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THE SCALAR CURVATURE OF THE TANGENT BUNDLE OF A FINSLER MANIFOLD

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Abstract. Let $\mathbb{F}^m = (M, F)$ be a Finsler manifold and G be the Sasaki–Finsler metric on the slit tangent bundle $TM^0 = TM \setminus \{0\}$ of M. We express the scalar curvature $\widetilde{\rho}$ of the Riemannian manifold (TM^0, G) in terms of some geometrical objects of the Finsler manifold $\mathbb{F}^m$. Then, we find necessary and sufficient conditions for $\widetilde{\rho}$ to be a positively homogenenous function of degree zero with respect to the fiber coordinates of TM^0. Finally, we obtain characterizations of Landsberg manifolds, Berwald manifolds and Riemannian manifolds whose $\widetilde{\rho}$ satisfies the above condition.

Introduction

The geometry of the tangent bundle TM of a Riemannian manifold (M, g) goes back to Sasaki [**10**], who constructed on TM a Riemannian metric G which in our days is called the Sasaki metric. Then, several papers on the interrelations between the geometries of (M, g) and (TM, G) have been published (see Gudmundsson and Kappos [**6**] for results and references). The extension of the study from Riemannian manifolds to Finsler manifolds is not an easy task. This is because a Finsler manifold $\mathbb{F}^m = (M, F)$ does not admit a canonical linear connection on M, that plays the role of the Levi–Civita connection on a Riemannian manifold. Recently, the first author (cf. [**3**]) has initiated a study of the interrelations between the geometries of both the tangent bundle and indicatrix bundle of a Finsler manifold on one side, and the geometry of the manifold itself, on the other side. The main tool in the study was the Vrănceanu connection induced by the Levi–Civita connection on (TM^0, G), where G is the Sasaki–Finsler metric on TM^0.

We study the geometry of a Finsler manifold $\mathbb{F}^m = (M, F)$ under the assumption that the scalar curvature $\widetilde{\rho}$ of (TM^0, G) is a positively homogeneous function of degree zero with respect to the fiber coordinates (y^i) of TM^0. In the first part

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we present some geometric objects from the geometries of $\mathbb{F}^m$ and (TM^0, G) and following [**3**] we give some structure equations which relate the curvature tensor fields of the Levi–Civita connection and the Vrănceanu connection on (TM^0, G). In the second part we express $\widetilde{\rho}$ in terms of some geometric objects of the Finsler manifold $\mathbb{F}^m$ (cf. Theorem 2.1) and obtain necessary and sufficient conditions for $\widetilde{\rho}$ to be positively homogeneous of degree zero with respect to (y^i) (cf. Theorem 2.2). In particular, we prove that such an $\mathbb{F}^m$ is locally Minkowskian, provided M is a compact connected boundaryless manifold (cf. Corollary 2.1). Finally, we show that if $\mathbb{F}^m$ is a Berwald manifold (cf. Corollary 2.4) or a Riemannian manifold (cf. Corollary 2.5) and $\widetilde{\rho}$ satisfies the above condition, then $\mathbb{F}^m$ must be locally Minkowskian or locally Euclidean, respectively. In case of a Riemannian manifold, our result improves a well known result of Musso–Tricerri [**9**].

1. Preliminaries

Let $\mathbb{F}^m = (M, F)$ be an m-dimensional Finsler manifold, where F is the fundamental function of $\mathbb{F}^m$ that is supposed to be of class C^∞ on the slit tangent bundle $TM^0 = TM \setminus \{0\}$. Denote by (x^i, y^i), $i \in \{1, \dots, m\}$, the local coordinates on TM, where (x^i) are the local coordinates of a point $x \in M$ and (y^i) are the coordinates of a vector $y \in T_xM$. Then, F is positively homogeneous of degree 1 with respect to (y^i) and the functions

$$g_{ij} = \frac{1}{2}\,\frac{\partial^2 F^2}{\partial y^i \partial y^j},$$

define a symmetric Finsler tensor field of type $(0, 2)$ on TM^0. We suppose that the $m \times m$ matrix $[g_{ij}]$ is positive definite and denote its inverse by $[g^{ij}]$.

