ON A MODEL OF A HYDROMECHANICAL CELL WITH TWO DIPOLES

by

N. M. OBRADOVIĆ and K. P. WORONETZ (Beograd)

This paper was induced by biologists[1]) who asked the authors to find out a hydromechanical explanation and a mathematical expression for phenomena taking place in the living organic cell during division, when characteristic flows occur in its cytoplasm.

We have tried to conceive a kinematic model in the idealized hydromechanical cell with two dipoles as well as to express by hydromechanical and mathematical means the regularity of flow and directions of flow in such a cell and the inevitability of its division into two equal streamline bodies. This body conceived by hydromechanical means has an imaginary impenetrable wall which keeps back the flow within the encircled space. It was further assumed that this streamline body takes different shapes varying with physical parameters which define the flow in this idealized cell.

Let us assume that the wall of our model has the form of a revolving body. This assumption enables the flow in this imaginary cell to be considered axially symmetrical, i. e. identical in all meridional planes. This requires the dipoles of the cell to be of equal moments and symmetrically distributed on the axis of symmetry, therefore their number must be even. Otherwise the streamline body could not divide into two equal parts. We have assumed the simplest possibility so that there are only two dipoles in our cell model. This represents the required idealization. It is now necessary to effect the dipole distribution in the streamline body. Symmetry requires this to be done according to the scheme $-+$, $-+$. We shall now superpose a parallel flow from right to left over the flow produced by the dipoles. In our dipole distribution, the outer source is discharging

[1]) Prof. Dr. B. Milojević and Dr. Filipović

the fluid into the outer sink, and the inner source into the inner sink. Due to the acceleration of each of these flows a reaction takes place contrary to the direction of flow. The dipoles will therefore move along the axis of the streamline body increasing their distance.

Thus our problem is reduced to the investigation of the axially symmetrical flow of an ideal fluid in the field of steady flow with two equally disposed dipoles. Let L denote the dipole moment which equals the product of the source intensity ε and the distance l between source and sink. The dipole moment is a finite quantity even when $\varepsilon \to \infty$ and $l \to 0$.

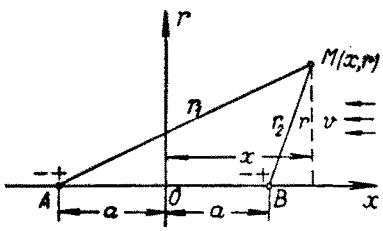

Fig. 1.

Superposition of a steady flow of velocity v_0 (Fig. 1) in the negative x-axis direction over the flow around the dipoles yields the flow potential φ [1]

$$\varphi = -\frac{L}{4\pi}\left(\frac{x+a}{r_1^3} + \frac{x-a}{r_2^3} + \frac{4x}{c^3}\right),$$

where a represents the distance of the dipoles from the center of symmetry, $c^3 = L/\pi v_0$, while r_1 and r_2 are defined by the expressions

$$r_1^2 = (x+a)^2 + r^2, \qquad r_2^2 = (x-a)^2 + r^2 .$$

Introducing Stokes' stream function ψ for axially symmetrical flow [2]

$$v_x = \frac{\partial\varphi}{\partial x} = -\frac{1}{r}\frac{\partial\psi}{\partial r}, \qquad v_r = \frac{\partial\rho}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial x}$$

and the non-dimensional quantities

$$\frac{a}{c} = b, \quad \frac{x}{c} = y, \quad \frac{r}{c} = z, \quad \frac{r_1}{c} = R_1, \quad \frac{r_2}{c} = R_2,$$

yields the following expression for the functions φ and ψ

$$\varphi = -\frac{cv_0}{4}\left(\frac{y+b}{R_1^3} + \frac{y-b}{R_2^2} + 4y\right),$$

$$\psi = -\frac{c^2 v_0 z^2}{4}\left(\frac{1}{R_1^3} + \frac{1}{R_2^3} - 2\right), \tag{1}$$

which enables the calculation of the ratios v_x and v_r (projections of v on the x-axis and r-axis):

$$\frac{v_x}{v_0} = \frac{1}{4}\left[\frac{2(y+b)^2 - z^2}{R_1^5} + \frac{2(y-b)^2 - z^2}{R_2^5} - 4\right],$$

$$\frac{v_r}{v_0} = \frac{3z}{4}\left(\frac{y+b}{R_1^5} + \frac{y-b}{R_2^5}\right). \tag{2}$$

By analysing equations (1) and (2) it is possible to deduce the following on the flow in the cell.

