ON SOME CHARACTERISTICS OF THE FREQUENCY EQUATION OF TORSIONAL VIBRATIONS OF LIGHT SHAFTS WITH SEVERAL DISKS

by

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Consider a light shaft whose mass we neglect and with several attached disks of moment of inertia J_k (Fig. 1). Then the Lagrange differential equation of torsional vibrations for three adjacent disks may be written

$$J_k \ddot{\Theta}_k - c_{k-1} \Theta_{k-1} + (c_{k-1} + c_k) \Theta_k - c_k \Theta_{k+1} = 0 , \tag{1}$$

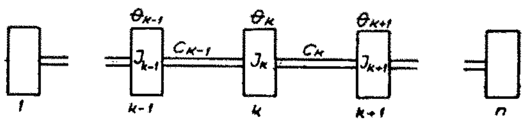

Fig. 1

where c_k are the torsional rigidities of portions of the shaft between two adjacent disks k and $k+1$ [1]. The above system of differential equations can be correlated to a system of algebraic equations of the form

$$-c_{k-1} A_{k-1} + (c_{k-1} + c_k - J_k \omega^2) A_k - c_k A_{k+1} = 0 , \quad k=1, \ldots , n, \tag{2}$$

where ω denotes the circular frequency of vibration.

If all disks are of the same moment of inertia ($J_k = J$) and if the rigidities of the different portions of the shaft are equal too ($c_k = c$), equations (2) may be written

$$A_{k-1} - (2 - u^2) A_k + A_{k+1} = 0 , \tag{3}$$

where $u = J \omega^2/c$.

In the three characteristic cases: free shaft (Fig. 2a), one end clamped shaft (Fig. 2b) and both ends clamped shaft (Fig. 2c), the frequency equation becomes

$$f(\omega^2) = f(z) = B_n z^n + B_{n-1} z^{n-1} + \ldots + B_1 z + B_0 \ . \tag{4}$$

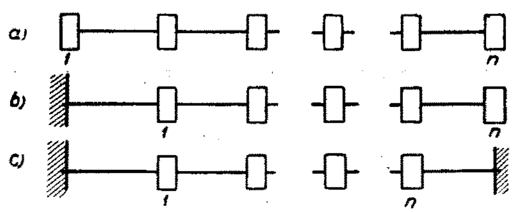

Fig. 2

It is possible immediately to percieve the following characteristics regarding the number and values of coefficients of equations

	1 (s)	2 (k)	3 (u)
N	n	$n+1$	$n+1$
B_n	0	1	1
B_{n-1}	1	$-(2n-1)p$	$-2np$
B_0	$(-1)^{n-1} n\, p^{n-1}$	$(-1)^n p^n$	$(-1)^n \cdot (n+1)\, p^n$

where n is the number of disks and $p = c/J$.

It can be seen from equations (4) that the coefficients B_r of the diagonal row form a series of numbers (Table I) whose differences are

$$\Delta^r = (-1)^r \cdot (2p)^r = \text{const.}, \qquad \Delta^{r+1} = 0 \, .$$

These coefficients may therefore be computed by the formula

$$B_r = (1+\Delta)^{n-r} B_0^{(n-r)} =$$
$$= B_0^{(n-r)} + \binom{n-r}{1} \Delta B_0^{(n-r)} + \binom{n-r}{2} \Delta^2 B_0^{(n-r)} + \ldots + \binom{n-r}{r-1} \Delta^r B_0^{(n-r)} , \tag{5}$$

