BENDING OF A RECTANGULAR PLATE WEAKENED BY A HOLE

by

DRAGOŠ RADENKOVIĆ (Beograd)

In the practice of reinforced concrete design we frequently meet with the problem of bending of a rectangular plate weakened by a hole whose edges are parallel to the sides of the plate. A. and L. Föppl in their popular book *Drang und Zwang*[1]) drew attention to this problem as far back as 1920, but to our knowledge, the problem has not yet been solved. Nevertheless, the problem can be solved in an elementary way, and this is done in the present paper. For the sake of simplicity the discussion is limited to the case of a square plate with a symmetrical hole, under uniformly distributed load. The more general case of a rectangular plate with a hole in arbitrary position and under arbitrary loading can be treated in the same manner, but this would require more numerical calculations. As the solution is of interest for applications in reinforced concrete structures we take Poisson's ratio $\nu = 0$.

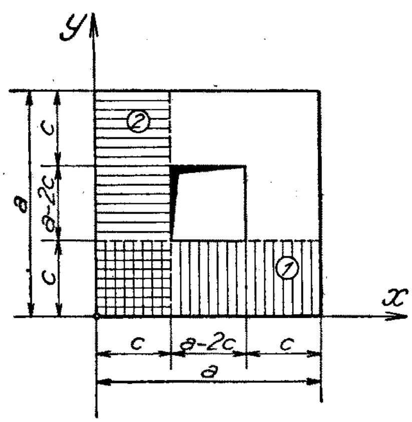

Fig. 1

Let us consider the plate shown in fig. 1, with simply supported outer boundary and with free edges of the hole, loaded with uniformly distributed load q. For the bending of this plate we have the differential equation

$$\Delta\Delta\, w = q/D \tag{1}$$

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[1]) First edition, p. 215. A fairly complete reference-list for the problem of bending of plate weakened by a hole is given in the paper of B. R. Seth in the *Quarterly Journal of Mech. and Appl. Math.* (Vol II, Part 2, June 1949, pp. 177—181).

with boundary conditions:

$$w = 0 \text{ and } \frac{\partial^2 w}{\partial \nu^2} = 0 \text{ on the outer boundary,} \tag{2}$$

$$\frac{\partial^2 w}{\partial \nu^2} = 0 \text{ and } \frac{\partial^3 w}{\partial \nu^3} + 2\frac{\partial^3 w}{\partial \tau^2 \partial \nu} = 0 \text{ on the inner boundary,} \tag{3}$$

$$\frac{\partial^2 w}{\partial \tau \partial \nu} = 0 \text{ in the corners of the hole,} \tag{4}$$

where w denotes the deflection of the plate, $d\nu$ an element of the normal and $d\tau$ an element of the tangent to the contour. The condition (4) requires that there should not exist any concentrated forces in the corners of the hole.

If the plate were without opening, then for its bending we would have the well-known Nádai's solution. Taking the coordinate axes as shown in fig. 1, this solution has the form:

$$w_q = \frac{4qa^4}{D\pi^5} \sum_{k=1,3,5\ldots}^{\infty} \frac{1}{k^5} \Bigg\{ 1 - \Bigg[\operatorname{ch} \alpha_k y - \frac{\alpha_k y}{2} \operatorname{sh} \alpha_k y - $$
$$- \left(\operatorname{th} \frac{\alpha_k a}{2} - \frac{\alpha_k a}{4} \operatorname{ch}^{-2} \frac{\alpha_k a}{2} \right) \operatorname{sh} \alpha_k y + \frac{\alpha_k y}{2} \operatorname{th} \frac{\alpha_k a}{2} \operatorname{ch} \alpha_k y \Bigg] \Bigg\} \sin \alpha_k x\,, \tag{5}$$

where $\alpha_k = \dfrac{k\pi}{a}$.

