ENERGY CRITERION OF ELASTIC STABILITY FOR THIN SHELLS

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SUMMARY

In the first six paragraphs the energetic criterion of elastic stability is discussed and the differential equation for the critical vector of displacement in orthogonal curvilinear coordinates is derived. Together with this equation, the boundary condition on those parts of the boundary is given, where the displacements are free. In the last two paragraphs the corresponding basic expressions for thin shells are determined, under the assumption that plane sections remain plane and that surface forces behave as hydrostatic pressure, when the body is passing over from its critical position of equilibrium to a neighbouring one.

Introduction

The determination of critical stresses in thin elastic bodies (rods, arches, plates, and shells) by the use of energetic considerations is very old. It is well known that it was first employed by G. H. Bryan in 1888 for the determination of critical stresses in a compressed plate (*Proc. Cambr. Phil. Soc.* VI.). Later this method was developed mostly by prof. S. Timoshenko in a series of papers, which are for the most part included in his well-known work on the theory of elastic stability [6]. H. Reissner improved this method theoretically in some detail [5], but it was E. Trefftz [8], who first gave a general theory of elastic stability based on energetic considerations, without taking into account the dimensions of the body. But the application of these methods to stability questions of shells is very uncertain and connected with great difficulties, because some quantities are given in Cartesian coordinates only and for other coordinates there are known only their first order terms. Their extension to curvilinear coordinates and to the second order terms is not always quite easy and unambiguous.

In the first part of our paper we shall therefore extend Trefftz's method to curvilinear orthogonal coordinates and establish general equations for critical stresses in elastic bodies. In the second part we shall show how to use the general theory to determine the critical loads in thin shells, which are obeying the Bernoulli-Navier rule of plane sections.

1. General considerations on the stability of elastic equilibrium.

Under the influence of external loads all elastic bodies change their shape. Usually, we are able to detemine the deformations of bodies under given loads and boundary conditions only approximately, according to rules of the strength of materials, because the corresponding problem of the theory of elasticity cannot be solved exactly. In our paper we suppose that these deformations of the body, after the loading has been applied, are determined and we shall call the position of the deformed body *the initial position.* If the deformations of the body under the influence of loads are small, there will be no great difference between the so defined initial position and the position before the loading was applied; but generally the deformations during the displacement of the body from its *natural position* (before the loading) to the initial position cannot be omitted, because by doing thus we would overlook some known phenomena of elastic instability, which are caused just by these deformations. A characteristic example for such an occurence is the instability of a round tube bent by two couples, where the initially circular cross section becomes more and more oval. We investigate the stability of the inital state of an elastic body in the following way.

We suppose that all particles of the body move from their initial position to another position, which we shall call for brevity the *neighbouring position.* The displacements of particles during this movement we suppose to be very small (but not infinitesimal), and to satisfy to all boundary conditions. In this paper we shall deal only with such displacements and therefore we call them simply displacements without any supplementary denomination. Let the external forces during the movement of the body from its initial position to the neighbouring one do the work ΔA, and let the increment of the elastic energy of the body be ΔE. The initial position of the body will be stable, if the work of the external forces is too small to cover the increase of the elastic energy for all possible displacements. But when the work of the external forces for one displacement at least is equal to the increase of the elastic energy, the initial position is indifferent and the loads have reached their critical value, because the body itself can move from the initial position to at least one neighbouring position without any external influence. Only those parts $\Delta^{(2)} A$ and $\Delta^{(2)} E$ of the work of external forces and of the elastic energy which are of the second order in displacements are to be considered, as the first variation of the potential energies of internal and external forces in the position of equilibrium is equal to zero.

When the loads have reached their critical values, such displacements $\vec{\rho}$ do therefore exist for which

$$\Delta^{(2)} E = \Delta^{(2)} A \quad ;$$

while for other displacements there still remains the inequality

$$\Delta^{(2)} E > \Delta^{(2)} A \quad .$$

It follows from these considerations that the functional

$$Q(u_i) \equiv \Delta^{(2)} E - \Delta^{(2)} (A) \tag{1}$$

reaches its minimum at the critical value of loads and its variation must therefore be zero for at least one set of displacements, which we shall call the critical displacements. From equation

$$\delta Q(u_i) = 0 \tag{2}$$

we get a system of differential equations for critical displacements, from which we can also derive the values of critical loads or of critical initial stresses. It is the purpose of the first part of our paper to establish the general form of the functional $Q(u_i)$ and to derive equations for the critical displacements and stresses from it.

2. Displacements

We designate the three curvilinear orthogonal coordinates by α_1, α_2, α_3. Let a point P of the elastic body in the initial state be given by the position vector

$$\vec{R} = \sum_{j=1}^{3} x_j \, i_j \ , \tag{3}$$

where i_j are the unit vectors of a Cartesian system of coordinates. The Cartesian coordinates x_1, x_2, x_3 of the point are functions of the three curvilinear coordinates:

$$x_j = x_j \ (\alpha_1, \alpha_2, \alpha_3) \ . \tag{3'}$$

Introducing three fundamental coefficients

$$H_i = + \sqrt{\frac{\partial \vec{R}}{\partial \alpha_i} \cdot \frac{\partial \vec{R}}{\partial \alpha_i}}, \qquad (i=1, 2, 3) \tag{4}$$

and fundamental unit vectors along the coordinate lines

$$\vec{E}_i = \frac{1}{H_i} \frac{\partial \vec{R}}{\partial \alpha_i}, \qquad (i=1, 2, 3) \tag{5}$$

we obtain for the vector $d\vec{R}$, connecting the point P $(\alpha_1, \alpha_2, \alpha_3)$ with the point Q $(\alpha_1 + d\alpha_1,\ d_2 + d\alpha_2,\ \alpha_3 + d\alpha_3)$, the form

$$d\vec{R} = \sum_{i=1}^{3} \frac{\partial \vec{R}}{\partial \alpha_i} \cdot d\alpha_i = \sum_{i=1}^{3} H_i \vec{E}_i \, d\alpha_i \ . \tag{6}$$

As nearly all summations in this paper are to be taken from one to three, we shall usually drop the designation of the summation limits and shall mark them only if the sum is not to be taken between those limits.

For applications, it is very important to know the projections of the derivatives of the fundamental unit vectors $\vec{E}_i$ with respect to the coordinates α_1, α_2, α_3. These derivates are given by the equations (see e. g. Lagally, Vektorrechung, pp. 94—96) of the type

$$\begin{aligned} \frac{\partial \vec{E}_1}{\partial \alpha_1} &= -\frac{1}{H_2} \cdot \frac{\partial H_1}{\partial \alpha_2} \vec{E}_2 - \frac{1}{H_3} \frac{\partial H_1}{\partial \alpha_3} \vec{E}_3 \ , \\ \frac{\partial \vec{E}_1}{\partial \alpha_2} &= \frac{1}{H_1} \frac{\partial H_2}{\partial \alpha_1} \vec{E}_2 \ , \\ \frac{\partial \vec{E}_1}{\partial \alpha_3} &= \frac{1}{H_1} \frac{\partial H_3}{\partial \alpha_1} \vec{E}_3 ; \end{aligned} \tag{7}$$

the equations for derivatives of the other two vectors follow immediately through cyclic permutation of the indices.

Let the point P come into the position P_1 during its movement from the initial to the neighbouring state and let the displacement be given by the vector

$$\vec{\rho} = \sum_i u_i \vec{E}_i , \tag{8}$$

where the components u_1, u_2, u_3 are functions of the curviliniear coordinates α_1, α_2, α_3:

$$u_i = u_i(\alpha_1, \alpha_2, \alpha_3) , \quad (i=1, 2, 3) . \tag{8'}$$

The position vector to the point P_1 will be

$$\vec{r} = \vec{R} + \vec{\rho} , \tag{9}$$

so that the former vector $\overrightarrow{PQ} = d\vec{R}$ changes to $d\vec{r} = \overrightarrow{P_1Q_1}$ where

$$d\vec{r} = d\vec{R} + \sum_i \frac{\partial \vec{\rho}}{\partial \alpha_i} d\alpha_i = \sum_i H_i \left(\vec{E}_i + \frac{1}{H_i} \frac{\partial \vec{\rho}}{\partial \alpha_i} \right) d\alpha_i .$$

Introducing three new vectors

$$\vec{e}_i = \vec{E}_i + \frac{1}{H_i} \frac{\partial \vec{\rho}}{\partial \alpha_i} , \qquad (i=1, 2, 3) \tag{10}$$

which we call net vectors and which are nearly equal to the unit vectors $\vec{E}_i$ if the displacements are small, we can write the equation for $d\vec{r}$ also in the form:

$$d\vec{r} = \sum H_i \cdot \vec{e}_i \cdot d\alpha_i , \tag{11}$$

where the right hand term has the same form as the expression for $d\vec{R}$ in (6).

