CONTRIBUTION TO THE RESEARCH OF INFLUENCE OF ROTATORY INERTIA AND SHEARING FORCE ON THE LATERAL VIBRATIONS OF PRISMATIC BARS

by

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The problem of lateral vibrations of prismatical bars taking into consideration the influence of rotatory inertia and shearing force conducts to a partial differential equation of the fourth order

$$a^2 \frac{\partial^4 y}{\partial z^4} + \frac{\partial^2 y}{\partial t^2} - i_x^2 \left(1 + E\beta\right) \frac{\partial^4 y}{\partial z^2 \, \partial t^2} + i_x^2 \rho \, \beta \frac{\partial^4 y}{\partial t^4} = 0 \,, \tag{1}$$

where

$$a^2 = \frac{E i_x^2}{\rho} \quad ; \quad i_x^2 = \frac{I_x}{A} \quad ; \quad \beta = \frac{\varkappa}{G} \quad ; \quad \rho = \frac{\gamma}{g} \, ;$$

E – modulus of elasticity,
G – shear modulus of elasticity,
$\varkappa$ – numerical factor depending on the shape of the cross section,
i_x – radius of gyration of the cross section,
I_x – moment of inertia,
A – cross sectional area,
γ – weight per unit volume,
g – gravitational acceleration,
l – length of the bar,
$y(z,t)$ – deflection of the bar.

Particular solutions of the differential equation (1) may be obtained by the Daniel Bernoulli method of particular solutions, i. e. by trying to find these solutions in the form

$$y\,(z,t) = T(t) \cdot Z(z) = (A \cos \omega t + B \sin \omega t)\, Z(z) \,. \tag{2}$$

This yields a linear homogenous differential equation of the fourth order with constant coefficients

$$a^2 \frac{d^4 Z}{dz^4} + i_x^2 \omega^2 (1+E\beta) \frac{d^2 Z}{dz^2} - \omega^2 (1 - i_x^2 \rho \beta \omega^2) Z = 0 , \tag{3}$$

whose characteristic equation is

$$A_4 \lambda^4 + A_2 \lambda^2 - A_0 = 0 , \tag{4}$$

where

$$A_4 = a^2 , \qquad A_2 = i_x^2 (1+E\beta) \omega^2 , \qquad A_0 = \omega^2 (1 - i_x^2 \rho \beta \omega^2) .$$

The roots of (4) are

$$\lambda_{1,2} = \pm m \quad , \quad \lambda_{3,4} = \pm m_1 i,$$

where

$$m = \sqrt{\frac{1}{2A_4}(-A_2 + \sqrt{A_2^2 + 4A_0A_4})} = \sqrt{-c\omega^2 + \sqrt{\omega^4(c^2-d)+e\omega^2}},$$

$$m_1 = \sqrt{\frac{1}{2A_4}(A_2 + \sqrt{A_2^2 + 4A_0A_4})} = \sqrt{c\omega^2 + \sqrt{\omega^4(c^2-d)+e\omega^2}}, \tag{5}$$

$$c = \frac{\rho(1+E\beta)}{2E} , \quad d = \frac{\rho^2\beta}{E} , \quad e = \frac{1}{a^2} .$$

Since

$$c^2 - d = \frac{\rho^2}{4E^2}(1-E\beta)^2 > 0 ,$$

m_1 is always a real number. To make m a real number the condition $e\omega^2 > d\omega^4$ should be satisfied, or

$$\omega^2 < \frac{e}{d} = \frac{1}{i_x^2 \rho \beta} .$$

We find that

$$\frac{1}{\rho\beta} = \frac{gG}{\gamma \varkappa} \sim 8{,}38 \cdot 10^{10}$$

for $G = 8 \cdot 10^5$ kg/cm², $\varkappa = 1{,}2$, $\gamma = 7{,}8 \cdot 10^{-3}$ kg/cm² and $g = 981$ cm/sec² .

Hence, the condition

$$\omega < \frac{1}{i_x} 2{,}89 \cdot 10^5$$

should be satisfied, where i_x is given in cm. This is practically always the case in view of the nature of the problem.