I. Zero Streamline

1. — The zero streamline is defined by equation $z^2 = 0$, or

$$[(y+b)^2+z]^{-3/2} + [(y-b)^2+z^2]^{-3/2} - 2 = 0 \tag{3}$$

The zero streamline consists therefore of the closed curve (3) and parts of the x-axis from $\pm\infty$ up to the intersection with that curve. The shape of the curve depends from the parameter b which is proportional to the distance of the dipoles from the center of symmetry. Since the zero streamline represents the boundary for the flow in the cell it follows that the cell wall is impenetrable for the fluid within and outside of the cell.

2. — The zero streamline intersects the r-axis at the point where $z^2 = 1 - b^2$, thus only for values of the parameter b within the interval $0 \leqslant b^2 \leqslant 1$ does the streamline body represent a single body. For greater values of the parameter b the streamline body would decompose into two separate bodies. Corresponding values of z^2 are enclosed within the interval $1 \geqslant z_1{}^2 \geqslant 0$, the value $b^2 = 0$ corresponding to the initial phase when both dipoles are in the origin of the system of coordinates, i. e. in the center of symmetry. The value $b^2 = 1$ corresponds to the final phase of the division prior to the cell separation. In this case there is also a third point of intersection of the curve (3) with the x-axis at the origin of the coordinate system. For greater values of b^2 we would obtain two separate bodies (four points of intersection of the zero streamline and the x-axis). A further increase of the parameter b^2 would enlarge the distance between the bodies while their shape would approach to that of the sphere.

3. — The zero streamline has its extreme at the point of intersection with the r-axis since in this point the projection of the velocity $v_r = 0$.

As for $x = 0$

$$\left(\frac{d^2r}{dx^2}\right)_{extr} = \frac{(5b^2 - 1)\sqrt{1 - b^2}}{c\,(1 - b^2)} ,$$

the zero streamline has its maximum for $b^2 < 1/5$ and its minimum for $b^2 > 1/5$. In the latter case the zero streamline must have at least one more maximum in each quadrant, which is also evident from equation $v_r = 0$.

Fig. 2 shows the shapes of the zero streamline for different values of the parameter b^2.

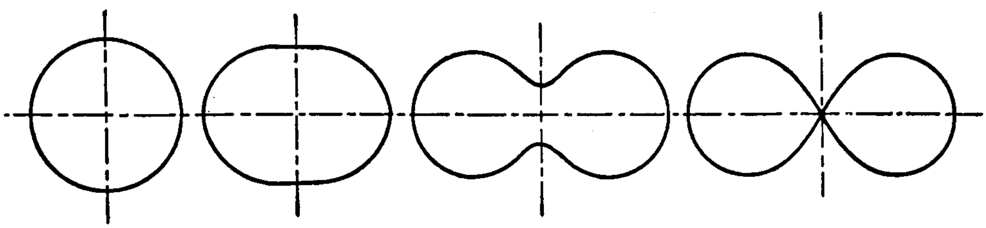

Fig. 2.

II. Velocity

1. — The velocity projection v_r is zero on the x-axis, on the r-axis and in the points of the curve defined by equation $v_r = 0$. The zero streamline does not intersect this curve for all values of the parameter b^2. Thus it has only a maximum or a maximum and a minimum in a quadrant.