It follows from the orthogonality of the fundamental unit vectors, that the square of the line segment $\overrightarrow{PQ}$ is given by

$$d\vec{R}^2 = \sum_i H_i^2 \, d\alpha_i^2 = \sum_i ds_i^2 , \tag{12}$$

if we denote for brevity

$$ds_i = H_i \, d\alpha_i . \tag{12'}$$

For the square of the line segment $\overline{P_1Q_1}$ we get from (11)

$$\vec{dr}^2 = \sum_i \sum_j H_i H_j \vec{e}_i \cdot \vec{e}_j \, d\alpha_i \, d\alpha_j = \sum_\mu \sum_\nu g_{ij} \, ds_i \, ds_j \tag{13}$$

and for the coefficients g_{ij} of the last quadratic form we obtain from (10) the expression

$$g_{ij} = \vec{e}_i \cdot \vec{e}_j = \vec{E}_i \cdot \vec{E}_j + \frac{1}{H_i} \frac{\partial \vec{\rho}}{\partial \alpha_i} \cdot \vec{E}_j + \frac{1}{H_j} \frac{\partial \vec{\rho}}{\partial \alpha_j} \cdot \vec{E}_i + \frac{1}{H_i H_j} \frac{\partial \vec{\rho}}{\partial \alpha_i} \cdot \frac{\partial \vec{\rho}}{\partial \alpha_j} \; . \tag{14'}$$

If we designate the coefficients of the corresponding quadratic form for $\overline{PQ}^2$ by G_{ij}, we get from (12)

$$G_{ij} = \vec{E}_i \cdot \vec{E}_j = \delta_{ij} \, , \tag{14''}$$

where $\delta_{ij} = 1$ for $i = j$, and $\delta_{ij} = 0$ for $i \neq j$. The variation of coefficients of the square of the line segment $\overline{PQ}$ during its movement from the initial to the neighbouring position is therefore

$$\Delta g_{ij} = g_{ij} - G_{ij} = \frac{1}{H_i} \frac{\partial \vec{\rho}}{\partial \alpha_i} \cdot \vec{E}_j + \frac{1}{H_j} \frac{\partial \vec{\rho}}{\partial \alpha_j} \cdot \vec{E}_i + \frac{1}{H_i H_j} \frac{\partial \vec{\rho}}{\partial \alpha_i} \cdot \frac{\partial \vec{\rho}}{\partial \alpha_j} \; . \tag{15}$$

The first two right-hand terms thereby are evidently linear in the displacement $\vec{\rho}$, while the last one is of the second order.

3. Stresses and elastic energy.

We decompose the stress vector $\vec{p}_h$, acting across the surface element $H_i H_j \, d\alpha_i \, d\alpha_j$ $(i, j \neq h)$, into components in the directions of the net vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$

$$\vec{p}_h = \sum_i p_{hi} \vec{e}_i \tag{16}$$

and we define $\vec{p}_h$ as the quotient of the force, divided by the inital area of the element $H_i. H_j. d\alpha_i. d\alpha_j$.

When the displacements are not infinitesimal, the vectors $\vec{e}_i$ are not unit vectors and so the quantities p_{hi} are not the components of the stress in the usual sense of word. But we shall see that by introducing these components, we obtain relatively simple expressions for all quantities; moreover, as these components are approaching the values of the stress components in the usual sense, if the neighbouring state is approaching the initial state, there can be no serious objections against their introduction.

Let us consider a curved element of the elastic body, bounded by the surfaces $\alpha_1 \pm \frac{1}{2} d\alpha_1 = \text{const.}$, $\alpha_2 \pm \frac{1}{2} d\alpha_2 = \text{const.}$, $\alpha_3 \pm \frac{1}{2} d\alpha_3 = \text{const.}$ The force across a curvilinear element of the surface $\alpha_1 = \text{const.}$, whose sides are given by the two vectors $H_2 \vec{e}_2 d\alpha_2$ and $H_3 \vec{e}_3 d\alpha_3$, is $H_2 H_3 \vec{p}_1 \, d\alpha_2 \, d\alpha_3$. force across an element of the surface $\alpha_1 - \frac{1}{2} d\alpha_1$ is then

$$-\left[H_2 H_3 \vec{p}_1 - \frac{\partial (H_2 H_3 \vec{p}_1)}{\partial \alpha_1} \frac{d\alpha_1}{2} \right] d\alpha_2 \, d\alpha_3$$

and the vector to its center is

$$\vec{r} - H_1 \vec{e}_1 \cdot \frac{d\alpha_1}{2} .$$

The moment of this force about the center of the volume element is

$$+ H_1 \vec{e}_1 \frac{d\alpha_1}{2} \times \left[H_2 H_3 \vec{p}_1 - \frac{\partial (H_2 H_3 \vec{p}_1)}{\partial \alpha_1} \cdot \frac{d\alpha_1}{2} \right] d\alpha_2 \, d\alpha_3$$

and the moment of the opposite force about the same point is;

$$+ H_1 \vec{e}_1 \frac{d\alpha_1}{2} \times \left[H_2 H_3 \vec{p}_1 + \frac{(H_2 H_3 \vec{p}_1)}{\partial \alpha_1} \cdot \frac{d\alpha_1}{2} \right] d\alpha_2 \, d\alpha_3 \quad .$$

The expressions for moments of forces across other surfaces of the elementary body are similar.

Summing up the contributions of forces across all sides of the considered body element, we obtain the following moment condition of equilibrium, if neglecting small terms of higher order

$$H_1 H_2 H_3 (\vec{e}_1 \times \vec{p}_1) + H_1 H_2 H_3 (\vec{e}_2 \times \vec{p}_2) + H_1 H_2 H_3 (\vec{e}_3 \times \vec{p}_3) = 0$$

and from there we obtain the well known relation between the stress components

$$p_{ij} = p_{ji} , \qquad (i, j = 1, 2, 3). \tag{17}$$

In fact, the moment condition can be written in the form:

$$\vec{e}_1 \times \vec{p}_1 + \vec{e}_2 \times \vec{p}_2 + \vec{e}_3 \times \vec{p}_3 = 0 \ ;$$

or, decomposing the stress vectors into their components, and omitting such vector products as $\vec{e}_1 \times p_{11} \vec{e}_1$, which vanish, we get

$$\vec{e} \times (p_{12} \vec{e}_2 + p_{13} \vec{e}_3) + \vec{e}_2 \times (p_{21} \vec{e}_1 + p_{23} \vec{e}_3) + \vec{e}_3 \times (p_{31} \vec{e}_1 + p_{32} \vec{e}_2) = 0$$

or

$$\vec{e}_2 \times \vec{e}_3 (p_{23} - p_{32}) = 0 \, ,$$
$$\vec{e}_3 \times \vec{e}_1 (p_{31} - p_{13}) = 0 \, ,$$
$$\vec{e}_1 \times \vec{e}_2 (p_{12} - p_{21}) = 0 \, ,$$

from where equations (17) follow immediately.

In a similar way we obtain that the resultant traction force across the surfaces $\alpha_1 \pm \frac{d\alpha_1}{2} = \text{const.}$, $\alpha_2 \pm \frac{d\alpha_2}{2} = \text{const.}$, and $\alpha_3 \pm \frac{d\alpha_3}{2} = \text{const.}$ is given by the expression

$$d\vec{R}_p = \left[\frac{\partial (H_2 H_3 \vec{p}_1)}{\partial \alpha_1} + \frac{\partial (H_3 H_1 \vec{p}_2)}{\partial \alpha_2} + \frac{\partial (H_1 H_2 \vec{p}_3)}{\partial \alpha_3} \right] d\alpha_1 \, d\alpha_2 \, d\alpha_3 \ . \tag{18}$$

The work of the two traction forces across the surfaces $\alpha_1 \pm \frac{1}{2} d\alpha_1$ of the elementary body for the displacement $\delta \vec{r}$ of its center is given by

$$\delta A_1 = - \left[H_2 H_3 \vec{p}_1 - \frac{\partial (H_2 H_3 \vec{p}_1)}{\partial \alpha_1} \frac{d\alpha_2}{2} \right] d\alpha_2 \, d\alpha_3 \cdot \left(\delta \vec{r} - \frac{\partial \delta \vec{r}}{\partial \alpha_1} \frac{d\alpha_1}{2} \right) +$$
$$+ \left[H_2 H_3 \vec{p}_1 + \frac{\partial (H_2 H_3 \vec{p}_1)}{\partial \alpha_1} \frac{d\alpha_1}{2} \right] d\alpha_2 \, d\alpha_3 \cdot \left(\delta \vec{r} + \frac{\partial \delta \vec{r}}{\partial \alpha_1} \frac{d\alpha_1}{2} \right)$$

or to the third order terms in the differentials $d\alpha_1$, $d\alpha_2$, $d\alpha_3$

$$\delta A_1 = \left[\frac{\partial (H_2 H_3 \vec{p}_1)}{\partial \alpha_1} \cdot \delta \vec{r} + H_2 H_3 \vec{p}_1 \cdot \frac{\partial \delta \vec{r}}{\partial \alpha_1} \right] d\alpha_1 \, d\alpha_2 \, d\alpha_3 \ .$$

