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ON SUPER QUASI EINSTEIN MANIFOLD

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Abstract. We study super quasi Einstein manifold and viscous fluid super quasi Einstein spacetime. The existence of super quasi Einstein manifold and viscous fluid super quasi Einstein spacetime are shown by two closely related examples. Also, some results involving super quasi Einstein manifold, pseudo quasi Einstein manifold and quasi Einstein manifold are established. Finally, the bounds of the cosmological constant in a viscous fluid super quasi Einstein spacetime are deduced.

Introduction

The notion of quasi Einstein manifold was introduced by Chaki and Maity [**4**]. According to them, a nonflat n-dimensional Riemannian manifold (M_n, g), $n \geqslant 3$ is said to be a quasi Einstein manifold if its Ricci tensor of type $(0,2)$ is not identically zero and satisfies the condition

$$\mathrm{Ric}(X,Y) = ag(X,Y) + bA(X)A(Y),$$

where a, b are associated scalars of which $b \neq 0$ and A is a nonzero associated 1-form, metrically equivalent to the unit vector field U, i.e., for all vector fields X

$$g(X,U) = A(X), \quad g(U,U) = 1,$$

U is usually called the generator of the manifold. Such an n-dimensional manifold is denoted by the symbol $(QE)_n$.

Subsequently different authors generalized the concept of $(QE)_n$ having different expressions of the Ricci tensor like:

i) Generalized quasi Einstein manifold [**5**], denoted by $G(QE)_n$.

ii) Mixed super quasi Einstein manifold [**1**], denoted by $MS(QE)_n$.

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iii) Super quasi Einstein manifold [**6**], denoted by $S(QE)_n$ when the Ricci tensor is given by

$$\text{Ric}(X,Y) = ag(X,Y) + bA(X)A(Y) + c[A(X)B(Y) + B(X)A(Y)] + fD(X,Y) \tag{0.1}$$

where $b \neq 0$, A, B are two nonzero 1-forms such that

$$g(X,U) = A(X),\ g(X,V) = B(X),\ g(U,U) = 1,$$
$$g(V,V) = 1,\ g(U,V) = A(V) = B(U) = 0,\ \text{for all } X, \tag{0.2}$$

U, V being mutually orthogonal unit vector fields, D is a symmetric $(0,2)$ tensor which satisfies the conditions

$$D(X,U) = 0,\ \operatorname{trace} D = 0 \tag{0.3}$$

for all X and in such a case a, b, c, f are called the associated scalars, A, B are respectively called the associated main and auxiliary 1-forms, metrically equivalent to the main and auxiliary generators U, V and $(0,2)$ type symmetric tensor D is called the associated tensor of the manifold. The $S(QE)_n$ is a particular case of $MS(QE)_n$.

iv) Pseudo quasi Einstein manifold [**2**] is denoted by $P(QE)_n$, when the Ricci tensor is given by

$$\text{Ric}(X,Y) = ag(X,Y) + bA(X)A(Y) + fD(X,Y)$$

where a, b, f are nonzero scalars and A is a nonzero 1-form such that $g(X,U) = A(X)$ for all X, U being a unit vector field called the generator of the manifold, D is a symmetric (0,2) tensor with zero trace which satisfies $D(X,U) = 0$, for all X.

Here we study $S(QE)_n$ and show its existence with an example. Further, the existence of a $S(QE)_4$ spacetime has been established with an example. Also, some results involving $S(QE)_n$, $P(QE)_n$ and $(QE)_n$ have been deduced. The study of $S(QE)_n$, $P(QE)_n$ and $(QE)_n$ becomes meaningful due to their applications in the general theory of relativity and cosmology. It is found that a viscous fluid spacetime obeying the Einstein equation with cosmological constant λ is a 4-dimensional Lorentzian $S(QE)_4$. Further, the bounds of the cosmological constant in a viscous fluid $S(QE)_n$ spacetime are established.

The importance of a $S(QE)_n$ lies in the fact that such a 4-dimensional Lorentzian manifold is relevant to the study of a general relativistic viscous fluid spacetime admitting heat flux and it represents the earlier stage of the universe. Many authors are generalizing the expression of the Ricci tensor for the study of spacetimes admitting fluid viscosity and electromagnetic fields.