Table I

	n: s	n: u	B_8	B_7	B_6	B_5	B_4	B_3	B_2	B_1	B_0'	n: s	n: u
	2	1								1	$-2p$	2	1
	3	2							1	$-4p$	$3p^2$	3	2
1	4	3						1	$-6p$	$10p^2$	$-4p^3$	4	3
(s)	5	4				$r=0$	1	$-8p$	$21p^2$	$-20p^3$	$5p^4$	5	4
3	6	5				1	$-10p$	$36p^2$	$-56p^3$	$35p^4$	$-6p^5$	6	5
(u)	7	6			1 $r=1$	$-12p$	$55p^2$	$-120p^3$	$126p^4$	$-56p^5$	$7p^6$	7	6
	8	7		1	$-14p$	$78p^2$	$-220p^3$	$330p^4$	$-252p^5$	$84p^6$	$-8p^7$	8	7
	9	8	1	$-16p$	$105p^2$	$-364p^3$	$715p^4$	$-792p^5$	$462p^6$	$-120p^7$	$9p^8$	9	8
	$\Delta=0$		$\Delta^1=-2p$	$\Delta^2=4p^2$	$\Delta^3=-8\gamma^3$	$\Delta^4=16p^4$	$\Delta^5=-32p^5$	$\Delta^6=64p^6$	$\Delta^7=-128p^7$				
	1									1	$-p$	1	
	2								1	$-3p$	p^2	2	
	3							1	$-5p$	$6p^2$	$-p^3$	3	
2	4						1	$-7p$	$15p^2$	$-10p^3$	p^4	4	
(k)	5					1	$-9p$	$28p^2$	$-35p^3$	$15p^4$	$-p^5$	5	
	6				1	$-11p$	$45p^2$	$-84p^3$	$70p^4$	$-21p^5$	p^6	6	
	7			1	$-13p$	$66p^2$	$-165p^3$	$210p^4$	$-126p^5$	$28p^6$	$-p^7$	7	
	8		1	$-15p$	$91p^2$	$-286p^3$	$495p^4$	$-462p^5$	$210p^6$	$-36p^7$	p^8	8	
	$\Delta=0$		$\Delta^1=-2p$	$\Delta^2=4p^2$	$\Delta^3=-8p^3$	$\Delta^4=16p^4$	$\Delta^5=-32p^5$	$\Delta^6=64p^7$					

where in the first case (Fig. 2a) the index $r-1$ should be taken instead of the index r, because $\omega_0=0$ is not the root of the frequency equation but only shows that the momentum of the system is constant $(\Sigma J\ddot{\Theta}=0)$.

From (5) we obtain the coefficients for the quoted characteristic cases

$$B^s_{n-2}=-2(n-1)p, \qquad B^s_{n-3}=[3+(n-3)(2n-1)]p^2,$$

$$B^s_{n-4}=-\left[4+16\binom{n-4}{1}+20\binom{n-4}{2}+8\binom{n-4}{3}\right]p^3;$$

$$B^k_{n-1}=-(2n-1)p, \qquad B^k_{n-2}=\left[1+5\binom{n-2}{1}+4\binom{n-2}{2}\right]p^2,$$

$$B^k_{n-3}=-\left[1+9\binom{n-3}{1}+16\binom{n-3}{2}+8\binom{n-3}{3}\right]p^3,$$

$$B^k_{n-4}=\left[1+14\binom{n-4}{1}+41\binom{n-4}{2}+44\binom{n-4}{3}+16\binom{n-4}{4}\right]p^4;$$

$$B^u_{n-1}=-2pn; \qquad B^u_{n-2}=\left[3+7\binom{n-2}{1}+4\binom{n-2}{2}\right]p^2,$$

$$B^u_{n-3}=-\left[4+16\binom{n-3}{1}+20\binom{n-3}{2}+8\binom{n-3}{3}\right]p^3,\ldots$$

Equations (4) may then be written

$$z^{n-1}-\binom{2n-2}{1}pz^{n-2}+\binom{2n-3}{2}p^2z^{n-3}-\ldots\pm\binom{2n-(r+1)}{r}p^rz^{n-(r+1)}+\ldots$$
$$\ldots+(-1)^n n p^{n-1}=0,$$

$$z^n-\binom{2n-1}{1}pz^{n-1}+\binom{2n-2}{2}p^2z^{n-2}-\ldots\pm\binom{2n-r}{r}p^rz^{n-r}+\ldots$$
$$\ldots+(-1)^n p^n=0, \tag{6}$$

$$z^n-\binom{2n}{1}pz^{n-1}+\binom{2n-1}{2}p^2z^{n-2}-\ldots\pm\binom{2n-(r-1)}{r}p^rz^{n-r}+\ldots$$
$$\ldots+(-1)^n(n+1)p^n=0.$$

The Lagrange determinants of these vibrations are

$$\Delta_n^s(\omega^2)=\begin{vmatrix} a & -c & & & \\ -c & b & -c & & \\ & -c & b & \ddots & \\ & & \ddots & b & -c \\ & & & -c & a \end{vmatrix}=0\,, \qquad \Delta_n^k(\omega^2)=\begin{vmatrix} b & -c & & & \\ -c & b & -c & & \\ & -c & b & \ddots & \\ & & \ddots & b & -c \\ & & & -c & a \end{vmatrix}=0\,, \tag{7}$$

$$\Delta_n^u(\omega^2)=\begin{vmatrix} b & -c & & & \\ -c & b & -c & & \\ & -c & b & \ddots & \\ & & \ddots & b & -c \\ & & & -c & b \end{vmatrix}=0\,,$$

where $b=2\,c-J\,\omega^2$, $a=c-J\,\omega^2=b-c$.