It is well known that the first right side term in this equation represents the contribution of the traction forces across the surfaces $\alpha_1 \pm \frac{d\alpha_1}{2}$, which are necessary for compensating the mass forces. (In fact, we can deduce from equation (18) that the corresponding surface force in the expression for δA_1 is equal and opposite to a part of the vector sum of the mass and inertia forces). The work corresponding to the second term is accumulated during the displacement $\delta \vec{r}$ in the volume element as elastic energy. If we designate therefore by δe the increment of the elastic energy per unit of the original volume $H_1 H_2 H_3 \, d\alpha_1 \, d\alpha_2 \, d\alpha_3$, we get from the last equation

$$(19) \qquad \delta e = \frac{1}{H_1} \vec{p}_1 \cdot \frac{\partial \delta \vec{r}}{\partial \alpha_1} + \frac{1}{H_2} \vec{p}_2 \cdot \frac{\partial \delta \vec{r}}{\partial \alpha_2} + \frac{1}{H_3} \vec{p}_3 \cdot \frac{\partial \delta \vec{r}}{\partial \alpha_3} \, .$$

From equation (11) we obtain: $\frac{\partial \vec{r}}{\partial \alpha_i} = H_i \vec{e}_i$ and so

$$\frac{\partial \delta \vec{r}}{\partial \alpha_i} = \delta \frac{\partial \vec{r}}{\partial \alpha_i} = \delta (H_i \vec{e}_i) = H_i \delta \vec{e}_i \; .$$

We can therefore write the expression for the change of the elastic energy per unit of the initial volume in the form

$$(20) \qquad \delta e = \vec{p}_1 \cdot \delta \vec{e}_1 + \vec{p}_2 \cdot \delta \vec{e}_2 + \vec{p}_3 \cdot \delta \vec{e}_3 \, ,$$

or, introducing the stress components p_{hi} from equation (16),

$$(20') \qquad \delta e = \sum_i \sum_j p_{ij} \cdot \vec{e}_j \cdot \delta \vec{e}_i \, .$$

But from equations (14') and (15) we deduce

$$\delta g_{ij} = \delta (g_{ij} - G_{ij}) = \vec{e}_i \cdot \delta \vec{e}_j + \vec{e}_i \cdot \delta \vec{e}_i = \delta g_{ji} \, ,$$

and so every two symmetrical terms in equation (20') can be written as follows

$$p_{ij} \vec{e}_j \cdot \delta \vec{e}_i + p_{ji} \vec{e}_i \cdot \delta \vec{e}_j = p_{ij} \frac{\delta g_{ij} + \delta g_{ji}}{2} = \frac{1}{2} (p_{ij} \, \delta g_{ij} + p_{ji} \, \delta g_{ji}) \, ,$$

because of the symmetry of stress components. The definitive expression for the increment of elastic energy per unit of volume is therefore

$$(20'') \qquad \delta e = \frac{1}{2} \sum_i \sum_j p_{ij} \, \delta g_{ij} \, .$$

4. The variation of elastic energy during the transition from the initial to the neighbouring position.

Each stress component p_{ij} in an intermediate position of the body can be written as a sum of the relative stress component σ_{ij} in the initial state and of the additional stress component τ_{ij}, caused by displacements u_1, u_2, u_3

(21) $$p_{ij}=\sigma_{ij}+\tau_{ij}\ .$$

Equation (20″) for the variation of the elastic energy per unit of volume during the transition from the initial position to a neighbouring one is therefore

(22) $$\Delta e=\int \delta e=\frac{1}{2}\int \sum_i\sum_j \sigma_{ij}\,\delta g_{ij}+\frac{1}{2}\int \sum_i\sum_j \tau_{ij}\,\delta g_{ij}\ .$$

But $\sigma_{ij}=\text{const.}$ during the transition period and so the first right-hand term in the last equation can be written

(23) $$\frac{1}{2}\int \sum_i\sum_j \sigma_{ij}\,\delta g_{ij}=\frac{1}{2}\sum_i\sum_j \sigma_{ij}\cdot\Delta g_{ij}=$$
$$=\frac{1}{2}\sum_i\sum_j \sigma_{ij}\left[\frac{1}{H_i}\frac{\partial\vec{\rho}}{\partial\alpha_i}\cdot\vec{E}_j+\frac{1}{H_j}\frac{\partial\vec{\rho}}{\partial\alpha_j}\cdot\vec{E}_i+\frac{1}{H_iH_j}\frac{\partial\vec{\rho}}{\partial\alpha_i}\cdot\frac{\partial\vec{\rho}}{\partial\alpha_j}\right].$$

The second right-hand term of eq. (22) represents the elastic energy of additional stresses τ_{ij} and here only first order approximations are necessary, since all third and higher powers of displacements can be omitted in Δe. Because of the orthogonality of the coordinates α_1, α_2, α_3 we obtain for this term in the case of three-dimensional state of stresses the well-known expression (see e. g. A. E. H. Love, Theory of elasticity, 4^{th} edition, p. 302):

(24) $$a=\frac{1}{2}\int \sum_i\sum_j \tau_{ij}\,\delta g_{ij}=\frac{E}{2(+\nu)}\left\{\frac{1-\nu}{1-2\nu}\Theta^2-2(\varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3+\varepsilon_3\varepsilon_1)+\right.$$
$$\left.+\frac{1}{2}(\gamma_{12}^2+\gamma_{23}^2+\gamma_{31}^2)\right\}.$$

In the bidimensional case we have

(25a) $$a=\frac{1}{2}\frac{E}{1-\nu^2}\left\{\varepsilon_1^2+\varepsilon_2^2+2\nu\,\varepsilon_1\varepsilon_2+\frac{1-\nu}{2}\gamma_{12}^2\right\},$$

and if all other stresses except τ_{11} and τ_{12} vanish, this expression becomes

(25b) $$a = \frac{E}{2}\left(\varepsilon_1^2 + \frac{1}{2(1+\nu)}\gamma_{12}^2\right)\cdot$$

In the above equations ε_1, ε_2, ε_3 are the dilatations in the direction of coordinate lines and γ_{12}, γ_{23}, γ_{31} are the corresponding shearing strains. From the definition of ε_i and γ_{ij}

$$\varepsilon_i = \frac{ds_i - ds_i}{ds_i} = \sqrt{g_{ij}} - 1 \quad \text{and} \quad \gamma_{ij} = \frac{1}{ds_i\ ds_j}\cdot \vec{dr_i}\cdot\vec{dr_j}$$

we obtain

(26a, b)
$$\varepsilon_i = \frac{1}{H_i}\frac{\partial\vec{\rho}}{\partial\alpha_i}\cdot\vec{E_i} = \frac{1}{2}\Delta^{(1)}g_{ii},$$
$$\gamma_{ij} = \frac{1}{H_i}\frac{\partial\vec{\rho}}{\partial\alpha_i}\cdot\vec{E_j} + \frac{1}{H_j}\frac{\partial\vec{\rho}}{\partial\alpha_j}\cdot\vec{E_i} + \frac{1}{H_i H_j}\cdot\frac{\partial\rho}{\partial\alpha_i}\frac{\partial\vec{\rho}}{\partial\alpha_j} = \Delta g_{ij}$$

where $\Delta^{(1)}g_{ii}$ denote first order differences between g_{ii} and G_{ii}.

The variation of the total elastic energy $\Delta E = \int e\cdot dV$ (where dV is the volume element and the integration is to be extended over the whole body) can be decomposed in a linear term $\Delta^{(1)}E$ and a second order term $\Delta^{(2)}E$ in the displacements and from eq. (22) and (23) we get

(27a) $$\Delta^{(1)}E = \frac{1}{2}\int\sum_i\sum_j\sigma_{ij}\left(\frac{1}{H_i}\frac{\partial\vec{\rho}}{\partial\alpha_i}\cdot E_j + \frac{1}{H_j}\frac{\partial\vec{\rho}}{\partial\alpha_j}\vec{E_i}\right)\cdot dV$$

and

(27b) $$\Delta^{(2)}E = \frac{1}{2}\int\sum_i\sum_j\frac{1}{H_i H_j}\cdot\sigma_{ij}\frac{\partial\vec{\rho}}{\partial\alpha_i}\cdot\frac{\partial\vec{\rho}}{\partial\alpha_j}dV + \int a\ dV\cdot$$

5. The work of external forces and energy criterion of stability.

On the elastic body two kinds of external forces are acting, the components of which are given by the equation

(28) $$\vec{X} = \sum_i X_i\vec{E_i}$$

for the body forces and by the equation

(29) $$\vec{F} = \sum_i F_i\vec{E_i}$$

for the surface forces. From the equilibrium condition on the surface of the body we obtain the well-known relation between the surface force $\vec{F}$ and the stresses $\vec{p}_i$

$$\vec{F} = \sum_i \vec{p}_i \cos(\vec{n}, \vec{E}_i), \tag{29'}$$

where $\vec{n}$ is the unit vector in the direction of the outer normal to the surface.