1. Some geometric properties of $S(QE)_n$

In this section, we consider a $S(QE)_n$, $(n \geqslant 3)$ with associated scalars a, b, c and f, associated main and auxiliary 1-forms A and B with corresponding main and auxiliary generators U, V and associated symmetric $(0,2)$ tensor D. We now prove the following theorem:

Theorem 1.1. *An $S(QE)_n$ is a $P(QE)_n$ if either of the generators is a parallel vector field.*

Proof. By the definition of the Riemannian curvature tensor, if U is a parallel vector field, then we find that

$$R(X,Y,U) = \nabla_X \nabla_Y U - \nabla_Y \nabla_X U - \nabla_{[X,Y]} U = 0$$

and consequently we get

$$\text{Ric}(X,U) = 0. \tag{1.1}$$

Again, putting $Y = U$ in equation (0.1) and applying (0.2) and (0.3), we get

$$\text{Ric}(X,U) = (a+b)\, g(X,U) + cg(X,V).$$

So, if U is a parallel vector field, by (1.1), we get

$$(a+b)\, g(X,U) + cg(X,V) = 0. \tag{1.2}$$

Now, putting $X = V$ in equation (1.2) and using (0.2), we get $c = 0$. So, if U is a parallel vector field in an $S(QE)_n$, then $c = 0$ i.e., the manifold is a $P(QE)_n$.

Again, if V is a parallel vector field, then $R(X,Y)V = 0$. Contracting, we get

$$\text{Ric}(Y,V) = 0. \tag{1.3}$$

Putting $X = V$ in equation (0.1) and applying (0.2), we get

$$\text{Ric}(Y,V) = ag(Y,V) + cA(Y)B(V) + fD(Y,V) = 0.$$

If, V is a parallel vector field then by equation (1.3), we get

$$ag(Y,V) + cA(Y)B(V) + fD(Y,V) = 0.$$

Putting $Y = U$ and using (1.3), (0.2), and (0.3), we get $c = 0$, i.e., in this case also the manifold is a $P(QE)_n$. □

As a consequence of the above theorem, we get the following:

Corollary 1.1. *If the main generator U of an $S(QE)_n$ is a parallel vector field, then $a + b = 0$.*

Proof. Putting $X = U$ into equation (1.2) and using equation (0.2), we directly get the required result. □

Theorem 1.2. *In a $S(QE)_n$, QU is orthogonal to U iff $a + b = 0$, where Q is the Ricci transformation i.e., $g(QX,Y) = \text{Ric}(X,Y)$.*

Proof. From equation (0.1) we get

$$\text{Ric}(X,U) = g(QX,U) = (a+b)\, g(X,U) + cg(X,V).$$

Now, putting $X = U$, we get $g(QU,U) = (a+b)$, which implies that QU is orthogonal to U iff $a + b = 0$. □

Theorem 1.3. *In an $S(QE)_n$, 0 is the eigenvalue of L in the direction of the eigenvector U, i.e., $LU = 0$, where L is the symmetric endomorphism of the tangent space at any point of the manifold corresponding to the structure tensor D, i.e., $g(LX,Y) = D(X,Y)$.*

PROOF. We have $g(LX,Y) = D(X,Y)$ for all vector fields X, Y. Putting $X = U$, we get $g(LU,Y) = D(U,Y) = 0$, for all vector fields Y. So, $LU = 0$, i.e., 0 is the eigenvalue of L in the direction of the eigenvector U. □

Next, we are going to prove some results on Ricci semi-symmetric $S(QE)_n$.