Next, we consider the *vertical bundle* VTM^0 over TM^0, which is the kernel of the differential of the projection map $\Pi : TM^0 \to M$. Denote by $\Gamma(VTM^0)$ the $\mathcal{F}(TM^0)$-module of sections of VTM^0, where $\mathcal{F}(TM^0)$ is the algebra of smooth functions on TM^0. The same notation will be used for any other vector bundle. Locally, $\Gamma(VTM^0)$ is spanned by the natural vector fields $\{\partial/\partial y^1, \dots, \partial/\partial y^m\}$. Then, we define the vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G_i^j\,\frac{\partial}{\partial y^j}\,, \quad i \in \{1, \dots, m\},$$

where we put

$$G_i^j = \frac{\partial G^j}{\partial y^i} \;\text{ and }\; G^j = \frac{1}{4}\,g^{jk}\left\{\frac{\partial^2 F^2}{\partial y^k \partial x^i}\,y^i - \frac{\partial F^2}{\partial x^k}\right\}.$$

Thus, we obtain the *horizontal bundle* HTM^0 over TM^0, which is locally spanned by $\{\delta/\delta x^1, \dots, \delta/\delta x^m\}$. Moreover, we have the decomposition

$$TTM^0 = HTM^0 \oplus VTM^0,$$

which enables us to define the *Sasaki–Finsler metric* G on TM^0 as follows (cf. Bejancu–Farran [**4**, p. 35])

$$G\left(\frac{\delta}{\delta x^j}\,,\frac{\delta}{\delta x^i}\right) = G\left(\frac{\partial}{\partial y^j}\,,\frac{\partial}{\partial y^i}\right) = g_{ij}(x, y),\; G\left(\frac{\delta}{\delta x^j}\,,\frac{\partial}{\partial y^i}\right) = 0. \tag{1.1}$$

Now, we define some geometric objects of Finsler type on TM^0. First, we express the Lie brackets of the above vector fields as follows:

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = \mathbf{R}^k{}_{ij}\,\frac{\partial}{\partial y^k}, \quad \left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right] = G_i{}^k{}_j\,\frac{\partial}{\partial y^k},$$

where we put

$$\mathbf{R}^k{}_{ij} = \frac{\delta G^k_i}{\delta x^j} - \frac{\delta G^k_j}{\delta x^i}, \quad G_i{}^k{}_j = \frac{\partial G^k_j}{\partial y^i}.$$

If $\mathbf{R}^k{}_{ij} = 0$ for all $i, j, k \in \{1, \dots, m\}$, we say that $\mathbb{F}^m$ is a *flat Finsler manifold*. This name is justified by the fact that in this case the flag curvature of $\mathbb{F}^m$ vanishes identically on TM^0. Also, the functions

$$F_i{}^k{}_j = \frac{1}{2}\,g^{kh}\left\{\frac{\delta g_{hi}}{\delta x^j} + \frac{\delta g_{hj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^h}\right\},$$

represent the local coefficients of Chern–Rund connection. Then, we define a Finsler tensor field of type $(1, 2)$ whose local components are given by $B_i{}^k{}_j = F_i{}^k{}_j - G_i{}^k{}_j$. Finally, the Cartan tensor field is given by its local components

$$C_i{}^k{}_j = \frac{1}{2}\,g^{kh}\,\frac{\partial g_{ij}}{\partial y^h}.$$

Next, we denote by h and v the projection morphisms of TTM^0 on HTM^0 and VTM^0, respectively. Then, by using the above Finsler tensor fields $\mathbf{R}^k{}_{ij}$, $C_i{}^k{}_j$ and $B_i{}^k{}_j$ we define the following adapted tensor fields:

$$\mathbf{R} : \Gamma(HTM^0) \times \Gamma(HTM^0) \to \Gamma(VTM^0), \quad \mathbf{R}(hX, hY) = \mathbf{R}^k{}_{ij} Y^i X^j\,\frac{\partial}{\partial y^k} \tag{1.2}$$

$$C : \Gamma(HTM^0) \times \Gamma(HTM^0) \to \Gamma(VTM^0), \quad C(hX, hY) = C_i{}^k{}_j Y^i X^j\,\frac{\partial}{\partial y^k}, \tag{1.3}$$

$$B : \Gamma(VTM^0) \times \Gamma(VTM^0) \to \Gamma(HTM^0), \quad B(vU, vW) = B_i{}^k{}_j W^i U^j\,\frac{\delta}{\delta x^k}, \tag{1.4}$$

where we set

$$hX = X^j\frac{\delta}{\delta x^j}, \; hY = Y^i\frac{\delta}{\delta x^i}, \; vU = U^j\frac{\partial}{\partial y^j}, \; vW = W^i\frac{\partial}{\partial y^i}.$$