To this solution we shall add in the region ①, between the straight lines $y = 0$, $y = c$ and $x = 0$, $x = a$ the biharmonical function

$$w_1 = \frac{1}{D} \sum_{n=1,3,5\ldots}^{\infty} \alpha_n^{-2} [A_n \operatorname{sh} \alpha_n y + \alpha_n y B_n \operatorname{ch} \alpha_n y] \sin \alpha_n x \tag{6}$$

and in the region ②, between $x = 0$, $x = c$ and $y = 0$, $y = a$ the function

$$w_2 = \frac{1}{D} \sum_{\nu=1,3,5\ldots}^{\infty} \alpha_\nu^{-2} [C_\nu \operatorname{sh} \alpha_\nu x + \alpha_\nu x D_\nu \operatorname{ch} \alpha_\nu x] \sin \alpha_\nu y\,. \tag{7}$$

In our case there is for the reason of symmetry $C_\nu = A_n$ and $D_\nu = B_n$ for $\nu = n$.

In order to fulfil the condition (4) we need a further function, biharmonical in the whole region between both boundaries, having one free parameter. This function is taken in the form of the known solution for the bending of a simply supported plate without a hole loaded by a force P at the center of the plate, the intensity P of the force being considered as the free parameter. In the region ① this solution has the form:

$$w_p = \frac{1}{D} \sum_{p=1,3,5\ldots}^{\infty} \alpha_p^{-2} [Q_p \operatorname{sh} \alpha_p y + \alpha_p y R_p \operatorname{ch} \alpha_p y] \sin \alpha_p x \,, \tag{8}$$

where

$$Q_p = (-1)^{\frac{p-1}{2}} \frac{P}{2 \alpha_p a \operatorname{ch} \frac{\alpha_p a}{2}} \left(1 + \frac{\alpha_p a}{2} \operatorname{th} \frac{\alpha_p a}{2}\right),$$

$$R_p = -(-1)^{\frac{p-1}{2}} \frac{P}{2 \alpha_p a \operatorname{ch} \frac{\alpha_p a}{2}}. \tag{9}$$

Accordingly, the solution of our problem is given in the region ① with

$$w = w_q + w_1 + w_p \tag{10}$$

and in the region ② with

$$w = w_q + w_2 + w_p. \tag{11}$$

All chosen functions, as it is readily seen, fulfil the conditions on the outer boundary, so that the coefficients A_n, B_n and P are determined by the conditions: 1) both functions w_1 and w_2 must be equal in the region common to the regions ① and ②; 2) the edges of the hole must be free, and 3) at the corners of the hole there should be no concentrated forces. Because of the symmetry it is enough if we fulfil these conditions on the part $0 < x < a/2$ of the straight line $y = c$.

The condition which requires w_1 and w_2 to be equal in the common region, will be fulfilled if we have:

$$\frac{\partial^2 w_1}{\partial y^2}=\frac{\partial^2 w_2}{\partial y^2} \quad \text{and} \quad \frac{\partial^3 w_1}{\partial y^3}+2\frac{\partial^3 w_1}{\partial x^2 \partial y}=\frac{\partial^3 w_2}{\partial y^3}+2\frac{\partial^3 w_2}{\partial x^2 \partial y} \tag{12}$$

on the part $0<x<a/2$ of the straight line $y=c$, because similar relations must exist then on the line $x=c$ for the reason of symmetry; and, with the chosen form of functions w_1 and w_2, the conditions

$$w_1=w_2, \quad \frac{\partial^2 w_1}{\partial v^2}=\frac{\partial^2 w_2}{\partial v^2} \tag{13}$$

are secured on the rest of the boundary; and moreover there is

$$\frac{\partial^2 w_1}{\partial x\, \partial y}=\frac{\partial^2 w_2}{\partial x\, \partial y} \tag{14}$$

in the point $x=c$, $y=c$, again for the reason of symmetry, for any A_n and $B_{..}$.[2])

[2]) That (12), (13) and (14) secure the equality of functions w_1 and w_2 in the common region can be seen from simple physical considerations. Namely, this region can be considered as an independent plate with two adjacent simply supported and two free edges. Let this plate at one time be loaded on the free edges with the bending moments $-D\frac{\partial^2 w_1}{\partial v^2}$ and with shearing forces $-D\left(\frac{\partial^3 w_1}{\partial v^3}+2\frac{\partial^3 w_1}{\partial \tau^2 \partial v}\right)$, and at the corner with a concentrated force $-2D\frac{\partial^2 w_1}{\partial \tau\, \partial v}$; and then let the same plate at another time be loaded in the same manner with bending moments $-D\frac{\partial^2 w_2}{\partial v^2}$, shearing forces $-D\left(\frac{\partial^3 w_2}{\partial v^3}+2\frac{\partial^3 w_2}{\partial \tau^2 \partial v}\right)$ and concentrated force $-2D\frac{\partial^2 w_2}{\partial \tau\, \partial v}$. The loading on the plate being the same in both cases, by (12) and (14), it is clear that in both cases the plate must have identical deflections i. e. $w_1=w_2$ in the whole region.