During the transition from the initial state the body forces (weight c. a.) usually do not change and we shall take therefore

$$\Delta\vec{X} = 0 \ . \tag{30}$$

The surface forces are often changing their direction, intensity or both during the transition phase. We shall decompose therefore the force $\vec{F}$ into its component $\vec{P}$ in the initial position and in the increment $\Delta\vec{F}$

$$\vec{F} = \vec{P} + \Delta\vec{F} = \sum_i P_i \vec{E}_i + \Delta\vec{F} \ . \tag{31}$$

If, for example, the components F_i change their direction from $\vec{E}_i$ to $\vec{e'}_i$, the variation of F is given by

$$\Delta\vec{F} = \sum_i P_i \ (\vec{e_i}' - \vec{E}_i) \ . \tag{31'}$$

The vectors $\vec{e'}_i - \vec{E}_i$ must be determined to the first order terms in displacements only, because in the expression for the work the increment of the force must be multiplied by the vector of the elementary displacement and so the product contains all terms of the second order.

The work of external forces during the transition from the initial position contains therefore terms of the first and second order

$$\Delta A = \Delta^{(1)} A + \Delta^{(2)} A \ , \tag{32}$$

with

$$\Delta^{(1)} A = \int \vec{X} \cdot \vec{\rho} \, dV + \int \vec{P} \cdot \vec{\rho} \, dS \tag{33a}$$

and

$$\Delta^{(2)} A = \int (\Delta\vec{F} \cdot \delta\vec{\rho}) \, dS \ , \tag{33b}$$

where dS signifies the element of the external surface of the body.

It is known from the principle of virtual work, and can be easily verified directly, that the difference between the first order terms of the work of external forces and the corresponding terms of the internal energy vanishes

$$(34) \qquad \Delta^{(1)} A - \Delta^{(1)} E = 0 .$$

In fact, by integrating partially the volume integral for $\Delta^{(1)} E$ in the expression (27a), we obtain a surface and a volume integral which appear also in (33a) as a first order approximation of ΔA. For the functional $Q(u_i)$ of equation (1) we obtain therefore the following expression

$$(35) \quad Q(u_i) = \frac{1}{2} \int \sum_i \sum_j \frac{1}{H_i H_j} \sigma_{ij} \frac{\partial \vec{\rho}}{\partial \alpha_i} \cdot \frac{\partial \vec{\rho}}{\partial \alpha_j} dV + \int a dV - \int \int \left[\Delta \vec{F} \cdot \delta \vec{\rho} \right] dS .$$

The quantity a is given by equation (24) and $\Delta \vec{F}$ is determined by eq. (32') under the condition that the force $\vec{F}$ on the surface is changing its direction only during the transition from the initial to the neighbouring position.

The above expression for the functional $Q(u_i)$ forms the starting point for deducing the differential equations of the critical displacements, which we shall deal with in the next paragraphs. Besides, it can be used immediately for an approximate determination of critical values for concentrated forces $\vec{Q}$, line loads $\vec{q}$ or surface pressures $\vec{p}$, following the well-known method developed mainly by prof. S. Timoshenko.

6. **Differential equations for critical stresses.**

In order to obtain the general form of differential equations for critical displacements we must take that the variation of $Q(u_i)$ in equation (35) is equal to zero. In the first right-hand term the only quantity that must be varied is the displacement vector $\vec{\rho}$ and its derivatives. Taking into account that

$$dV = H_i \cdot H_2 \cdot H_3 \cdot d\alpha_1 \cdot d\alpha_2 \cdot d\alpha_3$$

we obtain here nine integrals of the type

$$I_{12} = \int\int\int \sigma_{12} \frac{\partial \delta \vec{\rho}}{\partial \alpha_1} \cdot \frac{\partial \vec{\rho}}{\partial \alpha_2} H_3 \, d\alpha_1 \, d\alpha_2 \, d\alpha_3$$

and integrating by parts we get

$$I_{12}=\iiint\left\{\frac{\partial}{\partial\alpha_1}\left(H_3\sigma_{12}\cdot\delta\vec{\rho}\cdot\frac{\partial\vec{\rho}}{\partial\alpha_2}\right)-\frac{\partial}{\partial\alpha_1}\left(H_3\cdot\sigma_{12}\frac{\partial\vec{\rho}}{\partial\alpha_2}\right)\cdot\delta\vec{\rho}\right\}d\alpha_1\,d\alpha_2\,d\alpha_3=$$

$$=\iint\frac{1}{H_2}\sigma_{12}\frac{\partial\vec{\rho}}{\partial\alpha_2}\cdot\delta\vec{\rho}\cdot\cos(\vec{n},\vec{E}_1)\,dS-\iiint\frac{\partial}{\partial\alpha_1}\left(H_3\cdot\sigma_{12}\frac{\partial\vec{\rho}}{\partial\alpha_2}\right)\cdot\delta\vec{\rho}\cdot d\alpha_1\,d\alpha_1\,d\alpha_3$$

where $(\vec{n},\vec{E}_1)$ is the angle between the normal $\vec{n}$ to the surface of the elastic body and the vector $\vec{E}_1$ and the surface integral is to be taken over those parts of the surface, where no displacements are prescribed. By doing so for all nine summands of the first right - hand term of eq. (35) we have for the variation of this term the following expression

$$\delta Q_1(u_i)=\iint\left\{\sum_i\sum_j\sigma_{ij}\cdot\cos(\vec{n},\vec{E}_i)\frac{1}{H_j}\frac{\partial\vec{\rho}}{\partial\alpha_j}\right\}\cdot\delta\vec{\rho}\cdot dS-$$

$$-\iiint\sum_i\sum_j\frac{\partial}{\partial\alpha_i}\left(\sigma_{ij}\overline{H}_k\frac{\partial\vec{\rho}}{\partial\alpha_j}\right)\cdot\delta\vec{\rho}\cdot d\alpha_1\,d\alpha_3\,d\alpha_3$$

where

$$\overline{H}_k=H_k,\ k\neq i,\ j \text{ for } i\neq j \text{ and } \overline{H}_k=\frac{H_k H_l}{H_i},\ k,\ l\neq i,\ k\neq l \text{ for } i=j.$$

But we can replace the sum $\sum\sigma_{ij}\cdot\cos(\vec{u},\vec{E}_i)$ according to eq. (29′) by F_j and the definitive form of $\delta\,Q_1(u_i)$ is therefore

$$\delta\,Q_1(u_i)=\iint\left(\sum_j\frac{F_j}{H_j}\cdot\frac{\partial\vec{\rho}}{\partial\alpha_j}\right)\cdot\delta\vec{\rho}\cdot dS-\tag{36}$$

$$-\iiint\sum_i\sum_j\frac{\partial}{\partial\alpha_i}\left(\sigma_{ij}\overline{H}_k\frac{\partial\vec{\rho}}{\partial\alpha_j}\right)\cdot\delta\vec{\rho}\,d\alpha_1\,d\alpha_2\,d\alpha_3\,.$$

The variation $\delta\,Q_2(u_i)$ of the integral $\int a\,dV$ is known from the theory of elasticity (see, e. g., A. E. H. Love, Theory of elasticity, 4th ed., p. 167 – 168)

$$\delta\,Q_2(u_i)=\int\delta\,a\cdot dV=\int(\tau_{11}\,\delta\varepsilon_1+\cdots+\tau_{31}\,\delta\gamma_{31})\,\delta V\,.$$

But variations of the six strain components $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3,\ \gamma_{12},\ \gamma_{23},\ \gamma_{31}$, can be expressed, according to equations (26a, b), as variations of the coefficients g_{ij} and G_{ij}, where only first order terms are to be taken into account. Integrating the expression for $\delta_2 Q(u_i)$ by parts we obtain finally

$$\delta Q_2(u_i) = \iiint \left\{ \tau_{11} \frac{1}{H_1} \frac{\partial \delta \vec{\rho}}{\partial \alpha_1} \cdot \vec{E}_1 + \tau_{12} \left(\frac{1}{H_1} \frac{\partial \delta \vec{\rho}}{\partial \alpha_1} \cdot \vec{E}_2 + \frac{1}{H_2} \frac{\partial \delta \vec{\rho}}{\partial \alpha_2} \cdot \vec{E}_1 \right) + \right.$$

$$\left. + \cdots \right\} H_1 H_2 H_3 \, d\alpha_1 d\alpha_2 d\alpha_3 =$$

(37)

$$= \iint \{ [\tau_{11} \vec{E}_1 + \tau_{12} \vec{E}_2 + \cdots] \cos(\vec{n}, \vec{E}_1) + \cdots \} \cdot \delta \vec{\rho} \, dS -$$

$$- \iiint \left\{ \frac{\partial}{\partial \alpha_1} [H_2 H_3 (\tau_{11} \vec{E}_1 + \tau_{12} \vec{E}_2 + \cdots)] + \cdots \right\} \cdot \delta \vec{\rho} \, d\alpha_1 d\alpha_2 d\alpha_3 .$$

From equations (36) and (37) we obtain for the variation $\delta Q(u_i)$ of the functional $Q(u_i)$ a sum of a volume integral, which must be extended over the whole elastic body and of a surface integral over those parts of the surface where no displacements are prescribed

$$\delta Q(u_i) = \iint \left\{ \sum_j \frac{F_j}{H_j} \frac{\partial \vec{\rho}}{\partial \alpha_j} + \sum_i \sum_j \tau_{ij} \cos(\vec{n}, \vec{E}_i) E_j - \Delta \vec{F} \right\} \cdot \delta \vec{\rho} \cdot dS -$$

(38)

$$- \iiint \left\{ \sum_i \sum_j \frac{\partial}{\partial \alpha_i} \left(\overline{H}_k \cdot \sigma_{ij} \frac{\partial \vec{\rho}}{\partial \alpha_j} \right) + \sum_i \frac{(H_j H_k \vec{p}_i)}{\partial \alpha_i} \right\} \cdot \delta \vec{\rho} \cdot dS .$$

The vector $\vec{p}_i$ in the last sum signifies the vector of the resultant additional stress in the surface normal to the α_i – line, i. e.