For each of the above tensor fields $\mathbf{R}, C$ and B we define a twin (denoted by the same symbol) as follows:

$$\begin{aligned} &\mathbf{R} : \Gamma(HTM^0) \times \Gamma(VTM^0) \to \Gamma(HTM^0), \\ &g(\mathbf{R}(hX, vY), hZ) = G(\mathbf{R}(hX, hZ), vY), \end{aligned} \tag{1.5}$$

$$\begin{aligned} &C : \Gamma(HTM^0) \times \Gamma(VTM^0) \to \Gamma(HTM^0), \\ &G(C(hX, vY), hZ) = G(C(hX, hZ), vY), \end{aligned} \tag{1.6}$$

$$\begin{aligned} &B : \Gamma(HTM^0) \times \Gamma(VTM^0) \to \Gamma(VTM^0), \\ &G(B(hX, vY), vZ) = G(B(vY, vZ), hX). \end{aligned} \tag{1.7}$$

Locally, we have the following formulas:

$$(1.8)\qquad \begin{gathered}\text{(a) } \mathbf{R}\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right) = \mathbf{R}^k{}_{ij}\,\frac{\partial}{\partial y^k}, \quad \text{(b) } C\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right) = C_i{}^k{}_j\,\frac{\partial}{\partial y^k}, \\ \text{(c) } B\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i}\right) = B_i{}^k{}_j\,\frac{\delta}{\delta x^k},\end{gathered}$$

$$(1.9)\qquad \begin{gathered}\text{(a) } \mathbf{R}\left(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}\right) = \overline{\mathbf{R}}^k{}_{ij}\,\frac{\delta}{\delta x^k}, \quad \text{(b) } C\left(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}\right) = \overline{C}_i{}^k{}_j\,\frac{\delta}{\delta x^k}, \\ \text{(c) } B\left(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}\right) = \overline{B}_i{}^k{}_j\,\frac{\partial}{\partial y^k},\end{gathered}$$

$$(1.10)\qquad \text{(a) } \overline{\mathbf{R}}^k{}_{ij} = g_{ih}\,\mathbf{R}^h{}_{tj}\,g^{tk}, \quad \text{(b) } \overline{C}_i{}^k{}_j = C_i{}^k{}_j, \quad \text{(c) } \overline{B}_i{}^k{}_j = B_i{}^k{}_j.$$

Now, let $\widetilde{\nabla}$ be the Levi–Civita connection on (TM^0, G) and ∇ be the Vrănceanu connection induced by $\widetilde{\nabla}$ given by (cf. Ianuş [**7**])

$$\nabla_X Y = v\widetilde{\nabla}_{vX} vY + h\widetilde{\nabla}_{hX} hY + v[hX, vY] + h[vX, hY].$$

It is important to note that the Vrănceanu connection is locally given by the local coefficients of the classical Finsler connections as follows:

$$(1.11)\qquad \begin{aligned} &\nabla_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta x^i} = F_i{}^k{}_j\,\frac{\delta}{\delta x^k}, \quad && \nabla_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i} = C_i{}^k{}_j\,\frac{\partial}{\partial y^k}, \\ &\nabla_{\frac{\partial}{\partial y^j}}\frac{\delta}{\delta x^i} = 0, && \nabla_{\frac{\delta}{\delta x^j}}\frac{\partial}{\partial y^i} = G_i{}^k{}_j\,\frac{\partial}{\partial y^k}. \end{aligned}$$