If the plate were rectangular, then the condition (14) would not be identically satisfied. In this case, while constructing the solution, two biharmonical functions should be taken instead of one such function (8), and (14) would appear as an additional condition

The condition which requires the edges of the hole to be free, assumes in view of (10) on the part $c < x < a/2$ of the edge $y = c$ the form

$$\frac{\partial^2 w_1}{\partial y^2} = - \frac{\partial^2 w_q}{\partial y^2} - \frac{\partial^2 w_p}{\partial y^2} , \tag{15}$$

$$\frac{\partial^3 w_1}{\partial y^3} + 2 \frac{\partial^3 w_1}{\partial x^2 \partial y} = - \left(\frac{\partial^3 w_q}{\partial y^3} + 2 \frac{\partial^3 w_q}{\partial x^2 \partial y} \right) - \left(\frac{\partial^3 w_p}{\partial y^3} + 2 \frac{\partial^3 w_p}{\partial x^2 \partial y} \right) .$$

In this way the function w_1 of the region ①, which with chosen form already satisfies the conditions (2) on three edges of the region, has besides to satisfy on the fourth edge, in view of (12) and (15), the conditions[3]):

$$\left[\frac{\partial^2 w_1}{\partial y^2} \right]_{\substack{y=c \\ 0<x<a/2}} = \left[\frac{\partial^2 w_2}{\partial y^2} \right]_{\substack{y=c \\ 0<x<c}} - \left[\frac{\partial^2 w_q}{\partial y^2} \right]_{\substack{y=c \\ c<x<a/2}} - \left[\frac{\partial^2 w_p}{\partial y^2} \right]_{\substack{y=c \\ c<x<a/2}} \tag{16}$$

and

$$\left[\frac{\partial^3 w_1}{\partial y^3} + 2 \frac{\partial^3 w_1}{\partial x^2 \partial y} \right]_{\substack{y=c \\ 0<x<a/2}} =$$

$$= \left[\frac{\partial^3 w_2}{\partial y^3} + 2 \frac{\partial^3 w_2}{\partial x^2 \partial y} \right]_{\substack{y=c \\ 0<x<c}} - \left[\frac{\partial^3 w_q}{\partial y^3} + 2 \frac{\partial^3 w_q}{\partial x^2 \partial y} \right]_{\substack{y=c \\ c<x<a/2}} -$$

$$- \left[\frac{\partial^3 w_p}{\partial y^3} + 2 \frac{\partial^3 w_p}{\partial x^2 \partial y} \right]_{\substack{y=c \\ c<x<a/2}} . \tag{17}$$

The derivatives of the function w_1 appearing on the left side of equations (16) and (17) are given, as it is seen from (6), in the form of

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[3]) If the plate or the opening were rectangular, then we would have to satisfy separately conditions on $y = c$ for w_1 and on $x = c$ for w_2. In this case the coefficients of the series representing w_2 would appear as new unknowns in the conditions (16) and (17), and the coefficients A_n, B_n would enter as unknowns in two similar conditions on $x = c$.

trigonometrical series of sines with the period $2a$. Introducing

$$\overline{A}_n = A_n \ \text{sh}\, \alpha_n c \quad \text{and} \quad \overline{B}_n = B_n \ \text{ch}\, \alpha_n c \ ,$$

in order to simplify further numerical calculations, these derivatives are given with:

$$\left(\frac{\partial^2 w_1}{\partial y^2}\right)_{y=c} = \frac{1}{D} \sum_n \left[\overline{A}_n + \overline{B}_n \ (2 \ \text{th}\, \alpha_n c + \alpha_n c)\right] \ \sin \ \alpha_n x \ , \tag{18}$$

$$\left(\frac{\partial^3 w_1}{\partial y^3} + 2 \frac{\partial^3 w_1}{\partial x^2 \partial y}\right)_{y=c} = \tag{19}$$

$$= - \frac{\pi}{aD} \sum_u n \left[\overline{A}_n \, \text{cth}\, \alpha_n c + \overline{B}_n (-1 + \alpha_n c \ \text{th}\, \alpha_n c)\right] \sin \alpha_n x \, .$$