(38′)
$$\vec{p}_i = \sum_j \tau_{ij} \vec{E}_j .$$

When the initial stresses σ_{ij} reach their critical value and the displacement $\vec{\rho}$ is a critical one, the variation $\delta Q(u_i)$ must vanish. This involves that both the volume and the surface integral in (38) must be equal to zero. So we obtain a vectorial differential equation for the critical stresses

(39)
$$\sum_k \frac{(H_i H_j \vec{p}_k)}{\partial \alpha_k} + \sum_i \sum_k \frac{\partial}{\partial \alpha_j} \left(\overline{H}_i \, \sigma_{jk} \frac{\partial \vec{\rho}}{\partial \alpha_k} \right) = 0$$

and a vectorial condition on such parts of the surface, where no displacements are prescribed

$$(40) \qquad \sum_j \frac{F_j}{H_j} \frac{\partial \vec{\rho}}{\partial \alpha_j} + \sum_i \sum_j \tau_{ij} \cos(\vec{n}, \vec{E}_i) \vec{E}_j - \Delta \vec{F} = 0 .$$

In equation (39) we have as in the expression for $\delta Q_1(u_i)$

$$(41) \qquad \overline{H}_i = H_i, i \neq j, k, \quad \text{for } j \neq k; \quad \overline{H}_i = \frac{H_l H_m}{H_i},$$

$$l, m \neq i, l \neq m \text{ for } j = k .$$

Equation (39) gives three partial differential equations for components of the critical displacement and for critical initial stresses, while eq. (40) represents three boundary conditions for these quantities.

7. Displacements and deformations in thin shells.

Let the position vector to a point on the middle surface of the shell be denoted by $\mathfrak{X}$:

$$(42) \qquad \mathfrak{X} = \mathfrak{X}(\alpha_1, \alpha_2),$$

where α_1, α_2 are Gaussian coordinates along the lines of curvature on the surface. By using the coordinate lines in these directions, many of the following formulae will take a much more simple form than usually. Then, denoting by α_3 the distance of a point in the shell from the middle surface, we have for the position vector to this point

$$(43) \qquad \vec{R} = \mathfrak{X}(\alpha_1, \alpha_2) + \alpha_3 \cdot \vec{E}_3,$$

where $\vec{E}_3$ is the unit vector in the direction of the normal to this surface. Denoting the quantities H_1 and H_2 in the middle surface by A_1, A_2 we have

$$(44) \qquad \begin{aligned} \frac{\partial \mathfrak{X}}{\partial \alpha_1} &= A_1 \vec{E}_1, \qquad A_1 = \left(\frac{\partial \mathfrak{X}}{\partial \alpha_1} \cdot \frac{\partial \mathfrak{X}}{\partial \alpha_1} \right)^{\frac{1}{2}} \\ \frac{\partial \mathfrak{X}}{\partial \alpha_2} &= A_2 \vec{E}_2, \qquad A_2 = \left(\frac{\partial \mathfrak{X}}{\partial \alpha_2} \cdot \frac{\partial \mathfrak{X}}{\partial \alpha_2} \right)^{\frac{1}{2}}, \quad \vec{E}_3 = \vec{E}_1 \times \vec{E}_2 . \end{aligned}$$

Equations for the derivatives of the unit vectors $\vec{E}_1, \vec{E}_2, \vec{E}_3$ can be obtained for this special case without any trouble using the well-known formulae of Gauss and Weingarten

$$\begin{aligned}
&\frac{\partial \vec{E}_1}{\partial \alpha_1} = -\frac{1}{A_2}\frac{\partial A_1}{\partial \alpha_2}\vec{E}_2 + \frac{A_1}{R_1}\vec{E}_3 \;;\; \frac{\partial \vec{E}_1}{\partial \alpha_2} = \frac{1}{A_1}\frac{\partial A_2}{\partial \alpha_1}\vec{E}_2 \;;\; \frac{\partial \vec{E}_1}{\partial \alpha_3} = 0 \\
&\frac{\partial \vec{E}_2}{\partial \alpha_1} = \frac{1}{A_2}\frac{\partial A_1}{\partial \alpha_2}\vec{E}_1 \;,\; \frac{\partial \vec{E}_2}{\partial \alpha_2} = -\frac{1}{A_1}\frac{\partial A_2}{\partial \alpha_1}\vec{E}_1 + \frac{A_2}{R_2}\vec{E}_3 \;;\; \frac{\partial \vec{E}_2}{\partial \alpha_3} = 0 \\
&\frac{\partial \vec{E}_3}{\partial \alpha_1} = -\frac{A_1}{R_1}\vec{E}_1 \;,\; \frac{\partial \vec{E}_3}{\partial \alpha_2} = -\frac{A_2}{R_2}\vec{E}_2 \;,\; \frac{\partial \vec{E}_3}{\partial \alpha_3} = 0 \;,
\end{aligned} \tag{45}$$

where R_1 and R_2 are the two principal radiuses of curvature (positive, if the α_1- and α_2-lines have $\vec{E}_3$ on their concave side). With the above results the derivatives of the position vector $\vec{R}$ are

$$\frac{\partial \vec{R}}{\partial \alpha_1} = A_1\left(1 - \frac{\alpha_3}{R_1}\right)\vec{E}_1 \;,\; \frac{\partial \vec{R}}{\partial \alpha_2} = A_2\left(1 - \frac{\alpha_3}{R_2}\right)\vec{E}_2 \;,\; \frac{\partial \vec{R}}{\partial \alpha_3} = \vec{E}_3$$

and thus the coefficients H_1, H_2, H_3 are given by

$$H_1 = A_1\left(1 - \frac{\alpha_3}{R_1}\right) \;,\; H_2 = A_2\left(1 - \frac{\alpha_3}{R_2}\right) \;,\; H_3 = 1 \;. \tag{46}$$

It is sometimes advantageous to supersede some derivatives by equivalent forms which follow from the well-known theorems of Mainardi — Codazzi and of Gauss (theorema egregium) about certain mixed derivatives of higher order. As the coordinates α_1, α_2 are taken along the lines of curvature, those theorems have the following simple form

$$\begin{aligned}
&\frac{\partial}{\partial \alpha_2}\left(\frac{A_1}{R_1}\right) = \frac{1}{R_2}\frac{\partial A_1}{\partial \alpha_2} \;,\; \frac{\partial}{\partial \alpha_1}\left(\frac{A_2}{R_2}\right) = \frac{1}{R_1}\frac{\partial A_2}{\partial \alpha_1} \;, \\
&\frac{\partial}{\partial \alpha_2}\left(\frac{1}{A_2}\frac{\partial A_1}{\partial \alpha_2}\right) + \frac{\partial}{\partial \alpha_1}\left(\frac{1}{A_1}\frac{\partial A_2}{\partial \alpha_1}\right) = -\frac{A_1 A_2}{R_1 R_2} \;.
\end{aligned} \tag{47}$$

As in the case of beams and plates we suppose that the strains in a shell are sufficiently determined by deformations of its middle surface. To this purpose we assume with J. Bernoulli and Navier that plane cross sections of the shell remain plane and orthogonal to the middle surface after the deformation. Denoting by $\vec{\rho}_0 = \vec{\rho}_0\,(\alpha_1, \alpha_2)$ the vector of displacement of the point $\mathfrak{X}\,(\alpha_1, \alpha_2)$ in the middle surface, we have for the position vector of this point after deformation

$$\mathfrak{x} = \mathfrak{X} + \vec{\rho}_0 = \mathfrak{X} + \sum_i U_i\,\vec{E}_i\ . \tag{48}$$

