\pi t}{a}+\frac{a^3}{n\pi}(-1)^n\left(\frac{6}{n^2\pi^2}-1\right)\right]- \tag{11}$$

$$\left.-\frac{P}{J}(2bh+b^2)\frac{a}{n\pi}\left[(-1)^n-\cos\frac{n\pi t}{a}\right]\right\}\sin\frac{n\pi y}{a} \quad .$$

The multiplicator of the function Φ_2 from (4)

$$f_2(y)=\begin{cases}0 & \text{for} \quad -a<y<-t\\ \sin\frac{m\pi y}{t} & \text{for} \quad -t<y<+t\\ 0 & \text{for} \quad +t<y<+a\end{cases} \tag{12}$$

can be developed in Fourier's series

$$f_2(y)=\frac{(-1)^{m+1}\,2mt}{a\,\pi}\sum_{n=1,2}\frac{\sin\frac{n\pi t}{a}}{m^2-\left(\frac{nt}{a}\right)^2}\sin\frac{n\pi y}{a} \quad , \tag{13}$$

Condition at the boundary 3 - 4 and 5 - 6 from (11) and the coincidence of the functions $(\Phi_1)_{x=h}$ from (3) and $(\Phi_2)_{x=h}$ from (13) on the line 4 - 5 gives the following equation

$$\sum_{n=1,2}^{\infty}\left\{-\frac{a}{\pi}\frac{P}{J}\left[(h+b)^2-\frac{\mu}{1+\mu}\frac{a^2}{3}\right]\frac{(-1)^n}{n}+A_n\right\}\sin\frac{n\pi y}{a}=$$

$$=\sum_{m=1,2}^{\infty}\left\{-\frac{1}{\pi}\frac{P}{J}\left[h^2 t-\frac{\mu}{1+\mu}\frac{t^3}{3}+(2bh+b^2)\,a\right]\frac{(-1)^m}{m}+C_m\, ch\,\frac{m\pi h}{t}\right\}\cdot$$

$$\cdot(-1)^{m+1}\frac{2mt}{a\pi}\sum_{n=1,2}^{\infty}\frac{\sin\frac{n\pi t}{a}}{m^2-\left(\frac{nt}{a}\right)^2}\sin\frac{n\pi y}{a}+\sum_{n=1,2}^{\infty}\delta_n\sin\frac{n\pi y}{a} \quad . \tag{14}$$

It follows from the equation (14) that the coefficients A_n and C_m must satisfy the condition

$$A_n = \frac{a}{\pi}\frac{P}{J}\left[(h+b)^2 - \frac{\mu}{1+\mu}\frac{a^2}{3}\right]\frac{(-1)^n}{n} + \delta_n +$$

$$+ \frac{2t}{a\pi^2}\frac{P}{J}\left[h^2 t - \frac{\mu}{1+\mu}\frac{t^3}{3} + (2bh+b^2)\,a\right]\sin\frac{n\pi t}{a}\sum_{m=1,2}^{\infty}\frac{1}{m^2-\left(\frac{nt}{a}\right)^2} -$$

$$- \frac{2t}{a\pi}\sin\frac{n\pi t}{a}\sum_{m=1,2}^{\infty}\frac{(-1)^m m\, ch\frac{m\pi h}{t}}{m^2-\left(\frac{nt}{a}\right)^2}C_m \quad . \tag{15}$$

Equally on the line 4—5 the value of the derivative of Φ_1 must coincide with the value of the derivative of Φ_2. The multiplicator of the function Φ_1 from (3)

$$f_3(y) = \sin\frac{n\pi y}{a} \qquad \text{for} \quad -t<y<+t \tag{16}$$

can be developed in Fourier's series

$$f_3(y) = -\frac{2}{\pi}\sin\frac{n\pi t}{a}\sum_{m=1,2}^{\infty}\frac{(-1)^m m}{m^2-\left(\frac{nt}{a}\right)^2}\sin\frac{m\pi y}{t} \quad . \tag{17}$$