According to this, in order to fulfil the conditions (16) and (17) by means of comparison of coefficients, we must represent also the right side of these equations in the form of similar trigonometrical series. This can be achieved if we develop in such series all members of series representing the derivatives of the functions w_2 , w_q and w_p on the straight line $y = c$. For this reason we represent the derivatives of w_2 as odd periodic functions with the period $2a$, having for $0 < x < c$ values prescribed by the form of w_2, for $c < x < a/2$ zero values and for $a/2 < x < a$ values symmetrical to those on the part $0 < x < a/2$. In the same manner the derivatives of w_q and w_p are represented as odd functions, having for $0 < x < c$ zero values and for $c < x < a/2$ values prescribed by w_q and w_p.

In this way we have:

$$\left[\frac{\partial^2 w_2}{\partial y^2}\right]_{\substack{y=c \\ 0<x<c}} = - \frac{1}{D} \sum_\nu \left(\overline{A}_\nu \frac{\text{sh}\, \alpha_\nu x}{\text{sh}\, \alpha_\nu c} + \overline{B}_\nu \frac{\alpha_\nu x \ \text{ch}\, \alpha_\nu x}{\text{ch}\, \alpha_\nu c}\right) \sin \ \alpha_\nu c =$$

$$= - \frac{1}{D} \sum_\nu \left[\overline{A}_\nu \left(\sum_n b_{\nu n} \sin \alpha_n x\right) + \overline{B}_\nu \left(\sum_n c_{\nu n} \sin \alpha_n x\right)\right] \ \sin \ \alpha_\nu c \ ,$$

where $b_{\nu n}$ and $c_{\nu n}$ are Fourier's coefficients:

$$b_{\nu n}=\frac{4}{a\,\mathrm{sh}\,\alpha_\nu c}\int_0^c \mathrm{sh}\,\alpha_\nu x\,\sin\alpha_n x\,dx\ ,$$

$$c_{\nu n}=\frac{4\,\alpha_\nu}{a\,\mathrm{ch}\,\alpha_\nu c}\int_0^c x\,\mathrm{ch}\,\alpha_\nu x\,\sin\alpha_n x\,dx\ .$$

After the interchange of the order of summation there is

$$\left[\frac{\partial^2 w_2}{\partial y^2}\right]_{\substack{y=c\\0<x<c}}=\frac{1}{D}\sum_n\left[\left(\sum_\nu \bar{A}_\nu\, b_{n\nu}\sin\alpha_\nu c\right)+\left(\sum_\nu \bar{B}_\nu c_{n\nu}\sin\alpha_\nu c\right)\right]\sin\alpha_n x\,. \quad (20)$$

Likewise:

$$\left[\frac{\partial^3 w_2}{\partial y^3}+2\,\frac{\partial^3 w_2}{\partial x^2\,\partial y}\right]_{\substack{y=c\\0<x<c}}= \quad (21)$$

$$=\frac{1}{D}\sum_\nu \alpha_\nu\,\{\bar{A}_\nu[\sum_n b_{\nu n}\sin\alpha_n x]+\bar{B}_\nu[\sum_n(4\,b_{\nu n}\,\mathrm{th}\,\alpha_\nu c+c_{\nu n})\sin\alpha_n x]\}\cos\alpha_n c=$$

$$=\frac{\pi}{aD}\sum_n\{[\sum_\nu \nu\,\bar{A}_\nu b_{n\nu}\cos\alpha_\nu c]+[\sum_\nu \nu\bar{B}_\nu\cos\alpha_\nu c\,(4\,b_{n\nu}\,\mathrm{th}\,\alpha_\nu c+c_{n\nu})]\}\sin\alpha_n x\,.$$