Moreover, the curvature tensor field $\widetilde{R}$ of the Levi–Civita connection $\widetilde{\nabla}$ is completely determined by the curvature tensor field R of the Vrănceanu connection on (TM^0, G) and the adapted tensor fields $\mathbf{R}, C$ and B (cf. Bejancu [**3**]). We recall here only the following relations:

$$(1.12)\qquad \begin{aligned}\widetilde{R}(hX, hY, hZ) &= R(hX, hY, hZ) + B(hZ, \mathbf{R}(hX, hY)) \\ &\quad + C(hZ, \mathbf{R}(hX, hY)) + \frac{1}{2}\,\mathbf{R}(hZ, \mathbf{R}(hX, hY)) \\ &\quad - \mathcal{A}_{(hX,hY)}\Big\{(\nabla_{hX}C)(hY, hZ) + \frac{1}{2}(\nabla_{hX}\mathbf{R})(hY, hZ) \\ &\qquad + B(hX, C(hY, hZ)) + \frac{1}{2}\,B(hX, \mathbf{R}(hY, hZ)) \\ &\qquad + C(hX, C(hY, hZ)) + \frac{1}{2}\,C(hX, \mathbf{R}(hY, hZ)) \\ &\qquad + \frac{1}{2}\,\mathbf{R}(hX, C(hY, hZ)) + \frac{1}{4}\,\mathbf{R}(hX, \mathbf{R}(hY, hZ))\Big\},\end{aligned}$$

$$\begin{aligned}\widetilde{R}(hX, vY, vZ) &= R(hX, vY, vZ) - (\nabla_{hX} B)(vY, vZ) \\ &\quad - (\nabla_{vY} B)(hX, vZ) - (\nabla_{vY} C)(hX, vZ) - \frac{1}{2}(\nabla_{vY} \mathbf{R})(hX, vZ) \\ &\quad + C(hX, B(vY, vZ)) + \frac{1}{2}\mathbf{R}(hX, B(vY, vZ)) + B(vY, B(hX, vZ)) \\ &\quad - C(C(hX, vZ), vY) - \frac{1}{2} C(\mathbf{R}(hX, vZ), vY) \\ &\quad - \frac{1}{2}\mathbf{R}(C(hX, vZ), vY) - \frac{1}{4}\mathbf{R}(\mathbf{R}(hX, vZ), vY) \\ &\quad - B(C(hX, vZ), vY) - \frac{1}{2} B(\mathbf{R}(hX, vZ), vY),\end{aligned} \tag{1.13}$$

$$\begin{aligned}\widetilde{R}(vX, vY, vZ) &= R(vX, vY, vZ) - \mathcal{A}_{(vX,vY)}\Big\{(\nabla_{vX} B)(vY, vZ) \\ &\quad + C(B(vY, vZ), vX) + \frac{1}{2}\mathbf{R}(B(vY, vZ), vX) + B(B(vY, vZ), vX)\Big\},\end{aligned} \tag{1.14}$$

where $\mathcal{A}_{(hX,hY)}$ means that in the expression that follows this symbol we interchange hX and hY, and then subtract, as in the following formula

$$\mathcal{A}_{(hX,hY)}\{f(hX, hY)\} = f(hX, hY) - f(hY, hX).$$

In a similar way, we use the symbol $\mathcal{A}_{(vX,vY)}$. Finally, we present some local components of the curvature tensor field of the Vrănceanu connection on (TM^0, G):

$$\begin{aligned}&\text{(a)} \quad R\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^i} = K_i{}^h{}_{jk}\ \frac{\delta}{\delta x^h}, \\ &\text{(b)} \quad R\left(\frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^j}\right)\frac{\partial}{\partial y^i} = S_i{}^h{}_{jk}\ \frac{\partial}{\partial y^h}\end{aligned} \tag{1.15}$$

where we set

$$\begin{aligned}&\text{(a)} \quad K_i{}^h{}_{jk} = \frac{\delta F_i{}^h{}_j}{\delta x^k} - \frac{\delta F_i{}^h{}_k}{\delta x^j} + F_i{}^t{}_j\, F_t{}^h{}_k - F_i{}^t{}_k\, F_t{}^h{}_j, \\ &\text{(b)} \quad S_i{}^h{}_{jk} = \frac{\partial C_i{}^h{}_j}{\partial y^k} - \frac{\partial C_i{}^h{}_k}{\partial y^j} + C_i{}^t{}_j\, C_t{}^h{}_k - C_i{}^t{}_k\, C_t{}^h{}_j.\end{aligned} \tag{1.16}$$

We note that (1.16a) and (1.16b) give the local components of the hh-curvature and vv-curvature tensor fields of the Chern–Rund connection and Cartan connection, respectively.