THEOREM 1.4. *In a Ricci semi-symmetric $S(QE)_n$, $(a+b)A(R(X,Y)Z) = A(R(X,Y)QZ)$ holds for all vector fields X, Y, Z, where $g(QX,Y) = \mathrm{Ric}(X,Y)$.*

PROOF. A Riemannian manifold is Ricci semi-symmetric if $R(X,Y).\,\mathrm{Ric} = 0$ for all vector fields X and Y, where $R(X,Y)$ denotes the curvature operator. Thus, we have

$$(R(X,Y).\,\mathrm{Ric})(Z,W) = -\,\mathrm{Ric}(R(X,Y)Z,W) - \mathrm{Ric}(R(X,Y)W,Z).$$

Using (0.1), (0.2) and putting $W = U$, we get

$$(R(X,Y).\,\mathrm{Ric})(Z,U) = b[A(R(X,Y)Z)]+c[B(R(X,Y)U)A(Z)]+fD(R(X,Y)U,Z)$$

Since the manifold is Ricci semi-symmetric, we have

$$b[A(R(X,Y)Z)] + c[B(R(X,Y)U)A(Z)] + fD(R(X,Y)U,Z) = 0. \tag{1.4}$$

Again, we have

$$A(R(X,Y)QZ) = -\,\mathrm{Ric}(R(X,Y)U,Z),$$

i.e.,

$$A(R(X,Y)QZ) = aA(R(X,Y)Z) - cB(R(X,Y)U)A(Z) - fD(R(X,Y)U,Z).$$

Now, using equation (1.4), we get,

$$(a+b)A(R(X,Y)Z) = A(R(X,Y)QZ).$$ □

Let us now investigate, whether a $S(QE)_n$ can be projectively flat or not. If possible, then the Riemannian curvature tensor is given by

$$R(X,Y,Z,W) = \tfrac{1}{n-1}\big[\,\mathrm{Ric}(Y,Z)g(X,W) - \mathrm{Ric}(X,Z)g(Y,W)\big].$$

Contracting and putting $W = U$, we get $bA(X) = 0$. Again putting $X = U$, we get $b = 0$, contradicting the definition of the manifold. Thus we can state the following:

THEOREM 1.5. *An $S(QE)_n$ can never be projectively flat.*

2. Conformally flat and conformally conservative $S(QE)_n$

Let us now state and prove some results on conformally flat $S(QE)_n$.

THEOREM 2.1. *If the main generator of a conformally flat $S(QE)_n$ is a parallel vector field, then it is a $(QE)_n$ with sum of the associated scalars is zero.*

Before proving the theorem, we consider the following lemma:

LEMMA 2.1. *The scalar curvature r of an $S(QE)_n$ is given by $r = na + b$.*

PROOF. Putting $X = Y = e_i$, in equation (0.1) [where $\{e_i\}$, $i = 1, 2, \ldots, n$, is an orthonormal basis at each point of the tangent space of the manifold,] and summing over i and using trace $D = 0$, we get $r = na + b$. □

PROOF OF THE THEOREM. If a manifold is conformally flat, then the (0,4) type Riemannian curvature tensor is given by

$$R(X,Y,Z,W) = \tfrac{1}{n-2}\big[\operatorname{Ric}(Y,Z)g(X,W) - \operatorname{Ric}(X,Z)g(Y,W) + \operatorname{Ric}(X,W)g(Y,Z) - \operatorname{Ric}(Y,W)g(X,Z)\big] - \tfrac{r}{(n-1)(n-2)}\big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big].$$

By help of equation (0.1) and Lemma 2.1 and putting $Z = U$, we get

$$\begin{aligned} R(X,Y)U = {} & \frac{a+b}{n-1}[A(Y)X - A(X)Y] \\ & + \frac{c}{n-2}[B(Y)X - B(X)Y + B(X)A(Y)U - B(Y)A(X)U] \\ & + \frac{f}{n-2}[A(Y)LX - A(X)LY], \end{aligned} \tag{2.1}$$

where $g(LX,Y) = D(X,Y)$. If U is a parallel vector field, then $R(X,Y)U = 0 = a + b = c$ [by Theorem 1.1 and Corollary 1.1], and thus by (2.1) we have

$$f[A(Y)LX - A(X)LY] = 0.$$

Now, putting $X = U$, we get $f[A(Y)LU - A(U)LY] = 0$, i.e., $fLY = 0$, [as $LU = 0$, by Theorem 1.3] for all vector field Y. But, LY can not be zero for every vector field Y. Therefore $f = 0$.