In the same manner we have:

$$\left[\frac{\partial^2 w_q}{\partial y^2}\right]_{\substack{y=c\\c<x<a/2}}=-\frac{qa^2}{D\pi^2}\sum_k\frac{1}{k^2}\left[\left(\mathrm{ch}^{-2}\frac{\alpha_k a}{2}-2\frac{c}{a}\right)\mathrm{sh}\,\alpha_k c+2\frac{c}{a}\ \mathrm{th}\,\frac{\alpha_k a}{2}\mathrm{ch}\,\alpha_k c\right]\sin\alpha_k x=$$

$$=-\frac{qa^2}{D\pi^2}\sum_k M_k\sin\alpha_k x=-\frac{qa^2}{D\pi^2}\sum_k M_k\,(\sum_n a_{kn}\sin\alpha_n x)\,,$$

where

$$a_{kn} = \frac{4}{a} \int_c^{a/2} \sin \alpha_k x \sin \alpha_n x \, dx \ .$$

Changing the order of summation we get:

$$\left[\frac{\partial^2 w_q}{\partial y^2} \right]_{\substack{y=c \\ c<x<a/2}} = - \frac{qa^2}{D\pi^2} \sum_n \left(\sum_k M_k a_{nk} \right) \sin \alpha_n x \ . \tag{22}$$

Likewise:

$$\left[\frac{\partial^3 w_q}{\partial y^3} + 2 \frac{\partial^3 w_q}{\partial x^2 \partial y} \right]_{\substack{y=c \\ c<x<a/2}} = \frac{qa}{D\pi} \sum_k \frac{1}{k} \left[\left(\mathrm{ch}^{-2} \frac{\alpha_k a}{2} - \frac{6}{\alpha_k a} \mathrm{th} \frac{\alpha_k a}{2} - \frac{c}{a} \right) \mathrm{ch}\, \alpha_k c + \right.$$

$$\left. + \left(\frac{c}{a} \mathrm{th} \frac{\alpha_k a}{2} + \frac{6}{\alpha_k a} \right) \mathrm{sh}\, \alpha_k c \right] \sin \alpha_k x =$$

$$= \frac{\pi}{aD} \frac{qa^2}{\pi^2} \sum_k N_k \sin \alpha_k x =$$

$$= \frac{\pi}{aD} \frac{qa^2}{\pi^2} \sum_k N_k \left(\sum_n a_{kn} \sin \alpha_n x \right) =$$

$$= \frac{\pi}{aD} \frac{qa^2}{\pi^2} \sum_n \left(\sum_k N_k a_{nk} \right) \sin \alpha_n x \ . \tag{23}$$

Finally for the derivatives of the function w_p we have:

$$\left[\frac{\partial^2 w_p}{\partial y^2} \right]_{\substack{y=c \\ c<x<a/2}} = \frac{P}{D} \sum_p \frac{(-1)^{\frac{p-1}{2}}}{2 \alpha_p a \, \mathrm{ch} \frac{\alpha_p a}{2}} \left[\left(-1 + \frac{\alpha_p a}{2} \mathrm{th} \frac{\alpha_p a}{2} \right) \mathrm{sh}\, \alpha_p c - \alpha_p c \, \mathrm{ch} \alpha_p c \right] \sin \alpha_p x =$$

$$= \frac{P}{4D} \sum_p K_p \sin \alpha_p x = \frac{P}{4D} \sum_n \left(\sum_p K_p a_{np} \right) \sin \alpha_n x \tag{24}$$

and

$$\left[\frac{\partial^3 w_p}{\partial y^3} + 2\frac{\partial^3 w_p}{\partial x^2 \partial y}\right]_{\substack{y=c \\ c<x<a/2}} =$$

$$= -\frac{P}{D}\sum_p \frac{(-1)^{\frac{p-1}{2}}}{2a\,\mathrm{ch}\frac{\alpha_p a}{2}}\left[\left(2+\frac{\alpha_p a}{2}\,\mathrm{th}\,\frac{\alpha_p a}{2}\right)\mathrm{ch}\,\alpha_p c - \alpha_p c\,\mathrm{sh}\,\alpha_p c\right]\sin\alpha_p x =$$

$$= -\frac{\pi}{aD}\frac{P}{4}\sum_p L_p \sin\alpha_p x = -\frac{\pi}{aD}\frac{P}{4}\sum_n \left(\sum_p L_p a_{np}\right)\sin\alpha_n x\ . \tag{25}$$