Evaluation gives:

a) $$\Delta_n^s=b^{n-1}-\binom{n-2}{1}b^{n-3}c^2+\binom{n-3}{2}b^{n-5}c^4-\binom{n-4}{3}b^{n-7}c^6+\ldots+B_0=0, \tag{8}$$

$N=n/2$ if n is even, while $N=(n+1)/2$ if n is odd;

$$B_0=(-1)^{\frac{n-1}{2}}c^{n-1} \text{ for odd values of } n,$$

$$B_0=(-1)^{\frac{n-2}{2}}\frac{n}{2}c^{n-2} \text{ for even } n;$$

b) $$\Delta_n^k=b^n-b^{n-1}c-\binom{n-1}{1}b^{n-2}c^2+\binom{n-2}{1}b^{n-3}c^3+\binom{n-2}{2}b^{n-4}c^4-$$
$$-\binom{n-3}{2}b^{n-5}c^5-\binom{n-3}{3}b^{n-6}c^6+\binom{n-4}{3}b^{n-7}c^7+\binom{n-4}{4}b^{n-8}c^8-\ldots=0\,,$$

$N=n+1$;

c) $$\Delta_n^u=b^n-\binom{n-1}{1}b^{n-2}c^2+\binom{n-2}{2}b^{n-4}c^4-\binom{n-3}{3}b^{n-6}c^6+\ldots+B_0=0,$$

$N=(n+2)/2$ if n is even, and $N=(n+1)/2$ if n is odd;

$$B_0=(-1)^{\frac{n}{2}}c^n \text{ for even values of } n,$$

$$B_0=(-1)^{\frac{n-1}{2}}\frac{n+1}{2}b\,c^{n-1} \text{ for odd } n\,.$$

These determinants yield

$$\Delta_n^s = \Delta_{n-1}^u \, , \qquad \Delta_n^k = \Delta_n^u - c\,\Delta_{n-1}^u = \Delta_{n+1}^s + c\,\Delta_n^s \, . \tag{9}$$

We therefore conclude that clamping of both ends (Fig. 2c) replaces one disk of the free shaft, hence the coefficients of frequency equations are $B_{n-r}^u = B_{n+1-r}^s$. The first case is therefore the basic one, since the frequency equations of the other two cases may be deduced from the equation of the first case.

Equation (3) is similar to the Clapeyron three moments equation of an unloaded continuous beam of constant cross section and equal spans [2]. The method of finite-difference equations of second order may be used here. In the case of oscillatory movement [3] is $|u| < 2$ yielding

$$A_\nu = C_\nu \, e^{\nu\beta l} \, ;$$

hence

$$u = 2 \sin \frac{\beta}{2}, \qquad \omega_\nu = 2\sqrt{p} \sin \frac{\beta}{2} \, . \tag{10}$$

The coefficient β must be such as to satisfy the boundary conditions

$$a) \left| \begin{array}{l} A_2 - (1-u^2)\,A_1 = 0, \\ A_{n-1} - (1-u^2)\,A_n = 0, \end{array} \right. \quad b) \left| \begin{array}{l} A_2 - (2-u^2)\,A_1 = 0, \\ A_{n-1} - (1-u^2)\,A_n = 0, \end{array} \right. \quad c) \left| \begin{array}{l} A_2 - (2-u^2)\,A_1 = 0, \\ A_{n-1} - (2-u^2)\,A_n = 0, \end{array} \right.$$

hence

$$\beta = \frac{\nu\pi}{n} \, , \quad \beta = \frac{(2\nu - 1)\,\pi}{2\,n+1} \, , \quad \beta = \frac{\nu\,\pi}{n+1} \, .$$

From (10) the circular frequencies in all three cases are given by

$$\omega_\nu = 2\sqrt{p} \sin \frac{\nu\pi}{2n}, \quad \omega_\nu = 2\sqrt{p} \sin \frac{(2\nu-1)\,\pi}{2\,(2n+1)}, \quad \omega_\nu = 2\sqrt{p} \sin \frac{\nu\pi}{2\,(n+1)} \, . \tag{11}$$

$$\nu = 1, \ldots, n-1 \qquad\qquad \nu = 1, \ldots, n \qquad\qquad \nu = 1, \ldots, n$$

The roots of the frequency equation (4) are given in the Table II.