So, if U is a parallel vector field in a conformally flat $S(QE)_n$, then $a+b = c = f = 0$. Therefore, $\operatorname{Ric}(X,Y) = ag(X,Y) + bA(X)A(Y)$ i.e., the manifold becomes a $(QE)_n$ with sum of the associated scalars is zero. □

THEOREM 2.2. *A necessary condition for a $S(QE)_n$ to be conformally conservative is $(d((n-2)a + (2n-3)b))(V) = 2(n-1)(dc)(U)$.*

The following lemma is necessary to prove the above result.

LEMMA 2.2. *In an $S(QE)_n$, the Ricci tensor satisfies the conditions $\operatorname{Ric}(U,U) = a + b$ and $\operatorname{Ric}(U,V) = c$.*

PROOF. Putting $X = Y = U$ and $X = U, Y = V$ respectively in equation (0.1) and using equation (0.2) and (0.3), we get the required result. □

PROOF OF THE THEOREM. Now, Riemannian manifold is said to be conformally conservative if the divergence of its conformal curvature tensor is zero [**7**], i.e.,

$$(\nabla_X \operatorname{Ric})(Y,Z) - (\nabla_Z \operatorname{Ric})(Y,X) = \frac{1}{2(n-1)}[(dr)(X)g(Y,Z) - (dr)(Z)g(X,Y)].$$

Now, putting $X = Y = U$ and $Z = V$, we get

$$(\nabla_U \operatorname{Ric})(U,V) - (\nabla_V \operatorname{Ric})(U,U) = \frac{1}{2(n-1)}[(dr)(U)g(U,V) - (dr)(V)g(U,U)].$$

Using equation (0.2) and Lemma 2.2 we have

$$(dc)(U) - (d(a+b))(V) = -\frac{(dr)(V)}{2(n-1)} = -\frac{n(da)(V) + (db)(V)}{2(n-1)}.$$

On simplification with the help of Lemma 2.1, this reduces to

$$(d((n-2)a + (2n-3)b))(V) = 2(n-1)(dc)(U). \qquad \square$$

3. General relativistic viscous fluid $S(QE)_n$ spacetime

It is well known fact that a semi-Riemannian manifold with metric tensor g is a Lorentzian manifold if the index of g is 1, i.e., with signature $(+,+,+,\dots,+,-)$.

A spacetime is a time oriented 4-dimensional (M_4, g) manifold with Lorentz metric g and index 1, i.e., with signature $(+,+,+,-)$ and the viscous fluid spacetime is a spacetime whose matter content is a viscous fluid. Further, a viscous fluid super quasi Einstein spacetime is a viscous fluid spacetime which is also super quasi Einstein i.e., whose Ricci tensor is of the form

$$\text{Ric}(X,Y) = ag(X,Y) + bA(X)A(Y) + c[A(X)B(Y) + B(X)A(Y)] + fD(X,Y), \tag{3.1}$$

where $a, b\,(\neq 0), c$, and f are the associated scalars, U is the unit speed time like velocity vector field which is everywhere tangent to the flow lines of the viscous fluid, V is the unit heat flux vector field and D is the symmetric anisotropic pressure tensor field of the fluid satisfying the conditions

$$g(U,U) = -1, \quad g(V,V) = 1, \quad g(U,V) = 0$$
$$g(X,U) = A(X), \quad g(X,V) = B(X),$$
$$\operatorname{trace} D = 0, \quad D(X,U) = 0$$

for all vector fields X.

It is to be noted that the basic geometric features of $S(QE)_n$ are also being maintained in the Lorentzian manifold which is necessarily a semi-Riemannian manifold. This can be justified with the help of the following proposition (without proof):

PROPOSITION 3.1. i) *A Lorentzian super quasi Einstein manifold* $(LS(QE)_n)$ *is a Lorentzian pseudo quasi Einstein manifold* $(LP(QE)_n)$ *if either of the generator is a parallel vector field.*

ii) *If the main generator* U *of* $LS(QE)_n$ *is a parallel vector field then,* $a = b$.