With (18) – (25) the conditions (16) and (17) become:

$$\frac{1}{D}\sum_n\left[\bar{A}_n + \bar{B}_n\,(2\,\mathrm{th}\,\alpha_n c + \alpha_n c)\right]\sin\alpha_n x =$$

$$= -\frac{1}{D}\sum_n\left[\left(\sum_\nu \bar{A}_\nu b_{n\nu}\sin\alpha_\nu c\right) + \left(\sum_\nu \bar{B}_\nu c_{n\nu}\sin\alpha_\nu c\right)\right]\sin\alpha_n x -$$

$$-\frac{P}{4D}\sum_n\left(\sum_p K_p a_{np}\right)\sin\alpha_n x + \frac{qa^2}{D\pi^2}\sum_n\left(\sum_k M_k a_{nk}\right)\sin\alpha_n x \tag{26}$$

and

$$-\frac{\pi}{aD}\sum_n n\left[\bar{A}_n\,\mathrm{cth}\,\alpha_n c + \bar{B}_n(-1 + \alpha_n c\,\mathrm{th}\,\alpha_n c)\right]\sin\alpha_n x =$$

$$= \frac{\pi}{aD}\sum_n\left\{\left[\sum_\nu \bar{A}_\nu \nu\, b_{n\nu}\cos\alpha_\nu c\right] + \left[\sum_\nu \bar{B}_\nu \nu\cos\alpha_\nu c\,(4\,b_{n\nu}\,\mathrm{th}\,\alpha_\nu c + c_{n\nu})\right]\right\}\sin\alpha_n x +$$

$$+\frac{\pi}{aD}\frac{P}{4}\sum_n\left(\sum_p L_p a_{np}\right)\sin\alpha_n x - \frac{\pi}{aD}\frac{qa^2}{\pi^2}\sum_n\left(\sum_k N_k \alpha_{nk}\right)\sin\alpha_n x\ . \tag{27}$$

These conditions will be satisfied only if for every $n\,(n=1,3,5\ldots)$:

$$\sum_\nu \bar{A}_\nu k_{n\nu} + \sum_\nu \bar{B}_\nu \varkappa_{n\nu} + a_n\frac{P}{4} = b_n\frac{qa^2}{\pi^2}\,, \tag{28}$$

$$\sum_\nu \bar{A}_\nu l_{n\nu} + \sum_\nu \bar{B}_\nu \lambda_{n\nu} + c_n\frac{P}{4} = d_n\frac{qa^2}{\pi^2}\,, \tag{29}$$

where the meaning of the coefficients $k_{n\nu}$, $\varkappa_{n\nu}$ etc. is obvious from (26) and (27). With (28) and (29) we have obtained two infinite systems of algebraic linear equations with the unknowns A_n, B_n and P. It is seen that all unknowns appear in all equations of both systems. The coefficients $\bar{A}_n$ and $\bar{B}_n$ converge rapidly enough, so that in numerical calculations a relatively small number of terms has to be taken.

For the complete solution of the problem we need one more equation, which is given by the condition (4). This condition explicitely written has the form

$$\frac{\partial^2 w_1}{\partial x \partial y} + \frac{\partial^2 w_p}{\partial x \partial y} + \frac{\partial^2 w_q}{\partial x \partial y} = 0,$$

or with (6), (5) and (8):

$$\frac{1}{D} \sum_n \left[\bar{A} \operatorname{cth} \alpha_n c + \bar{B}_n (1 + \operatorname{th} \alpha_n c) \right] \cos \alpha_n c +$$

$$+ \frac{P}{D} \sum_p \frac{(-1)^{\frac{p-1}{2}}}{2\, \alpha_p \operatorname{ch} \frac{\alpha_p a}{2}} \left[\frac{\alpha_p a}{2} \operatorname{th} \frac{\alpha_p a}{2} \operatorname{ch} \alpha_p c - \alpha_p c \operatorname{sh} \alpha_p c \right] \cos \alpha_p c -$$

$$- \frac{qa^2}{D\pi^2} \sum_k \frac{2}{\pi k^3} \left[\left(1 + \alpha_k c \operatorname{th} \frac{\alpha_k a}{2} \right) \operatorname{sh} \alpha_k c - \right. \tag{30}$$

$$\left. - \left(\alpha_k c + \operatorname{th} \frac{\alpha_k a}{2} - \frac{\alpha_k a}{2} \operatorname{ch}^{-2} \frac{\alpha_k a}{2} \right) \operatorname{ch} \alpha_k c \right] \cos \alpha_k c = 0 \ .$$

Introducing here notations the meaning of which is seen from (30) we can write:

$$\sum_\nu \bar{A}_\nu m_\nu + \sum_\nu \bar{B}_\nu \mu_\nu + r_p \frac{P}{4} = r_q \frac{qa^2}{\pi^2} \ . \tag{31}$$

The total number of equations obtained from the conditions (28), (29) and (31) is equal to the number of unknown coefficients $\bar{A}_n$, $\bar{B}_n$ and P. In this way our problem is solved.