Taking into consideration the relation between the roots (11) and the coefficients of the frequency equation (4) we obtain

$$\frac{n-1}{2} = \sum_1^{n-1} \sin^2 \frac{\nu\pi}{2n}, \qquad \sum_1^{n-1} \cos \frac{\nu\pi}{n} = 0, \qquad \frac{n}{4^{n-1}} = \prod_1^{n-1} \sin^2 \frac{\nu\pi}{2n};$$

Table II

n		$z=\omega^2$							
s	u								
2	1								$2\,p$
3	2							p	$3\,p$
4	3						$(2-\sqrt{2})\,p$	$2\,p$	$(2+\sqrt{2})\,p$
5	4					$0{,}3819\,p$	$1{,}3890\,p$	$2{,}6180\,p$	$3{,}6181\,p$
6	5				$(2-\sqrt{3})\,p$	$1\,p$	$2\,p$	$3\,p$	$(2+\sqrt{3})\,p$
7	6			$0{,}1980\,p$	$0{,}7534\,p$	$1{,}5550\,p$	$2{,}4465\,p$	$3{,}2486\,p$	$3{,}8019\,p$
8	7		$0{,}1522\,p$	$0{,}5888\,p$	$1{,}2346\,p$	$2\,p$	$2{,}7653\,p$	$3{,}4142\,p$	$3{,}8478\,p$
9	8	$0{,}1206\,p$	$0{,}4679\,p$	$1\,p$	$1{,}6526\,p$	$2{,}3475\,p$	$3\,p$	$3{,}5319\,p$	$3{,}8790\,p$

n	$z=\omega^2$							
k								
1								p
2							$\frac{1}{2}(3-\sqrt{5})\,p$	$\frac{1}{2}(3+\sqrt{5})\,p$
3						$0{,}1980\,p$	$1{,}5575\,p$	$3{,}2467\,p$
4					$0{,}1206\,p$	$1\,p$	$2{,}3475\,p$	$3{,}5319\,p$
5				$0{,}0811\,p$	$0{,}6905\,p$	$1{,}7158\,p$	$2{,}8314\,p$	$3{,}6829\,p$
6			$0{,}0581\,p$	$0{,}4747\,p$	$1{,}2905\,p$	$2{,}2400\,p$	$3{,}1361\,p$	$3{,}7708\,p$
7		$0{,}0437\,p$	$0{,}3919\,p$	$1\,p$	$1{,}7909\,p$	$2{,}6180\,p$	$3{,}3882\,p$	$3{,}8271\,p$
8	$0{,}0335\,p$	$0{,}2996\,p$	$0{,}8503\,p$	$1{,}4529\,p$	$2{,}1847\,p$	$2{,}8918\,p$	$3{,}4702\,p$	$3{,}8640\,p$

$$\frac{2n-1}{4}=\sum_1^n \sin^2\frac{(2\nu-1)\pi}{2(2n+1)},\quad \sum_1^n \cos\frac{(2\nu-1)\pi}{2n+1}=\frac{1}{2},\quad \frac{1}{4^n}=\prod_1^n \sin^2\frac{(2\nu-1)\pi}{2(2n+1)};$$

$$\frac{n}{2}=\sum_1^n \sin^2\frac{\nu\pi}{2(n+1)},\qquad \sum_1^n \cos\frac{\nu\pi}{n+1}=0,\qquad \frac{n+1}{4^n}=\prod_1^n \sin^2\frac{\nu\pi}{2(n+1)}. \quad (12)$$

These expressions may be directly proved mathematically.