2. — The projection of the velocity v_x on the r-axis is given by the expression

$$\frac{v_x}{v_0} = \frac{1}{2}\left[\frac{2\,b^2 - z^2}{(b^2 + z^2)^{5/2}} - 2\right] ,$$

thus in the origin of the coordinate system

$$\frac{v_x}{v_0} = \frac{1 - b^3}{b^3} \geqslant 0 ,$$

and in the intersection of the zero streamline and the r-axis

$$\frac{v_x}{v_0} = \frac{3}{2}(b^2 - 1) \leqslant 0 .$$

The velocity therefore changes its direction on the r-axis in the streamline body, i. e. it passes somewhere through zero. The point with zero velocity corresponds to the value of the z-root of the equation

$$2\, b^2 - z^2 = 2\, (b^2 + z^2)^{5/2}.$$

3. — The projection of the velocity v_x on the r-axis equals v_0 for $z^2 = 2\, b^2$ and in infinity, within these limits the velocity must reach its extreme (in absolute value). This point corresponds to equation $z^2 = 4\, b^2$ and the maximum velocity

$$\frac{v_r}{v_0} = -\left(1 + \frac{\sqrt{5}}{125}\frac{1}{b^3}\right).$$

The zero streamline passes through the point of maximum velocity when $b^2 = 1/5$ and $z^2 = 4/5$. For $b^2 > 1/5$ the point with the greatest velocity emerges from the streamline body, but for $b^2 < 1/5$ it again moves back into it. Since lower pressures correspond to greater velocities and viceversa, it is possible to deduce from the characteristics of the flow defined by 2) and 3) some interesting conclusions about the pressure distribution in and outside of the cell during division.

Conclusion

The analysis has shown that the shape of the streamline body depends from the parameter b which is proportional to the distance between the dipoles and the center of symmetry of the revolving body. When $b^2 = 0$ both dipoles are located in the center of symmetry and the streamline body is a sphere. An increase in the parameter b^2 enlarges the dipole distance from the center of symmetry and the streamline body gradually elongates in the direction of the x-axis and flattens in the r-direction. This process continues until the parameter b^2 equals 1/5. Then the streamline body begins to contract normal to the x-axis and for $b^2 = 1$ the neck joins on the x-axis. The streamline body is then composed of two separate bodies which have a common point in the center of symmetry. A further increase of the parameter would bring about a separation and parting of the bodies while retaining their symmetrical shape. It is not our task to investigate the reasons causing the separation of the dipoles. We might mention that the pressure is increasing from the center of symmetry of the body up to the point where it reaches its maximum. This point is always located within the body and the velocity in it equals zero. Further in the direction of the r-axis the pressure is decreasing and reaches its minimum for $z^2 = 4b^2$.

It might be interesting to perform direct measurements of the dimensions of a living cell during division, for instance by means of motion picture, and compare our results with actual. This would enable the determination of the hydromechanic parameters we introduced, in case we wish to make an analogy between our model and the living organic cell.

Our work might incite biologists to endeavour to explain the cell division by means of the hydromechanical principles we laid down, which in our opinion must govern the division of an idealized cell. Moreover, there are analogies between our model and the organic cell during division. These analogies are the following:

1. — Before the division both the hydromechanical cell as well as the organic cell are spherically shaped.

2. — The fluid contained within the boundaries of our model might correspond to the fluid and colloid state of the cytoplasm.

3. — The zero streamline of the hydromechanical cell could correspond to the cytoplasm wall of the cell.

4. — The dipoles of our model could correspond to the polar forms (centrosomes), however it is still unsettled either these are also dipoles or they produce the same directions of flow we established during the analysis of the model.

5. — During the division of the hydromechanical cell the dipoles are moving from the center towards the surface of the sphere while the centrosomes have a similar movement during cell division.

6. — The movement of the dipoles elongates the hydromechanical cell towards the poles and at the same time brings about a contraction in the equatorial plane. Both these phenomena do not take place simultaneously from the very beginning of the division of the living cell.

7. — Upon completed division the hydromechanical cell too consists of two equal bodies with separate boundaries.

REFERENCES

[1] H. Lamb — Lehrbuch der Hydrodynamik, 1931, p. 65

[2] G. Stokes — On the Steady Motion of Incompressible Fluids. *Camb. Trans*, **VII** (1842).