This condition, according to (17), gives the following equation

$$\left(\frac{\partial\Phi_1}{\partial x}\right)_{x=h} = -\sum_{n=1,2}^{\infty}\frac{n\pi}{a}B_n\frac{2}{\pi}\sin\frac{n\pi t}{a}\sum_{m=1,2}^{\infty}\frac{(-1)^m m}{m^2-\left(\frac{nt}{a}\right)^2}\sin\frac{m\pi y}{t} =$$

$$= \left(\frac{\partial\Phi_2}{\partial x}\right)_{x=h} = \sum_{m=1,2}^{\infty}\frac{m\pi}{t}C_m\, sh\frac{m\pi h}{t}\sin\frac{m\pi y}{t} \quad . \tag{18}$$

It follows from the equation (14) that the coefficients B_n and C_m must satisfy the condition

$$C_m = -\frac{1}{sh\frac{m\pi h}{t}}\cdot\frac{2t(-1)^m}{a\pi}\sum_{n=1,2}^{\infty}\frac{n\sin\frac{n\pi t}{a}}{m^2-\left(\frac{nt}{a}\right)^2}B_n \quad . \tag{19}$$

Replacing in the expression (19) the index n by i and vice versa, and introducing (19) in (15) and (9), we obtain

$$B_n + \frac{4\,t^2}{a^2\pi^2} \sin\frac{n\pi t}{a} \operatorname{cth}\frac{n\pi b}{a} \sum_{i=1,2}^{\infty} i \sin\frac{i\pi t}{a} B_i \sum_{m=1,2}^{\infty} \frac{m \operatorname{cth}\dfrac{m\pi h}{t}}{\left[m^2-\left(\dfrac{nt}{a}\right)^2\right]\left[m^2-\left(\dfrac{it}{a}\right)^2\right]} =$$

$$= -\frac{P}{J}\,\frac{\mu}{1+\mu}\,\frac{2\,a^3}{\pi^3}\,\frac{(-1)^n}{n^3}\,\frac{1}{sh\,\dfrac{n\pi b}{a}} -$$

$$-\frac{a}{\pi}\,\frac{P}{J}\left[(h+b)^2-\frac{\mu}{1+\mu}\,\frac{a^2}{3}\right]\frac{(-1)^n}{n} \operatorname{cth}\frac{n\pi b}{a} - \delta_n \operatorname{cth}\frac{n\pi b}{a} -$$

$$-\frac{2\,t}{a\pi^2}\,\frac{P}{J}\left[h^2 t-\frac{\mu}{1+\mu}\,\frac{t^3}{3}+(2bh+b^2)\,a\right] \sin\frac{n\pi t}{a} \operatorname{cth}\frac{n\pi b}{a} \sum_{m=1,2}^{\infty} \frac{1}{m^2-\left(\dfrac{nt}{a}\right)^2}\,.$$

The coefficients B_n can be calculated from this system of linear equations, after which equations (9) and (19) determine explicitely the values of A_n and C_m.

The whole computation is illustrated by the following example.

Let us take

$$a=4{,}5, \quad b=2, \quad h=4, \quad t=1.$$

The system of linear equations becomes

$$+1{,}01\,B_1 +0{,}04\,B_2 +0{,}08\,B_3 +0{,}10\,B_4 +0{,}11\,B_5 = -29{,}86\,\frac{P}{J}$$

$$+0{,}02\,B_1 +1{,}07\,B_2 +0{,}13\,B_3 +0{,}17\,B_4 +0{,}17\,B_5 = -\ 9{,}09\,\frac{P}{J}$$

$$+0{,}02\,B_1 +0{,}08\,B_2 +1{,}16\,B_3 +0{,}21\,B_4 +0{,}22\,B_5 = -\ 3{,}65\,\frac{P}{J}$$

$$+0{,}02\,B_1 +0{,}08\,B_2 +0{,}16\,B_3 +1{,}22\,B_4 +0{,}23\,B_5 = -\ 0{,}57\,\frac{P}{J}$$

$$+0{,}02\,B_1 +0{,}07\,B_2 +0{,}13\,B_3 +0{,}19\,B_4 +1{,}22\,B_5 = +\ 0{,}30\,\frac{P}{J}.$$