iii) *In a* $LS(QE)_n$, QU *is orthogonal to* U *iff* $a = b$, *where* Q *is the Ricci transformation i.e.,* $g(QX,Y) = \text{Ric}(X,Y)$.

iv) *In a* $LS(QE)_n$, 0 *is the eigenvalue of* L *in the direction of the eigenvector* U, *i.e.,* $LU = 0$, *where* L *is the symmetric endomorphism of the tangent space at any point of the manifold corresponding to the structure tensor* D, *i.e.,* $g(LX,Y) = D(X,Y)$.

v) *In a Ricci semi-symmetric* $LS(QE)_n$, $(a-b)A(R(X,Y)Z) = A(R(X,Y)QZ)$ *holds for all vector fields* X, Y, Z, *where* $g(QX,Y) = \text{Ric}(X,Y)$.

vi) *If the main generator of a conformally flat $LS(QE)_n$ is a parallel vector field, then it is a Lorentzian quasi Einstein manifold ($L(QE)_n$) with equal associated scalars.*

vii) *The scalar curvature r of a $LS(QE)_n$ is given by $r = na - b$.*

In physical cosmology, the cosmological constant λ was proposed by Albert Einstein as a modification of his original theory of general relativity to achieve a stationary universe. Einstein abandoned the concept after the observation of the Hubble redshift which indicated that the universe might not be stationary, as he had based his theory on the idea that the universe is unchanging. However, the discovery of cosmic acceleration in the 1990's has renewed the interest in cosmological constant.

Ricci tensor controls the geometry of the spacetime whereas energy momentum tensor T, signifies the physical aspects of the spacetime and in general relativity they are related [**3**] by

$$\mathrm{Ric}(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = kT(X,Y) \tag{3.2}$$

for all vector fields X, Y, where Ric is the Ricci tensor of type $(0,2)$, r is the scalar curvature, λ is the cosmological constant and k is the gravitational constant.

Let T be the energy-momentum tensor of type $(0,2)$ describing the matter distribution of a viscous fluid. Then it can be expressed in the following form:

$$T(X,Y) = pg(X,Y) + (\sigma + p)A(X)A(Y) + [A(X)B(Y) + A(Y)B(X)] + D(X,Y),$$

where σ and p are the energy density and isotropic pressure respectively.

Thus by virtue of the above equation, equation (3.2) may be written as

$$\begin{aligned}\mathrm{Ric}(X,Y) = {}& \Big(kp + \frac{r}{2} - \lambda\Big)g(X,Y) + k(\sigma + p)A(X)A(Y) \\ & + k[A(X)B(Y) + A(Y)B(X)] + kD(X,Y).\end{aligned} \tag{3.3}$$

So, we have the following:

THEOREM 3.1. *A viscous fluid spacetime admitting heat flux and satisfying Einstein's equation with cosmological constant is a 4-dimensional connected Lorentzian $S(QE)_4$.*

Now, comparing equations (3.1) and (3.3), we get

$$a = kp + \frac{r}{2} - \lambda, \quad b = k(\sigma + p),; \; c = f = k. \tag{3.4}$$

Solving (3.4) we get

$$\sigma = \frac{2a + b - 2\lambda}{2k}, \quad p = \frac{2\lambda + b - 2a}{2k}. \tag{3.5}$$

Now, in a viscous fluid spacetime $\sigma > 0$ and $p > 0$, thus by (3.5) we get $\frac{2a+b}{2} > \lambda$ and $\lambda > \frac{2a-b}{2}$. So, we can state that

THEOREM 3.2. *A viscous fluid $S(QE)_4$ spacetime obeying Einstein's equation with cosmological constant λ satisfies the relation $\frac{2a+b}{2} > \lambda > \frac{2a-b}{2}$.*