The Lagrange's formula gives

$$\frac{1}{2}+\sum_1^n \cos\nu x=\frac{\sin(n+1/2)\,x}{2\sin(x/2)};$$

thus

$$\sum_1^n \cos\nu x=0 \qquad \text{for} \qquad x=\pi/(n+1).$$

Since

$$\sum_1^n \sin^2\frac{(2\nu-1)\pi}{2(2n+1)}=\frac{n}{2}-\frac{1}{2}\sum \cos\frac{(2\nu-1)\pi}{2n+1},$$

we obtain

$$\sum e^{(2\nu-1)zi}=e^{zi}\,\frac{1-e^{2nzi}}{1-e^{2zi}},\qquad \sum \cos(2\nu-1)\,z=\frac{1}{2};$$

thus

$$\frac{2n-1}{4}=\sum \sin^2\frac{(2\nu-1)\pi}{2(2n+1)}.$$

Since

$$z^{2n}-1=(z-1)\prod_1^{2n-1}(z-e^{\nu i\pi/n}),$$

we obtain

$$1+z+\cdots+z^{2n-1}=-\prod_1^{2n-1}(e^{\nu i\pi/n}-z).$$

For $z=1$ we have

$$2n\prod e^{-\nu\pi i/2n}=-(2i)^{2n-1}\prod \sin\frac{\nu\pi}{2n}=(-1)^n\,2ni;$$

thus

$$\frac{n}{4^{n-1}}=\prod_1^{2n-1}\sin\frac{\nu\pi}{2n}=\prod_1^{n-1}\sin^2\frac{\nu\pi}{2n}.$$

Since

$$\Gamma(1+x)=x\,\Gamma(x), \qquad \Gamma(1+x)\,\Gamma(1-x)=\frac{\pi x}{\sin\pi x},$$

$$\Gamma(2-x)=(1-x)\;\Gamma(1-x)\;,$$

the product becomes

$$\Gamma(1+x)\,\Gamma(2-x)=x\,(1-x)\,\frac{\pi}{\sin\pi x}\,.$$

For $x=\nu/(n+1)=\nu/N$ the product will be

$$\prod_1^n\left\{\Gamma\left(1+\frac{\nu}{N}\right)\Gamma\left(1+\frac{N-\nu}{N}\right)\right\}=\left\{\prod_1^n\left[\Gamma\left(1+\frac{\nu}{N}\right)\right]\right\}^2=$$

$$=\left[\Pi\left(\frac{\nu}{N}\right)\right]^2\frac{\pi^n}{\prod\sin\pi x}\;.$$

Taking the root from the above expression and taking into consideration the gamma function multipication theorem we obtain

$$(2\pi)^{\frac{N-1}{2}}N^{1/2-N}\,\frac{N!}{N}=\pi^{n/2}\cdot\frac{n!}{N^n}\,\frac{1}{[\prod\sin\pi x]^{1/2}}$$

giving

$$\frac{n+1}{2^n}=\prod\sin\pi x=2^n\prod\sin\frac{\nu\pi}{2N}\cos\frac{\nu\pi}{2N}\;;$$

thus

$$\frac{n+1}{4^n}=\prod_1^n\sin^2\frac{\nu\pi}{2\,(n-1)}\,.$$

Substitution of $n-1$ in place of n directly yields the third formula of the first group (12).

According to the known analogy between linear, torsional and electrical vibrations, the foregoing exsposition may be applied also to these

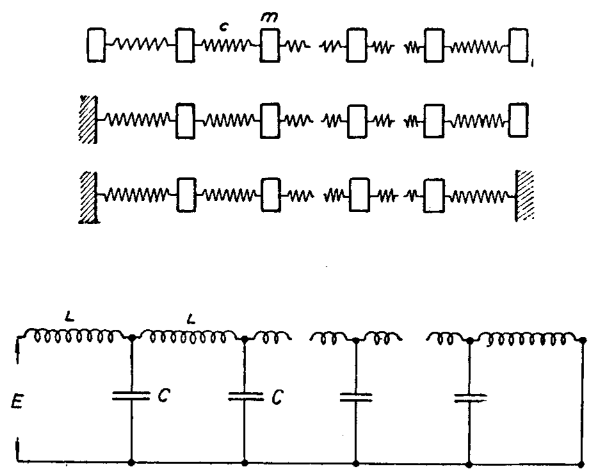

Fig. 3

other vibrations, provided the moment of inertia (J) is replaced by the mass (m) or the inductance (L), and the torsional rigidity (c) by the spring rigidity (c) or the reciprocal value of the capacitance $1/C$ (Fig. 3).

BIBLIOGRAPHY

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[2] Love. — A Treatise on the Mathematical Theory of Elasticity, IV ed., page 375.
[3] Kármán - Biot — Mathematical Methods in Engineering, 1940, Ch. XI.