Hence, the general solution of the differential equation (3) is:

$$Z(z) = Ae^{mz} + Be^{-mz} + Ce^{m_1 iz} + De^{-m_1 iz}\,, \tag{6}$$

or

$$Z(z) = C_1 \operatorname{Ch} mz + C_2 \operatorname{Sh} mz + C_3 \cos m_1 z + C_4 \sin m_1 z\,. \tag{7}$$

In order to facilitate the determination of the constants of integration $C_1, \ldots, C_4$, solution (7) may be expressed in a more convenient form

$$Z(z) = C_1 Z_1(z) + C_2 Z_2(z) + C_3 Z_3(z) + C_4 Z_4(z) \tag{8}$$

by introducing the functions

$$\begin{aligned} Z_1(z) &= \frac{1}{2}(\operatorname{Ch} mz + \cos m_1 z)\,, \\ Z_2(z) &= \frac{1}{2}(\operatorname{Sh} mz + \sin m_1 z)\,, \\ Z_3(z) &= \frac{1}{2}(\operatorname{Ch} mz - \cos m_1 z)\,, \\ Z_4(z) &= \frac{1}{2}(\operatorname{Sh} mz - \sin m_1 z)\,. \end{aligned} \tag{9}$$

The constants $C_1, \ldots, C_4$ of solution (8) differ from those of solution (7). The values of the functions Z_i $(i = 1, 2, 3, 4)$ and their derivatives, up to the third inclusive, for $z = 0$ are given in the following table:

	$Z_i(0)$	$Z_i'(0)$	$Z_i''(0)$	$Z_i'''(0)$
Z_1	1	0	$\frac{1}{2}(m^2 - m_1^2)$	0
Z_2	0	$\frac{1}{2}(m + m_1)$	0	$\frac{1}{2}(m^3 - m_1^3)$
Z_3	0	0	$\frac{1}{2}(m^2 + m_1^2)$	0
Z_4	0	$\frac{1}{2}(m - m_1)$	0	$\frac{1}{2}(m^3 + m_1^3)$

(10)

Solution (8) is also valid when we consider only the influence of rotatory inertia, neglecting the influence of shearing force on bar deflection when $\beta = 0$. Equations (5) defining m and m_1 then reduce to

$$m = m^* = \sqrt{-c^* \omega^2 + \sqrt{c^{*2}\omega^4 + e\omega^2}} \,,$$

$$m_1 = m_2^* = \sqrt{c^* \omega^2 + \sqrt{c^{*2}\omega^4 + e\omega^2}} \,, \tag{11}$$

where

$$c^* = \frac{\rho}{2E} \,.$$

Since

$$\sqrt{c^{*2}\omega^4 + e\omega^2} > c^* \omega^2 \,,$$

it follows that m^* and $m_1{}^*$ are always real numbers.

1. Bar with Hinged Ends

The end conditions are

$$(Z)_{z=0} = 0 \,, \quad (a) \qquad (Z)_{z=l} = 0 \,, \quad (c)$$

$$\left(\frac{d^2 Z}{dz^2}\right)_{z=0} = 0 \,, \quad (b) \qquad \left(\frac{d^2 Z}{dz^2}\right)_{z=l} = 0 \,. \quad (d)$$

From the condition (a) we obtain

$$C_1 = 0 \,, \qquad (e)$$

and from condition (b)

$$C_3 = 0 \,. \qquad (f)$$

Use of conditions (c) and (d), while taking into consideration (e) and (f), gives

$$C_2 (\text{Sh}\, ml + \sin m_1 l) + C_4 (\text{Sh}\, ml - \sin m_1 l) = 0 \,,$$

$$C_2 (m^2\, \text{Sh}\, ml - m_1^2 \sin m_1 l) + C_4 (m^2\, \text{Sh}\, ml + m_1^2 \sin m_1 l) = 0 \,,$$

which, after neglecting the trivial solution $C_2 = C_4 = 0$ yields the frequency equation in the following form

$$(m^2 + m_1^2) \text{ Sh } ml \sin m_1 l = 0 \, . \qquad (12)$$

Since $m^2 + m_1{}^2 \neq 0$ and $ml \neq 0$, it follows that

$$\sin m_1 l = 0,$$

or

$$m_1 = \frac{\pi}{l}, \frac{2\pi}{l}, \frac{3\pi}{l}, \ldots, \frac{n\pi}{l} \, . \qquad (13)$$

Substitution of (13) into (5) yields the frequency equation

$$\omega_n^4 - \frac{n^2 \pi^2 i_x^2(1 + E\beta) + l^2}{l^2 i_x^2 \beta \rho} \omega_n^2 + \frac{n^4 a^2 \pi^4}{l^4 i_x^2 \beta \rho} = 0 \quad (n = 1, 2, 3, \ldots) \qquad (14)$$

S. P. Timoshenko [1] obtained the same equation when trying to find the solution of (1) for the case of a bar with hinged ends in the following form

$$y(z,t) = C \sin \frac{n\pi z}{l} \cos \omega_n t \, , \quad (n = 1, 2, 3, \ldots) \, , \qquad (15)$$

since the function (15) satisfies the imposed end conditions.