2. Scalar curvature of (TM^0, G)

Let $\mathbb{F}^m = (M, F)$ be a Finsler manifold and (TM^0, G) be its slit tangent bundle endowed with the Sasaki–Finsler metric G given by (1.1). Consider the local orthonormal fields of frames $\{H_a\}$ and $\{V_a\}$, such that $H_a \in \Gamma(HTM^0)$ and $V_a \in \Gamma(VTM^0)$ for any $a \in \{1, \ldots, m\}$. Next, we set

$$H_a = H_a^i\ \frac{\delta}{\delta x^i} \quad \text{and} \quad V_a = V_a^i\ \frac{\partial}{\partial y^i}\cdot \tag{2.1}$$

Then, by using (1.1), we deduce that the inverse matrix of $[g_{ij}]$ has the entries given by

$$g^{ij} = \sum_{a=1}^{m} H_a^i H_a^j = \sum_{a=1}^{m} V_a^i V_a^j, \quad i,j \in \{1,\dots,m\}. \tag{2.2}$$

Now we denote by $\widetilde{\rho}$ the scalar curvature of the Riemannian manifold (TM^0, G). As $\{H_a, V_a\}$, $a \in \{1,\dots,m\}$, is a local orthonormal frame field on TM^0 with respect to G, we have

$$\widetilde{\rho} = \alpha + 2\beta + \gamma, \tag{2.3}$$

where we put

$$\begin{aligned} &\text{(a)} \quad \alpha = \sum_{a,b=1}^{m} \left\{ G\big(\widetilde{R}(H_a, H_b)H_b, H_a\big) \right\}, \\ &\text{(b)} \quad \beta = \sum_{a,b=1}^{m} \left\{ G\big(\widetilde{R}(H_a, V_b)V_b, H_a\big) \right\}, \\ &\text{(c)} \quad \gamma = \sum_{a,b=1}^{m} \left\{ G\big(\widetilde{R}(V_a, V_b)V_b, V_a\big) \right\}. \end{aligned} \tag{2.4}$$

In what follows we will express the above three functions α, β, γ in terms of the local components of some important Finsler tensor fields.

First, by using (1.5) and (1.6) and taking into account that $\mathbf{R}$ and C are skew-symmetric and symmetric adapted tensor fields respectively, we obtain

$$\begin{aligned} &\text{(a)} \quad G\big(C(H_a, C(H_b, H_b)), H_a\big) = G\big(C(H_a, H_a), C(H_b, H_b)\big), \\ &\text{(b)} \quad G\big(C(H_b, C(H_a, H_b)), H_a\big) = \|C(H_a, H_b)\|^2, \\ &\text{(c)} \quad G\big(C(H_b, \mathbf{R}(H_a, H_b)), H_a\big) = -G\big(\mathbf{R}(H_b, C(H_a, H_b)), H_a\big) \\ &\qquad\qquad = G\big(C(H_a, H_b), \mathbf{R}(H_a, H_b)\big), \\ &\text{(d)} \quad G\big(\mathbf{R}(H_a, C(H_b, H_b)), H_a\big) = G\big(\mathbf{R}(H_a, H_a), C(H_b, H_b)\big) = 0, \\ &\text{(e)} \quad G\big(\mathbf{R}(H_b, \mathbf{R}(H_a, H_b)), H_a\big) = -\|\mathbf{R}(H_a, H_b)\|^2, \\ &\text{(f)} \quad \sum_{a,b=1}^{m} \left\{ G(C(H_a, H_b), \mathbf{R}(H_a, H_b)) \right\} = 0, \end{aligned} \tag{2.5}$$

where the norm $\|\cdot\|$ is taken with respect to G. Then, by direct calculations using (2.4a), (1.12) and (2.5), we deduce that

$$\begin{aligned} \alpha &= \sum_{a,b=1}^{m} \left\{ G(h\widetilde{R}(H_a, H_b)H_b, H_a) \right\} \\ &= \sum_{a,b=1}^{m} \Big\{ G(R(H_a, H_b)H_b, H_a) - \frac{3}{4}\|\mathbf{R}(H_a, H_b)\|^2 + \|C(H_a, H_b)\|^2 \\ &\qquad - G(C(H_a, H_a), C(H_b, H_b)) \Big\}. \end{aligned} \tag{2.6}$$