Taking into account equations (45), we obtain for the first derivatives of $\mathfrak{x}$ the following equations

$$\frac{\partial \mathfrak{x}}{\partial \alpha_1} = \left(A_1 + \frac{\partial U_1}{\partial \alpha_1} + \frac{U_2}{A_2}\frac{\partial A_1}{\partial \alpha_2} - U_3\,\frac{A_1}{R_1}\right)\vec{E}_1 + \left(\frac{\partial U_2}{\partial \alpha_2} - \frac{U_1}{A_2}\frac{\partial A_1}{\partial \alpha_2}\right)\vec{E}_2 + \\ + \left(\frac{\partial U_3}{\partial \alpha_1} + U_1\,\frac{A_1}{R_1}\right)\vec{E}_3\ ,$$

$$\frac{\partial \mathfrak{x}}{\partial \alpha_2} = \left(\frac{\partial U_1}{\partial \alpha_1} - \frac{U_2}{A_1}\frac{\partial A_2}{\partial \alpha_1}\right)\vec{E}_1 + \left(A_2 + \frac{\partial U_2}{\partial \alpha_2} + \frac{U_1}{A_1}\cdot\frac{\partial A_2}{\partial \alpha_1} - U_3\,\frac{A_2}{R_2}\right)\vec{E}_2 + \\ + \left(\frac{\partial U_3}{\partial \alpha_2} + U_2\frac{A_2}{R_2}\right)\vec{E}_3\ .$$

With these expressions the position of the tangential plane to the middle surface in the neighbouring position can be determined. The quantities A_1, A_2 change over to A'_1, A'_2, which are given by their first order approximations

$$A'_1 = \left(\frac{\partial \mathfrak{x}}{\partial \alpha_1}\cdot\frac{\partial \mathfrak{x}}{\partial \alpha_1}\right)^{\frac{1}{2}} \approx A_1\left(1 + \frac{1}{A_1}\frac{\partial U_1}{\partial \alpha_1} + \frac{U_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} - \frac{U_3}{R_1}\right)$$

$$A'_2 = \left(\frac{\partial \mathfrak{x}}{\partial \alpha_2}\cdot\frac{\partial \mathfrak{x}}{\partial \alpha_2}\right)^{\frac{1}{2}} \approx A_2\left(1 + \frac{1}{A_2}\frac{\partial U_2}{\partial \alpha_2} + \frac{U_1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} - \frac{U_3}{R_2}\right)\ .$$

From the definition of the net vectors (equations 10) we obtain in the case of a shell

$$\vec{e}_1 = \left(1 + \frac{1}{A_1}\frac{\partial U_1}{\partial \alpha_1} + \frac{U_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} - \frac{U_3}{R_1}\right)\cdot\vec{E}_1 + \left(\frac{1}{A_1}\frac{\partial U_2}{\partial \alpha_1} - \frac{U_1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\right)\cdot\vec{E}_2 + \\ + \left(\frac{1}{A_1}\frac{\partial U_3}{\partial \alpha_1} + \frac{U_1}{R_1}\right)\cdot\vec{E}_3\ ,$$

$$\vec{e}_2 = \left(\frac{1}{A_2}\frac{\partial U_1}{\partial \alpha_2} - \frac{U_2}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}\right)\cdot\vec{E}_1 + \left(1 + \frac{1}{A_2}\frac{\partial U_2}{\partial \alpha_2} + \frac{U_1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} - \frac{U_3}{R_2}\right)\cdot\vec{E}_2 + \\ + \left(\frac{1}{A_2}\frac{\partial U_3}{\partial \alpha_2} + \frac{U_2}{R_2}\right)\cdot\vec{E}_3\ ,$$

and for the unit vectors $\vec{e}'_1, \vec{e}'_2$ in direction of the deformed lines of curvature we have as a first order approximation

$$\vec{e}'_1 = \frac{1}{A'_1}\frac{\partial \mathfrak{X}}{\partial \alpha_1} \approx \vec{E}_1 + \left(\frac{1}{A_1}\frac{\partial U_2}{\partial \alpha_1} - \frac{U_1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\right)\cdot \vec{E}_2 + \left(\frac{1}{A_1}\frac{\partial U_3}{\partial \alpha_1} + \frac{U_1}{R_1}\right)\cdot \vec{E}_3 ,$$

(49)

$$\vec{e}'_2 \approx \left(\frac{1}{A_2}\frac{\partial U_1}{\partial \alpha_2} - \frac{U_2}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}\right)\vec{E}_1 + \vec{E}_2 + \left(\frac{1}{A_2}\frac{\partial U_3}{\partial \alpha_2} + \frac{U_2}{R_2}\right)\vec{E}_3 .$$

The unit vector $\vec{e}'_3$ in the direction of the normal to the deformed middle surface is given by

$$\vec{e}'_3 = \frac{\partial \mathfrak{X}}{\partial \alpha_1} \cdot \frac{\partial \mathfrak{X}}{\partial \alpha_2} \Big/ (E' G' - F'^2) ,$$

where E', F', G' are the coefficients of the first fundamental quadratic form for the deformed middle surface. As a first order aproximation in displacements we obtain

$$\vec{e}'_3 = -\left(\frac{1}{A_1}\frac{\partial U_3}{\partial \alpha_1} + \frac{U_1}{R_1}\right)\vec{E}_1 - \left(\frac{1}{A_2}\frac{\partial U_3}{\partial \alpha_2} + \frac{U_2}{R_2}\right)\vec{E}_2 + \vec{E}_3 . \tag{49'}$$

With these preliminary results we can obtain the expression for the displacement of any point of the shell. Using the Bernoulli supposition we get for the position vector $\vec{r}$ to such a point

$$\vec{r} = \mathfrak{X} + \alpha_3 \vec{e}'_3 = \mathfrak{X} + \vec{\rho}_0 + \alpha_3 \vec{e}'_3$$

and the displacement vector of this point is

$$\vec{\rho} = \vec{r} - \vec{R} = \vec{r} - (\mathfrak{X} + \alpha_3 \vec{E}_3) ,$$

or

$$\vec{\rho} = \vec{\rho}_0 + \alpha_3 (\vec{e}'_3 - \vec{E}_3) . \tag{50}$$

Using equations (48) and (49′) for $\vec{\rho}_0$ and $\vec{e}'_3$ we obtain the components o $\vec{\rho}$ in the direction of $\vec{E}_1, \vec{E}_2, \vec{E}_3$

$$\vec{\rho} \approx \left[\left(1 - \frac{\alpha_3}{R_1}\right) U_1 - \frac{\alpha_3}{A_1}\frac{\partial U_3}{\partial \alpha_1}\right]\vec{E}_1 + \left[\left(1 - \frac{\alpha_3}{R_2}\right) U_2 - \frac{\alpha_3}{A_2}\frac{\partial U_3}{\partial \alpha_2}\right]\vec{E}_2 + U_3 \vec{E}_3 . \tag{50'}$$

But in thin shells the terms α_3/R_1 and α_3/R_2 are small against the unity and therefore we have approximately

$$(51) \qquad \vec{\rho} \approx \left(U_1 - \frac{\alpha_3}{A_1}\frac{\partial U_3}{\partial \alpha_1}\right)\cdot \vec{E}_1 + \left(U_2 - \frac{\alpha_3}{A_2}\frac{\partial U_3}{\partial \alpha_2}\right)\cdot \vec{E}_2 + U_3 \cdot \vec{E}_3$$

or decomposing this vector in components

$$(51') \qquad u_1 = U_1 - \frac{\alpha_3}{A_1}\frac{\partial U_3}{\partial \alpha_1}, \quad u_2 = U_2 - \frac{\alpha_3}{A_2}\frac{\partial U_3}{\partial \alpha_2}, \quad u_3 = U_3 .$$

8. The functional $Q(u_i)$ for thin shells

We shall give an expression for $Q(u_i)$ in thin shells with the use of equation (35), which will hold under the following assumptions

a) in the initial stage the stresses $\sigma_{11}, \sigma_{22}, \sigma_{12}$ are constant across the thickness of the shell and all other stress components are zero (membrane state),

b) the shell is thin and the quantities α_3/R_1 and α_3/R_2 can be neglected against unity,

c) the surface forces are of the hydrostatic type, i. e. they are always normal to the middle surface. During the transition from the initial to the neighbouring position the variation of $\vec{F}$ is therefore

$$(52) \qquad \Delta\vec{F} = p\,(\vec{e'}_3 - \vec{E}_3) = -p\left(\frac{U_1}{R_1} + \frac{1}{A_1}\frac{\partial U_3}{\partial \alpha_1}\right)\vec{E}_1 - p\left(\frac{U_2}{R_2} + \frac{1}{A_2}\frac{\partial U_3}{\partial \alpha_2}\right)\vec{E}_2$$

(if p has the direction of $\vec{E}_3$) and the work of this force can be calculated under the assumption, that the displacements are varying together proportionately

$$U_i = U_{i0}\cdot\lambda, \qquad 0 \leqq \lambda \leqq 1 \qquad \text{and } i = 1, 2, 3 .$$

(Here we denote provisionally by U_{i0} the displacements in the neighbouring position and by U_i those in an intermediate position!)