4. Example of a 4-dimensional super quasi Einstein manifold and super quasi Einstein spacetime

At first we construct a nontrivial concrete example of a super quasi Einstein manifold which is not a quasi Einstein manifold. Let us consider a Riemannian metric g on the 4-dimensional real number space M_4 by

$$ds^2 = g_{ij}dx^i dx^j = (e^{x^2})(dx^1)^2 + (x^1x^3)^2(dx^2)^2 + (dx^3)^2 + (dx^4)^2, \tag{4.1}$$

where $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3 and x^4 are the standard coordinates of M_4. Then the only nonvanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are

$$[1\,2, 1] = \frac{1}{2}e^{x^2}, \quad [2\,1, 2] = x^1(x^3)^2, \quad [2\,3, 2] = (x^1)^2x^3, \tag{4.2}$$

$$R_{1223} = R_{3221} = x^1x^3, \quad R_{1221} = \frac{(e^{x^2})}{4}, \quad R_{2113} = R_{3112} = -\frac{e^{x^2}}{2x^3}, \tag{4.3}$$

$$R_{11} = \frac{(e^{x^2})}{4(x^1x^3)^2}, \quad R_{22} = \frac{1}{4}, \quad R_{13} = \frac{1}{x^1x^3}, \quad R_{23} = -\frac{1}{2x^3}, \tag{4.4}$$

and the components which can be obtained from these by symmetric properties. Also it can be shown that the scalar curvature of the manifold is nonvanishing and nonconstant. We shall now show that this manifold is an $S(QE)_4$. Let us now define

$$a = \frac{1}{12(x^1x^3)^2}, \quad b = \frac{1}{6(x^1x^3)^2}, \quad c = -\frac{1}{\sqrt{2}x^1(x^3)^2}, \quad f = \frac{1}{(x^1x^3)^2},$$

the 1-form

$$A_i(x) = \begin{cases} x^1x^3, & \text{for } i = 2 \\ 0, & \text{otherwise.} \end{cases}$$

The corresponding vector field is given by

$$A^i(x) = \begin{cases} 1/x^1x^3, & \text{for } i = 2 \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad B_i(x) = \begin{cases} \sqrt{e^{x^2}/2}, & \text{for } i = 1, \\ 1/\sqrt{2}, & \text{for } i = 3, \\ 0, & \text{otherwise,} \end{cases}$$

the associated tensor as

$$D_{ij}(x) = \begin{cases} e^{x^2}/6, & \text{for } i = j = 1, \\ \frac{1}{2}(x^1)^2x^3\sqrt{e^{x^2}}, & \text{for } i = 1,\ j = 2, \\ -e^{x^2}/6, & \text{for } i = 3,\ j = 3, \\ 0, & \text{otherwise, at any point } x \in M. \end{cases}$$

Then we have

$$\begin{aligned} &\text{(i)} && R_{11} = ag_{11} + bA_1A_1 + 2cA_1B_1 + fD_{11}, \\ &\text{(ii)} && R_{22} = ag_{22} + bA_2A_2 + 2cA_2B_2 + fD_{22}, \\ &\text{(iii)} && R_{13} = ag_{13} + bA_1A_3 + c(A_1B_3 + A_3B_1) + fD_{13}, \\ &\text{(iv)} && R_{23} = ag_{23} + bA_2A_3 + c(A_2B_3 + A_3B_2) + fD_{23}. \end{aligned}$$

Since all the cases other than (i)–(iv) are trivial, we can say

$$R_{ij} = ag_{ij} + bA_iA_j + c(A_iB_j + A_jB_i) + fD_{ij}, \quad \text{for} \quad i,j = 1,2,3,4.$$

Also, we find that

$$\text{(v)}\ \ D_{ij}A^i = 0,\ j = 1,2,3,4,$$

$$\text{(vi)}\ \ g^{ij}A_iA_j = 1, \quad \text{(vii)}\ \ g^{ij}B_iB_j = 1, \quad \text{(viii)}\ \ g^{ij}A_iB_j = 0.$$

So, we can say that the manifold under consideration is an $S(QE)_4$. Thus we have the following theorem.

THEOREM 4.1. *Let $(M_4; g)$ be a Riemannian manifold endowed with the metric given by*

$$ds^2 = g_{ij}dx^idx^j = (e^{x^2})(dx^1)^2 + (x^1x^3)^2(dx^2)^2 + (dx^3)^2 + (dx^4)^2$$

where $i,j = 1,2,3,4$ and x^1, x^2, x^3, x^4 are the standard coordinates of M_4. Then it is an $S(QE)_4$ with nonzero and nonconstant scalar curvature.