Next, by using (1.5), (1.6) and (1.7), we obtain

$$\begin{aligned}
&\text{(a)} \quad G(B(V_b, B(H_a, V_b)), H_a) = \|B(H_a, V_b)\|^2,\\
&\text{(b)} \quad G(C(C(H_a, V_b), V_b), H_a) = \|C(H_a, V_b)\|^2,\\
&\text{(c)} \quad G(C(\mathbf{R}(H_a, V_b), V_b), H_a) + G(\mathbf{R}(C(H_a, V_b), V_b), H_a) = 0,\\
&\text{(d)} \quad G(\mathbf{R}(\mathbf{R}(H_a, V_b), V_b), H_a) = -\|\mathbf{R}(H_a, V_b)\|^2.
\end{aligned} \tag{2.7}$$

Then, taking into account (2.4b), (1.13) and (2.7), we infer that

$$\begin{aligned}
\beta = \sum_{a,b=1}^{m} \Big\{&\|B(H_a, V_b)\|^2 - \|C(H_a, V_b)\|^2 + \frac{1}{4}\|\mathbf{R}(H_a, V_b)\|^2\\
&- G\Big((\nabla_{H_a} B)(V_b, V_b) + (\nabla_{V_b} C)(H_a, V_b) + \frac{1}{2}(\nabla_{V_b}\mathbf{R})(H_a, V_b), H_a\Big)\Big\}.
\end{aligned} \tag{2.8}$$

Now, as a consequence of (1.7), we obtain

$$\begin{aligned}
&\text{(a)} \quad G(B(B(V_b, V_b), V_a), V_a) = G(B(V_a, V_a), B(V_b, V_b))\\
&\text{(b)} \quad G(B(B(V_a, V_b), V_b), V_a) = \|B(V_a, V_b)\|^2.
\end{aligned} \tag{2.9}$$

Then, by using (2.4c), (1.14) and (2.9), we deduce that

$$\gamma = \sum_{a,b=1}^{m} \big\{G(R(V_a, V_b)V_b, V_a) + \|B(V_a, V_b)\|^2 - G(B(V_a, V_a), B(V_b, V_b))\big\}. \tag{2.10}$$

Also, by using (2.1), (2.2), (1.8), (1.9) and (1.10), we obtain

$$\begin{aligned}
&\text{(a)} \quad \sum_{a,b=1}^{m} \|\mathbf{R}(H_a, V_b)\|^2 = \sum_{a,b=1}^{m} \|\mathbf{R}(H_a, H_b)\|^2 = g^{ik}\, g^{jh}\, g_{st}\, \mathbf{R}^s{}_{ij}\, \mathbf{R}^t{}_{kh},\\
&\text{(b)} \quad \sum_{a,b=1}^{m} \|C(H_a, V_b)\|^2 = \sum_{a,b=1}^{m} \|C(H_a, H_b)\|^2 = g^{ik}\, g^{jh}\, g_{st}\, C_i{}^s{}_j\, C_k{}^t{}_h,\\
&\text{(c)} \quad \sum_{a,b=1}^{m} \|B(H_a, V_b)\|^2 = \sum_{a,b=1}^{m} \|B(V_a, V_b)\|^2 = g^{ik}\, g^{jh}\, g_{st}\, B_i{}^s{}_j\, B_k{}^t{}_h.
\end{aligned} \tag{2.11}$$