We have then

$$\Delta\vec{F} = -p\lambda\left(\frac{U_{10}}{R_1} + \frac{1}{A_1}\frac{\partial U_{30}}{\partial \alpha_1}\right)\vec{E}_1 - p\,\lambda\left(\frac{U_{20}}{R_2} + \frac{1}{A_2}\frac{\partial U_{30}}{\partial \alpha_2}\right)\vec{E}_2$$

and

$$\vec{\delta\rho} = \vec{\delta\rho_0} + \alpha_3\,\delta\,(\vec{e}_3' - \vec{E}_3)\,,$$

where

$$\vec{\delta\rho_0} = \delta\lambda \cdot \sum_i U_{i0}\,\vec{E}_i\,.$$

The work of $\Delta\vec{F}$ during an infinitesimal displacement $\vec{\delta\rho}$ is

$$\Delta\vec{F}\cdot\vec{\delta\rho} = \Delta\vec{F}\cdot\vec{\delta\rho_0} + \alpha_3\,\Delta\vec{F}\cdot\delta\,(\vec{e}_3' - \vec{E}_3) = \Delta\vec{F}\cdot\vec{\delta\rho_0} + \alpha_3\,p\,(\vec{e}_3' - \vec{E}_3)\cdot\delta\,(\vec{e}_3' - \vec{E}_3)$$

and for the work during the whole transition we have, dropping thereby the index $_0$ for displacements in the final position

$$\int\Delta\vec{F}\cdot\vec{\delta\rho} = -\frac{p}{2}\left\{\left(\frac{U_1}{R_1} + \frac{1}{A_1}\frac{\partial U_3}{\partial\alpha_1}\right)\left[U_1\left(1 - \frac{\alpha_3}{R_1}\right) - \frac{\alpha_3}{A_1}\frac{\partial U_3}{\partial\alpha_1}\right] + \right.$$
$$\left. + \left(\frac{U_2}{R_2} + \frac{1}{A_2}\frac{\partial U_3}{\partial\alpha_2}\right)\left[U_2\left(1 - \frac{\alpha_3}{R_2}\right) - \frac{\alpha_3}{A_2}\frac{\partial U_3}{\partial\alpha_2}\right]\right\}.$$

As the quantities α_3/R_1 and α_3/R_2 are small against unity, we can write definitively

$$\int\Delta\vec{F}\cdot\vec{\delta\rho} = -\frac{p}{2}\left\{\left(\frac{U_1}{R_1} + \frac{1}{A_1}\frac{\partial U_3}{\partial\alpha_1}\right)\left(U_1 - \frac{\alpha_3}{A_1}\frac{\partial U_3}{\partial\alpha_1}\right) + \right.$$
$$\left. + \left(\frac{U_2}{R_2} + \frac{1}{A_2}\frac{\partial U_3}{\partial\alpha_2}\right)\cdot\left(U_2 - \frac{\alpha_3}{A_2}\frac{\partial U_3}{\partial\alpha_2}\right)\right\}.$$

If the pressure on the surface $\alpha_3 = -\frac{h}{2}$ (h = thickness of the shell) is denoted by p_1 and the pressure on the other side $\alpha_3 = +\frac{h}{2}$ is p_2, then

$$p = p_1 \text{ for } \alpha_3 = -\frac{h}{2} \text{ and } p = p_2 \text{ for } \alpha_3 = +\frac{h}{2}\,.$$

The last equation becomes therefore for the work of pressures on both sides

$$(53)\qquad \int\Delta\vec{F}\cdot\vec{\delta\rho} = -\frac{p_1 - p_2}{2}\left\{U_1\left(\frac{U_1}{R_1} + \frac{1}{A_1}\frac{\partial U_3}{\partial\alpha_1}\right) + U_2\left(\frac{U_2}{R_2} + \frac{1}{A_2}\frac{\partial U_3}{\partial\alpha_2}\right)\right\} -$$
$$-\frac{p_1 + p_2}{2}\cdot\frac{h}{2}\left\{\left(\frac{U_1}{R_1} + \frac{1}{A_1}\frac{\partial U_3}{\partial\alpha_1}\right)\frac{1}{A_1}\frac{\partial U_3}{\partial\alpha_1} + \left(\frac{U_2}{R_2} + \frac{1}{A_2}\frac{\partial U_3}{\partial\alpha_2}\right)\frac{1}{A_2}\frac{\partial U_3}{\partial\alpha_2}\right\}.$$

The difference $p_1 - p_2$ gives the resultant pressure p for both free surfaces of the shell and the corresponding term in (53) is the most important. For practical purposes the work of additional forces in the case of hydrostatic pressure is therefore given by the following equation

$$\int \Delta\vec{F} \cdot \delta\vec{\rho} = -\frac{p}{2}\left(\frac{U_1^2}{R_1} + \frac{U_2^2}{R_2} + \frac{U_1}{A_1}\frac{\partial U_3}{\partial \alpha_1} + \frac{U_2}{A_2}\frac{\partial U_3}{\partial \alpha_2}\right). \tag{53'}$$

As all displacements of the points in the shell are expressed by displacements of its middle surface, all volume integrals on the right side of equation (35) can be integrated over the shell thickness, if the derivatives of $\vec{\rho}$ with respect to α_1, α_2, α_3 are known. But from eq. (45) and (51) we obtain the following expressions for the derivatives of $\vec{\rho}$ with respect to α_1 and α_2

$$\begin{aligned}\frac{\partial\vec{\rho}}{\partial\alpha_1} &= \left\{\frac{\partial U_1}{\partial\alpha_1} + \frac{U_2}{A_2}\frac{\partial A_1}{\partial\alpha_2} - U_3\frac{A_1}{R_1} - \alpha_3\left[\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial U_3}{\partial\alpha_1}\right) + + \frac{1}{A_2^2}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial U_3}{\partial\alpha_2}\right]\right\}\vec{E}_1 + \\ &+ \left\{\frac{\partial U_2}{\partial\alpha_1} - \frac{U_1}{A_2}\frac{\partial A_1}{\partial\alpha_2} - \alpha_3\left[\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_2}\frac{\partial U_3}{\partial\alpha_2}\right) - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial U_3}{\partial\alpha_1}\right]\right\}\vec{E}_2 + \left\{\frac{\partial U_3}{\partial\alpha_1} + U_1\frac{A_1}{R_1}\right\}\vec{E}_3\,,\end{aligned} \tag{54}$$

$$\begin{aligned}\frac{\partial\vec{\rho}}{\partial\alpha_2} &= \left\{\frac{\partial U_1}{\partial\alpha_2} - \frac{U_2}{A_1}\frac{\partial A_2}{\partial\alpha_1} - \alpha_3\left[\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_1}\frac{\partial U_3}{\partial\alpha_1}\right) - \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial U_3}{\partial\alpha_2}\right]\right\}\vec{E}_1 \\ &+ \left\{\frac{\partial U_2}{\partial\alpha_2} + \frac{U_1}{A_1}\frac{\partial A_2}{\partial\alpha_1} - U_3\frac{A_2}{R_2} - \alpha_3\left[\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial U_3}{\partial\alpha_2}\right) + \frac{1}{A_1^2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial U_3}{\partial\alpha_1}\right]\right\}\vec{E}_1 \\ &+ \left\{\frac{\partial U_3}{\partial\alpha_2} + U_2\frac{A_2}{R_2}\right\}\vec{E}_3\,,\end{aligned} \tag{54'}$$

where in the last bracket of each equation the terms $\frac{\alpha_3}{R_1}\frac{\partial U_3}{\partial\alpha_1}$ and $\frac{\alpha_3}{R_2}\frac{\partial U_3}{\partial\alpha_2}$ respectively were omitted as being small when compared with the first ones.

The derivative of $\vec{\rho}$ with respect to α_3 is much simpler, because the unit vectors $\vec{E}_1$, $\vec{E}_2$, $\vec{E}_3$ are independent of α_3. Starting from equation (51) we obtain

$$\frac{\partial\vec{\rho}}{\partial\alpha_3} = -\frac{1}{A_1}\frac{\partial U_3}{\partial\alpha_1}\vec{E}_1 - \frac{1}{A_2}\frac{\partial U_3}{\partial\alpha_2}\vec{E}_2\,, \tag{55}$$

but if using equation (50′) we would get

(55′)
$$\frac{\partial \rho}{\partial \alpha_3} = -\left(\frac{U_1}{R_1} + \frac{1}{A_1}\frac{\partial U_3}{\partial \alpha_1}\right)\vec{E}_1 - \left(\frac{U_2}{R_2} + \frac{1}{A_2}\frac{\partial U_3}{\partial \alpha_2}\right)\vec{E}_2 .$$

We can omit here the terms U_1/R_1 and U_2/R_2 against the expressions on the right side of equation (55) for the same reason as in (50′) : if, for instance, U_1 be such that $\frac{\alpha_3}{R_1} U_1$ and $\frac{\alpha_3}{A_1}\frac{\partial U_3}{\partial \alpha_1}$ would be of the same order, then for thin shells the following first - order approximation would hold

$$\vec{\rho} = U_1 \vec{E}_1 + U_2 \vec{E}_2 + U_3 \vec{E}_3 = \vec{\rho}_0$$

This means that no substantial buckling would exist and such cases are not interesting. We shall therefore accept equation (55) as a sufficiently good approximation and in equations (54) we shall neglect the same terms as in (55′), where they appear together with terms from equation (55).