Solving it we obtain the values of the coefficients

$$A_1 = +26{,}64\,\frac{P}{J}; \qquad B_1 = -29{,}28\,\frac{P}{J}; \qquad C_1\,ch\,4\,\pi = -5{,}05\,\frac{P}{J};$$

$$A_2 = +\ 7{,}98\,\frac{P}{J}; \qquad B_2 = -\ 8{,}09\,\frac{P}{J}; \qquad C_2\,ch\,8\,\pi = +1{,}57\,\frac{P}{J};$$

$$A_3 = +\ 2{,}30\,\frac{P}{J}; \qquad B_3 = -\ 2{,}30\,\frac{P}{J}; \qquad C_3\,ch\,12\,\pi = -0{,}67\,\frac{P}{J};$$

$$A_4 = -\ 0{,}77\,\frac{P}{J}; \qquad B_4 = +\ 0{,}77\,\frac{P}{J}; \qquad C_4\,ch\,16\,\pi = +0{,}36\,\frac{P}{J}:$$

$$A_5 = -\ 1{,}27\,\frac{P}{J}; \qquad B_5 = +\ 1{,}27\,\frac{P}{J}: \qquad C_5\,ch\,20\,\pi = -0{,}22\,\frac{P}{J}.$$

The Fig. 2 gives the functions Φ_1 and Φ_2, the Fig. 3 gives the lines of equal shear stress τ_x and the Fig. 4 the lines of equal shear stress τ_y.

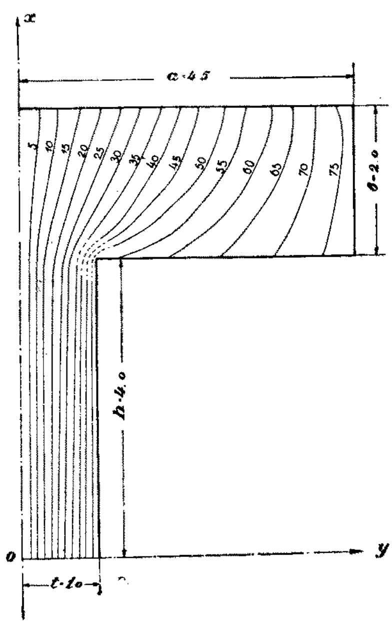

Fig. 2

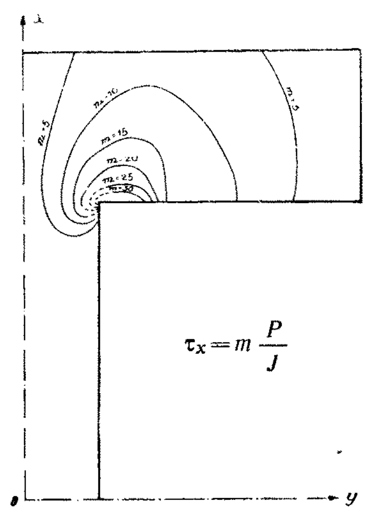

Fig.3

In Fig. 5 stress values τ_y are compared with values calculated by the usual elementary formula from the resistance of materials.

The Fig. 2, 3 and 4 agree with the results obtained by the soap-film method [2].

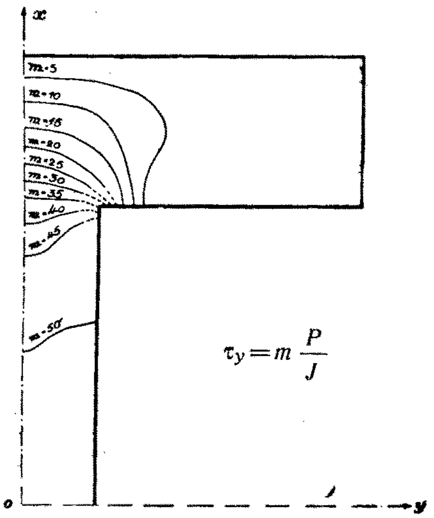

Fig. 4

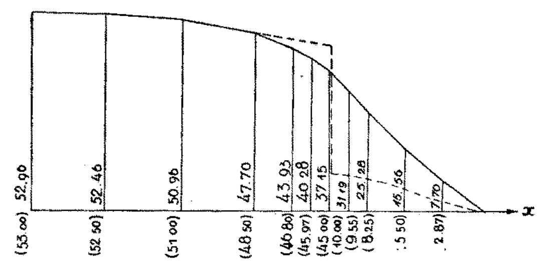

Fig. 5

REFERENCES

[1] Klitchieff J. — Torsion of I Rods. Eigth international congress on theoretical and applied mechanics.

[2] Timoshenko S. — Theory of Elasticity, New York 1934.