Finally, by using (2.3), (2.6), (2.8), (2.10) and (2.11), we deduce that

$$\begin{aligned}
\widetilde{\rho} = \sum_{a,b=1}^{m} \Big\{&G(R(H_a, H_b)H_b, H_a) + G(R(V_a, V_b)V_b, V_a)\\
&- \frac{1}{4}\|\mathbf{R}(H_a, H_b)\|^2 - \|C(H_a, H_b)\|^2 + 3\|B(V_a, V_b)\|^2\\
&- G\big(C(H_a, H_a), C(H_b, H_b)\big) - G\big(B(V_a, V_a), B(V_b, V_b)\big)\\
&-2G\Big((\nabla_{H_a} B)(V_b, V_b) + (\nabla_{V_b} C)(H_a, V_b) + \frac{1}{2}(\nabla_{V_b}\mathbf{R})(H_a, V_b), H_a\Big)\Big\}.
\end{aligned} \tag{2.12}$$

Next, we want to express the scalar curvature of (TM^0, G) in terms of some geometric objects of Finsler type of $\mathbb{F}^m$. First, by using (2.1), (2.2), (1.15), (1.8b) and

(1.8c), we deduce that

$$
\text{(2.13)}\quad
\begin{aligned}
&\text{(a)}\quad \sum_{a,b=1}^{m} \big\{G\big(R(H_a, H_b)H_b, H_a\big)\big\} = g^{ik}\, g^{jh}\, K_{ijkh},\\
&\text{(b)}\quad \sum_{a,b=1}^{m} \big\{G\big(R(V_a, V_b)V_b, V_a\big)\big\} = g^{ik}\, g^{jh}\, S_{ijkh},\\
&\text{(c)}\quad \sum_{a,b=1}^{m} \big\{G\big(C(H_a, H_a), C(H_b, H_b)\big)\big\} = g^{ik}\, g^{jh}\, g_{st}\, C_i{}^s{}_k\, C_j{}^t{}_h,\\
&\text{(d)}\quad \sum_{a,b=1}^{m} \big\{G\big(B(V_a, V_a), B(V_b, V_b)\big)\big\} = g^{ik}\, g^{jh}\, g_{st}\, B_i{}^s{}_k\, B_j{}^t{}_h.
\end{aligned}
$$

Then, by using (2.1), (2.2) and (1.11), we obtain

$$
\text{(2.14)}\quad
\begin{aligned}
&\text{(a)}\quad \sum_{a,b=1}^{m} \big\{G\big((\nabla_{H_a} B)(V_b, V_b), H_a\big)\big\} = g^{jh}\, B_j{}^i{}_{h|i},\\
&\text{(b)}\quad \sum_{a,b=1}^{m} \big\{G\big((\nabla_{V_b} C)(H_a, V_b), H_a\big)\big\} = g^{jh}\, C_h{}^i{}_{i\|j},\\
&\text{(c)}\quad \sum_{a,b=1}^{m} \big\{G\big((\nabla_{V_b} \mathbf{R})(H_a, V_b), H_a\big)\big\} = g^{jh}\, \overline{\mathbf{R}}^i{}_{hi\|j},
\end{aligned}
$$

where the covariant derivatives on the right side are defined by the Vrănceanu connection as follows

$$
\text{(2.15)}\quad
\begin{aligned}
&\text{(a)}\quad B_j{}^i{}_{h|i} = \frac{\delta B_j{}^i{}_h}{\delta x^i} + B_j{}^k{}_h\, F_k{}^i{}_i - B_k{}^i{}_h\, G_j{}^k{}_i - B_j{}^i{}_k\, G_h{}^k{}_i,\\
&\text{(b)}\quad C_h{}^i{}_{i\|j} = \frac{\partial C_h{}^i{}_i}{\partial y^j} - C_k{}^i{}_i\, C_h{}^k{}_j,\\
&\text{(c)}\quad \overline{\mathbf{R}}_h{}^i{}_{i\|j} = \frac{\partial \overline{\mathbf{R}}^i{}_{hi}}{\partial y^j} - \overline{\mathbf{R}}^i{}_{ki}\, C_h{}^k{}_j.
\end{aligned}
$$