With these approximations we first introduce the strain quantities of the middle surface

$$\varepsilon_{10} = \frac{1}{A_1}\frac{\partial U_1}{\partial \alpha_1} + \frac{U_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} - \frac{U_3}{R_1} ,$$

$$\varepsilon_{20} = \frac{1}{A_2}\frac{\partial U_2}{\partial \alpha_2} + \frac{U_1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} - \frac{U_3}{R_2} ,$$

(56′)
$$\gamma_{10} = \frac{1}{A_1}\frac{\partial U_2}{\partial \alpha_1} - \frac{U_1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} , \qquad \gamma_{20} = \frac{1}{A_2}\frac{\partial U_1}{\partial \alpha_2} - \frac{U_2}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} ,$$

$$\gamma_3' = \frac{1}{A_1}\frac{\partial U_3}{\partial \alpha_1} , \qquad \gamma_3'' = \frac{1}{A_2}\frac{\partial U_3}{\partial \alpha_2} ,$$

and the changes of curvature

$$\varkappa_1 = \frac{1}{A_1}\left[\frac{\partial}{\partial \alpha_1}\left(\frac{1}{A_1}\frac{\partial U_3}{\partial \alpha_1}\right) + \frac{1}{A_2^2}\frac{\partial A_1}{\partial \alpha_2}\frac{\partial U_3}{\partial \alpha_2}\right] ,$$

$$\varkappa_2 = \frac{1}{A_2}\left[\frac{\partial}{\partial \alpha_2}\left(\frac{1}{A_2}\frac{\partial U_3}{\partial \alpha_2}\right) + \frac{1}{A_1^2}\frac{\partial A_2}{\partial \alpha_1}\frac{\partial U_3}{\partial \alpha_1}\right] ,$$

(56″)

$$\lambda_2 = \frac{1}{A_1}\left[\frac{\partial}{\partial \alpha_1}\left(\frac{1}{A_2}\frac{\partial U_3}{\partial \alpha_2}\right) - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\frac{\partial U_3}{\partial \alpha_1}\right] ,$$

$$\lambda_2 = \frac{1}{A_2}\left[\frac{\partial}{\partial \alpha_2}\left(\frac{1}{A_1}\frac{\partial U_3}{\partial \alpha_1}\right) - \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}\frac{\partial U_3}{\partial \alpha_2}\right] .$$

Then the derivatives od $\vec{\rho}$ with respect to the coordinates α_1, α_2, α_3 can be written

$$\frac{1}{A_1}\frac{\partial\vec{\rho}}{\partial\alpha_1}=(\varepsilon_{10}-\alpha_3\varkappa_1)\vec{E}_1+(\gamma_{10}-\alpha_3\lambda_1)\vec{E}_2+\gamma_3{}'\vec{E}_3\,,$$

(57)
$$\frac{1}{A_2}\frac{\partial\vec{\rho}}{\partial\alpha_2}=(\gamma_{20}-\alpha_3\lambda_2)\vec{E}_1+(\varepsilon_{20}-\alpha_3\varkappa_2)\vec{E}_2+\gamma_3{}''\vec{E}_3\,,$$

$$\frac{1}{A_3}\frac{\partial\vec{\rho}}{\partial\alpha_3}=-\gamma'_3\vec{E}_1-\gamma_3{}''\vec{E}_2\ .$$

Neglecting in (46) the terms α_3/R_1 and α_3/R_2 against unity we have $H_1=A_1$, $H_2=A_2$, $H_3=1$ and we obtain for portions of the first right-hand term in (35) the following expressions

$$\frac{1}{H_1^2}\frac{\partial\vec{\rho}}{\partial\alpha_1}\cdot\frac{\partial\vec{\rho}}{\partial\alpha_1}=\varepsilon_{10}^2+\gamma_{10}^2+\gamma_3'^2-2\,\alpha_3(\varepsilon_{10}\varkappa_1+\gamma_{10}\lambda_1)+\alpha_3^2(\varkappa_1^2+\lambda_1^2)$$

and

$$\frac{1}{H_1H_2}\frac{\partial\vec{\rho}}{\partial\alpha_1}\cdot\frac{\partial\vec{\rho}}{\partial\alpha_2}=\varepsilon_{10}\gamma_{20}+\varepsilon_{20}\gamma_{10}+\gamma_3'\gamma_3''-$$

$$-\alpha_3(\varepsilon_{10}\lambda_2+\varepsilon_{20}\lambda_1+\gamma_{10}\varkappa_2+\gamma_{20}\varkappa_1)+\alpha_3^2(\varkappa_1\lambda_2+\varkappa_2\lambda_1)\,.$$

Integrating across the thickness the complete first right-hand term of (35) becomes

$$Q_1(u_i)=\frac{1}{2}\iiint\sum_i\sum_j H_iH_j\sigma_{ij}\frac{\partial\vec{\rho}}{\partial\alpha_i}\cdot\frac{\partial\vec{\rho}}{\partial\alpha_j}dV=$$

(58)
$$=\iint\left\{\frac{\sigma_{11}h}{2}\left[\varepsilon_{10}^2+\gamma_{10}^2+\gamma_3'^2+\frac{h^2}{12}(\varkappa_1^2+\lambda_1^2)\right]+\right.$$

$$+\frac{\sigma_{22}h}{2}\left[\varepsilon_{20}^2+\gamma_{20}^2+\gamma_3''^2+\frac{h^2}{12}(\varkappa_2^2+\lambda_2^2)\right]+$$

$$\left.+\sigma_{12}h\left[\varepsilon_{10}\gamma_{20}+\varepsilon_{20}\gamma_{10}+\gamma_3'\gamma_3''+\frac{h^2}{12}(\varkappa_1\lambda_2+\varkappa_2\lambda_1)\right]\right\}A_1A_2\,d\alpha_1\,d\alpha_3\,.$$

When forming the expression for the second right-hand volume integral of $Q(u_i)$ we may suppose that the stresses in thin shells are nearly plane. The elastic energy per unit volume of the additional stresses can be therefore calculated from equation (25 a), where according to equations (26 a, b)

$$\varepsilon_1 \approx \frac{1}{2}\Delta^{(1)} g_{11} = \frac{1}{H_1}\frac{\partial \vec{\rho}}{\partial \alpha_1}\cdot \vec{E}_1 = \varepsilon_{10} - \alpha_3 \varkappa_1 \tag{59'}$$

and

$$\gamma_{12} = \Delta g_{12} \approx \frac{1}{H_1}\frac{\partial \vec{\rho}}{\partial \alpha_1}\cdot \vec{E}_2 + \frac{1}{H_2}\frac{\partial \vec{\rho}}{\partial \alpha_2}\cdot \vec{E}_1 = \gamma_{10} + \gamma_{20} - \alpha_3(\lambda_1 + \lambda_2)\ , \tag{59''}$$

if we neglect the difference between H_1, H_2 and A_1, A_2. With these results we obtain for the second volume integral in the expression for $Q(u_i)$

$$\begin{aligned} Q_2(u_i) = \int a\, dV = \iint \frac{Eh}{2(1-\nu^2)}\Big\{ & \varepsilon_{10}^2 + \varepsilon_{20}^2 + 2\nu\,\varepsilon_{10}\,\varepsilon_{20} + \\ & + \frac{1-\nu}{2}(\gamma_{10}+\gamma_{20})^2 + \frac{h^2}{12}\Big[\varkappa_1^2 + \varkappa_2^2 + 2\nu\varkappa_1\varkappa_2 + \frac{1-\nu}{2}(\lambda_1+\lambda_2)^2\Big]\Big\} A_1 A_2\, d\alpha_1\, d\alpha_2\ . \end{aligned} \tag{60}$$

The functional $Q(u_i)$ for thin shells and for surface forces $\vec{F}$ of the hydrostatic pressure type can be therefore written as a surface integral of the form

$$Q(u_i) = Q_1(u_i) + Q_2(u_i) - \iint \frac{p}{2}\left(\frac{U_1^2}{R_1} + \frac{U_2^2}{R_2} + \frac{U_1}{A_1}\frac{\partial U_3}{\partial \alpha_1} + \frac{U_2}{A_2}\frac{\partial U_3}{\partial \alpha_2}\right) A_1 A_2\, d\alpha_1\, d\alpha_2, \tag{61}$$

where $Q_1(u_i)$ and $Q_2(u_i)$ are given by (58) and (60) respectively. The quantities in the last two equations can be expressed by components U_1, U_2, U_3 of the displacement of the middle surface through equations (56′) and (56″).

Forming the first variation of $Q(u_i)$ and integrating by parts we would obtain a surface and a line integral. The integrand of the first integral gives a set of three partial differential equations for the components U_1, U_2, U_3 of the displacement of the middle surface, while the second integrand gives a set of conditions on those parts of the boundary of the middle surface where the displacements are not prescribed. Such equations for general shapes of shells are too complicated to be written down here explicitely, but for some special types the expressions are not so intricate.

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