We can mention here that this solution is in principle applicable as well to the cases of rectangular plate with a hole in arbitrary position and under arbitrary loading, as to the different boundary conditions on the inner or the outer boundary. But this would require in some cases a fair amount of numerical work. Therefore it would be perhaps of interest to tabulate the distribution of bending moments in the plate for some special cases interesting in the practice.

As an example for the application of this solution we have done numerical calculations for the ratio $c/a = 1/3$. In the calculations we have taken six first members of the series representing w_1 and w_2. The results obtained are shown in the table 1 and on figs. 2 and 3. The table 1 gives the values of the calculated coefficients $\bar{A}_n$, $\bar{B}_n$ and P. Fig. 2 represents the deflections along $y=c$, compared with the deflections of a full plate along the same straight line (represented by the dotted line). Fig 3 shows the calculated values of bending moments M_x and M_y along $y=c$ (the bending moments M_x in the full plate along $y=c$ are represented by a dotted line for comparison). As it is seen from fig. 3 the calculated value of M_y for $x=c$ is equal to half the value of M_x on the same place. In fact, immediately left of this place there ought to be $M_y = M_x$ and immediately right $M_y = 0$. This appears because the function M_y, which at this place has a finite spring, is represented by a trigonometrical series.

n	$\bar{A}_n$	$\bar{B}_n$	
1	+3,411	−1,714	
3	−1,301	+0,206	
5	+0,605	−0,070	
7	−0,328	+0,035	
9	+0,204	−0,020	
11	−0,081	+0,007	
	$P/4=4{,}523$		$\times \frac{qa^2}{\pi^2}$

Table 1.

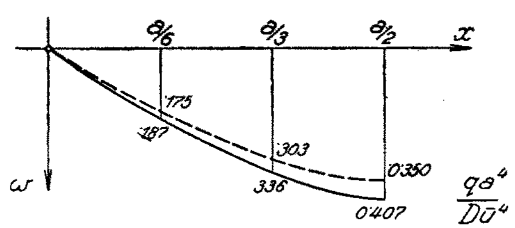

Fig. 2

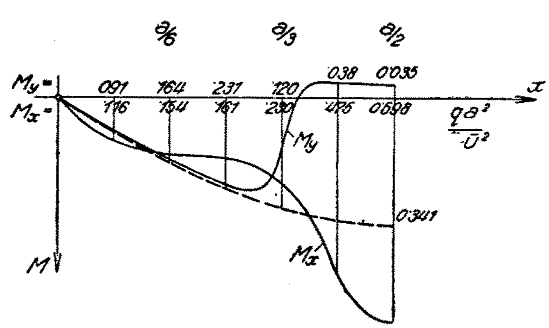

Fig. 3

It is interesting to mention here, that Föppl in the quoted place considered, that by the solution of this problem the corners of the hole ought to be rounded "as otherwise too great stresses should be expected at these places". In our calculations we did not get these great stresses, we rather obtained only a finite spring of bending moments and shearing forces. In fact, it appears that the stresses in the corners of the hole, which probably are very great, cannot at all be found by the approximate theory of plates, used in this solution, because the distribution of stresses in these places cannot be represented by stress-resultants and stress-couples. It appears therefore, that the stresses in the vicinity of these points could be found only by the exact theory of plates, which would be another interesting, but much more complicated problem.*

* While this paper was being printed I was notified that our problem was discussed by E. Wiedemann (*Ingenieurarchiv* Vol. VII (1936), № 1 and 3. pp 56—70 and 196—202). Wiedemann is making use of the finite differences method for the solution and in a numerical example takes the same ratio 1 : 3, as we do, between the lengths of the inner and the outer edges. It seems that our numerical results differ fundamentally. We shall refer to this question later on in a separate note.