Next, we are going to show that by a nominal modification of the metric given by equation (4.1), we can get a super quasi Einstein spacetime. Let us consider the Lorentzian metric g on the 4-dimensional real number space M_4 by

$$ds^2 = g_{ij}dx^idx^j = (e^{x^2})(dx^1)^2 + (x^1x^3)^2(dx^2)^2 + (dx^3)^2 - (dx^4)^2,$$

where $i,j = 1,2,3,4$ and x^1, x^2, x^3 and x^4 are the standard coordinates of M_4. Then the nonvanishing components of the Christoffel symbols, Riemann curvature tensor and Ricci tensor are same as equation (4.2), (4.3) and (4.4) respectively. Let us now define

$$a = \frac{3}{20(x^1x^3)^2}, \quad b = \frac{1}{10(x^1x^3)^2}, \quad c = -\frac{1}{\sqrt{2}x^1(x^3)^2}, \quad f = \frac{1}{10(x^1x^3)^2},$$

the 1-form

$$A_i(x) = \begin{cases} x^1x^3, & \text{for } i = 2, \\ \sqrt{2}, & \text{for } i = 4, \\ = 0, & \text{otherwise,} \end{cases}$$

the corresponding vector field is given by

$$A^i(x) = \begin{cases} 1/x^1x^3, & \text{for } i = 2, \\ -\sqrt{2}, & \text{for } i = 4, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad B_i(x) = \begin{cases} \sqrt{e^{x^2}/2}, & \text{for } i = 1, \\ 1/\sqrt{2}, & \text{for } i = 3, \\ 0, & \text{otherwise,} \end{cases}$$

the associated tensor as

$$D_{ij}(x) = \begin{cases} e^{x^2}, & \text{for } i = j = 1, \\ 5(x^1)^2x^3\sqrt{e^{x^2}}, & \text{for } i = 1,\ j = 2 \\ -e^{x^2}, & \text{for } i = j = 3 \\ 5x^1\sqrt{2e^{x^2}}, & \text{for } i = 1,\ j = 4 \\ 5x^1\sqrt{2}, & \text{for } i = 3,\ j = 4 \\ 0, & \text{otherwise, at any point } x \in M. \end{cases}$$

Then we have

$$\begin{aligned} &\text{(i)} \quad R_{11} = ag_{11} + bA_1A_1 + 2cA_1B_1 + fD_{11},\\ &\text{(ii)} \quad R_{22} = ag_{22} + bA_2A_2 + 2cA_2B_2 + fD_{22},\\ &\text{(iii)} \quad R_{13} = ag_{13} + bA_1A_3 + c(A_1B_3 + A_3B_1) + fD_{13},\\ &\text{(iv)} \quad R_{23} = ag_{23} + bA_2A_3 + c(A_2B_3 + A_3B_2) + fD_{23}.\end{aligned}$$

Since all the cases other than (i)–(iv) are trivial, we can say

$$R_{ij} = ag_{ij} + bA_iA_j + c(A_iB_j + A_jB_i) + fD_{ij}, \quad \text{for } i, j = 1, 2, 3, 4.$$

Also, we find that

$$\text{(v)} \ D_{ij}A^i = 0, \quad j = 1, 2, 3, 4,$$

$$\text{(vi)} \ g^{ij}A_iA_j = -1, \quad \text{(vii)} \ g^{ij}B_iB_j = 1, \quad \text{(viii)} \ g^{ij}A_iB_j = 0.$$

So, we can say that the manifold under consideration is a $S(QE)_4$ spacetime. Thus we have the theorem:

THEOREM 4.2. *Let $(M_4; g)$ be a 4-dimensional Lorentzian manifold endowed with the metric given by*

$$ds^2 = g_{ij}dx^idx^j = (e^{x^2})(dx^1)^2 + (x^1x^3)^2(dx^2)^2 + (dx^3)^2 - (dx^4)^2$$

where $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3, x^4 are the standard coordinates of M_4, is an $S(QE)_4$ spacetime with nonzero and nonconstant scalar curvature.

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