When we neglect the influence of the shearing force and consider only the influence of rotatory inertia, i. e. when $\beta = 0$, the frequency equation becomes

$$\omega_n^2 = \frac{n^2 a^2 \pi^4}{l^4} \cdot \frac{1}{1 + \frac{i_x^2 \pi^2}{l^2}} \, , \quad (n = 1, 2, 3, \ldots) \, . \qquad (16)$$

2. Bar with One End Built in and the Other Free

The end conditions are

$$(Z)_{z=0} = 0 \, , \quad (a) \qquad \left(\frac{d^2 Z}{dz^2}\right)_{z=l} = 0 \, , \quad (c)$$

$$\left(\frac{dZ}{dz}\right)_{z=0} = 0 \, , \quad (b) \qquad \left(\frac{d^3 Z}{dz^3}\right)_{z=l} = 0 \, . \quad (d)$$

From the condition (a) we obtain

$$C_1 = 0 \, , \qquad (e)$$

and from condition (b)

$$C_2 = \frac{m_1 - m}{m_1 + m} \, C_4 \, . \qquad (f)$$

Using the conditions (c) and (d) and taking into consideration (e) and (f) we obtain

$$C_3 \, (m^2 \operatorname{Ch} ml + m_1^2 \cos m_1 l) + C_4 \left(\frac{2 \, m^2 \, m_1}{m + m_1} \operatorname{Sh} ml + \frac{2 \, m \, m_1^2}{m + m_1} \sin m_1 l \right) = 0,$$

$$C_3 \, (m^3 \operatorname{Sh} ml - m_1^3 \sin m_1 l) + C_4 \left(\frac{2 \, m^3 \, m_1}{m + m_1} \operatorname{Ch} ml + \frac{2 \, m \, m_1^3}{m + m_1} \cos m_1 l \right) = 0,$$

from which, neglecting the trivial solution $C_3 = C_4 = 0$, we obtain the frequency equation

$$(m^2 - m_1^2) \operatorname{Sh} ml \sin m_1 l - 2 \, mm_1 \operatorname{Ch} ml \cos m_1 l = \frac{m^4 + m_1^4}{mm_1} \, . \qquad (17)$$

If in (5) we put $c = d = 0$, i. e. neglect both the influence of shearing force and rotatory inertia, we obtain $m = m_1 = \sqrt{\frac{\omega}{a}}$, thus the frequency equation (17) becomes

$$\operatorname{Ch} ml \cos ml = -1 ; \qquad (18)$$

this is the well-known equation wich in the case of a bar with one end built in and the other free was derived from the partial differential equation of the laterally vibrating prismatic bar

$$\frac{\partial^2 y}{\partial t^2} + a^2 \frac{\partial^4 y}{\partial z^4} = 0 \, . \qquad (19)$$

This equation was derived under the assumption that the cross sectional dimensions are small in comparison to the length of the bar and neglecting the influence of rotatory inertia and shearing force on bar deflection.

3. Bar with Free Ends

The end conditions are

$$\left(\frac{d^2Z}{dz^2}\right)_{z=0}=0, \quad (a) \qquad \left(\frac{d^2Z}{dz^2}\right)_{z=l}=0, \quad (c)$$

$$\left(\frac{d^3Z}{dz^3}\right)_{z=0}=0, \quad (b) \qquad \left(\frac{d^3Z}{dz^3}\right)_{z=l}=0. \quad (d)$$

From the condition (*a*) we obtain

$$C_1=\frac{m^2+m_1^2}{m_1^2-m^2}\,C_3\,, \qquad (e)$$

and from condition (*b*)

$$C_2=\frac{m^3+m_1^3}{m_1^3-m^3}\,C_4\,. \qquad (f)$$

Use of conditions (*c*) and (*d*) while taking into consideration (*e*) and (*f*) yields

$$C_3\left[\frac{2\,m_1\,m}{m_1^2-m^2}\,(\mathrm{Ch}\,ml-\cos m_1 l)\right]+C_4\left[\frac{2\,m_1\,m}{m_1^2-m^2}\,(m_1\,\mathrm{Sh}\,ml-m\sin m_1 l)\right]=0\,,$$

$$C_3\left[\frac{2\,m\,m_1}{m_1^2-m^2}\,(m\,\mathrm{Sh}\,ml+m_1\sin m_1 l\right]+C_4\left[\frac{2\,m^2\,m_1^2}{m_1^2-m^2}\,(\mathrm{Ch}\,ml-\cos m_1 l)\right]=0\,,$$

whence, neglecting the trivial solution $C_3=C_4=0$, we obtain the frequency equation

$$(m^2-m_1^2)\,\mathrm{Sh}\,ml\sin m_1 l+2\,mm_1\,\mathrm{Ch}\,ml\cos m_1 l=2\,mm_1\,. \qquad (20)$$

For $m=m_1$ the frequency equation (20) reduces to the well-known frequency equation

$$\mathrm{Ch}\,ml\cos ml=1\,. \qquad (21)$$

4. Bar with Both Ends Built-in

The end conditions are

$$(Z)_{z=0}=0\,, \qquad (a) \qquad\qquad (Z)_{z=l}=0\,, \qquad (c)$$

$$\left(\frac{dZ}{dz}\right)_{z=0}=0\,, \qquad (b) \qquad\qquad \left(\frac{dZ}{dz}\right)_{z=l}=0\,. \qquad (d)$$

From the condition (a) we obtain

$$C_1=0\,, \qquad (e)$$

and from condition (b)

$$C_2=\frac{m_1-m}{m+m_1}\,C_4\,. \qquad (f)$$

Use of conditions (c) and (d) while taking into consideration (e) and (f) yields

$$C_3\,(\mathrm{Ch}\,ml-\cos m_1 l)+C_4\left(\frac{2\,m_1}{m+m_1}\,\mathrm{Sh}\,ml-\frac{2\,m}{m+m_1}\,\sin m_1 l\right)=0\,,$$

$$C_3\,(m\,\mathrm{Sh}\,ml+m_1\sin m_1 l)+C_4\left(\frac{2\,m\,m_1}{m+m_1}\,\mathrm{Ch}\,ml-\frac{2\,m\,m_1}{m+m_1}\,\cos m_1 l\right)=0\,,$$

whence, neglecting the trivial solution $C_3=C_4=0$, we obtain the frequency equation

$$(m^2-m_1^2)\,\mathrm{Sh}\,ml\sin m_1 l+2\,m\,m_1\,\mathrm{Ch}\,ml\cos m_1 l=2\,m\,m_1\,. \qquad (22)$$

Thus for a bar with built-in ends we have obtained the same frequency equation as in the case of a bar with free ends.

5. Bar with One End Built-in and the Other Supported

The end conditions are

$$(Z)_{z=0}=0\,, \qquad (a) \qquad\qquad (Z)_{z=l}=0\,, \qquad (c)$$

$$\left(\frac{dZ}{dz}\right)_{z=0}=0\,, \qquad (b) \qquad\qquad \left(\frac{d^2Z}{dz^2}\right)_{z=l}=0\,. \qquad (d)$$

From the condition (a) we obtain

$$C_1 = 0\ , \qquad (e)$$

and from condition (b)

$$C_2 = \frac{m_1 - m}{m + m_1}\, C_4\ . \qquad (f)$$

Use of conditions (c) and (d) while taking into consideration (e) and (f) yields

$$C_3\,(\mathrm{Ch}\, ml - \cos\, m_1 l) + C_4\left(\frac{2\, m_1}{m + m_1}\,\mathrm{Sh}\, ml - \frac{2\, m}{m + m_1}\,\sin\, m_1 l\right) = 0\ ,$$

$$C_3\,(m^2\,\mathrm{Ch}\, ml + m_1^2 \cos\, m_1 l) + C_4\left(\frac{2\, m^2\, m_1}{m + m_1}\,\mathrm{Sh}\, ml + \frac{2\, m\, m_1^2}{m + m_1}\,\sin\, m_1 l\right) = 0\ ,$$

whence, neglecting the trivial solution $C_3 = C_4 = 0$, we obtain the frequency equation

$$\mathrm{Th}\, ml = \frac{m}{m_1}\,\mathrm{tg}\, m_1 l\ . \qquad (23)$$

For $m = m_1$ the frequency equation (23) reduces to the well-known frequency equation

$$\mathrm{Th}\, ml = \mathrm{tg}\, ml\ .$$

If we calculate the lowest frequencies of a steel bar of rectangular cross section with the following data: length $l = 100$ cm; height to length ratio $h/l = 1/5$; $\varkappa = 6/5$; $E = 2{,}1 \cdot 10^6$ kg/cm^2; $G = 8 \cdot 10^5$ kg/cm^2; $\gamma = 7{,}8 \cdot 10^{-3}$ kg/cm^3 and $g = 981$ cm/sec^2, for the following end conditions: hinged ends, free ends (or built-in ends), one end built-in and the other supported, we obtain from frequency equations (14), (20) and (23) frequencies which are lower by 1,94%, ~ 5,85% and ~ 2,05% respectively, than those calculated by the differential equation (19) which neglects the influence of rotatory inertia and shearing force.

If we consider the influence of rotatory inertia and neglect the influence of shearing force on bar deflection, i. e. taking $\beta = 0$, we obtain frequencies which are lower by 1,63%, ~ 5,04% and ~ 1,95%, respectively.

In the case of a cantilever bar the influence of rotatory inertia and shearing force can be neglected, i. e. we obtain practically the same frequencies from frequency equations (17), (18) and from (17) for $\beta = 0$.

REFERENCE

[1] S. P. Timoshenko — Vibration Problems in Engineering. D. van Nostrand, New York, 1947, p. 341, eq. (149).