Thus, by using (2.11), (2.13) and (2.14) into (2.12), we deduce that the scalar curvature of (TM^0, G) is given by

$$
\text{(2.16)}\quad
\begin{aligned}
\widetilde{\rho} = g^{ik}\, g^{jh}\Big\{&K_{ijkh} + S_{ijkh} - \frac{1}{4}\, g_{st}\, \mathbf{R}^s{}_{ij}\, \mathbf{R}^t{}_{kh} - g_{st}\, C_i{}^s{}_j\, C_k{}^t{}_h\\
&+ 3g_{st}\, B_i{}^s{}_j\, B_k{}^t{}_h - g_{st}\, C_i{}^s{}_k\, C_j{}^t{}_h - g_{st}\, B_i{}^s{}_k\, B_j{}^t{}_h\Big\}\\
&- 2g^{jh}\Big\{B_j{}^i{}_{h|i} + C_h{}^i{}_{i\|j} + \frac{1}{2}\, \overline{\mathbf{R}}^i{}_{hi\|j}\Big\}.
\end{aligned}
$$

An interesting formula for S_{ijkh} was given by Matsumoto [**8**, p. 114]:

$$
\text{(2.17)}\quad S_{ijkh} = g_{st}\{C_i{}^s{}_h\, C_j{}^t{}_k - C_i{}^s{}_k\, C_j{}^t{}_h\}.
$$

Then, by direct calculations, using (2.17) and (2.15b), we obtain

$$g^{ik}\,g^{jh}\{S_{ijkh}-g_{st}\,C_i{}^s{}_j\,C_k{}^t{}_h-g_{st}\,C_i{}^s{}_k\,C_j{}^t{}_h\}-2g^{jh}\,C_h{}^i{}_{i\|j}=-2g^{jh}\,\frac{\partial C_h}{\partial y^j},\tag{2.18}$$

where we put

$$C_h=C_h{}^i{}_i=g^{ki}\,C_{hki}.\tag{2.19}$$

Taking into account of (2.18) into (2.16), we can state the following.

Theorem 2.1. *Let $\mathbb{F}^m=(M,F)$ be a Finsler manifold. Then, the scalar curvature $\widetilde{\rho}$ of the Riemannian manifold (TM^0,G) is given by*

$$\begin{aligned}\widetilde{\rho}=g^{ik}\,g^{jh}\Big\{K_{ijkh}+3g_{st}\,B_i{}^s{}_j\,B_k{}^t{}_h-g_{st}\,B_i{}^s{}_k\,B_j{}^t{}_h-\frac{1}{4}\,g_{st}\mathbf{R}^s{}_{ij}\,\mathbf{R}^t{}_{kh}\Big\}\\-2g^{jh}\Big\{B_j{}^i{}_{h|i}+\frac{\partial C_h}{\partial y^j}+\frac{1}{2}\,\overline{\mathbf{R}}^i{}_{hi\|j}\Big\}.\end{aligned}\tag{2.20}$$

Next, following Matsumoto [**8**, p. 176], we call

$$C^i=g^{ih}\,C_h,\tag{2.21}$$

the *torsion vector field* of $\mathbb{F}^m$. Then, we can prove the following.

Theorem 2.2. *Let $\mathbb{F}^m=(M,F)$ be a Finsler manifold. Then, the scalar curvature of (TM^0,G) is a positively homogeneous function of degree zero with respect to (y^i) if and only if the following conditions are satisfied:*

(i) *$\mathbb{F}^m$ is a flat Finsler manifold.*

(ii) *The torsion vector field of $\mathbb{F}^m$ satisfies*

$$\frac{\partial C^i}{\partial y^i}+2g_{jk}\,C^j\,C^k=0.\tag{2.22}$$

Proof. First, we express (2.20) as follows

$$\widetilde{\rho}=A+B+C,\tag{2.23}$$

where

$$\begin{aligned}&\text{(a)}\quad A=g^{ik}\,g^{jh}\{K_{ijkh}+3g_{st}\,B_i{}^s{}_j\,B_k{}^t{}_h-g_{st}\,B_i{}^s{}_k\,B_j{}^t{}_h\}\\&\qquad\qquad-2g^{jh}\Big\{B_j{}^i{}_{h|i}+\frac{1}{2}\,\overline{\mathbf{R}}^i{}_{hi\|j}\Big\},\\&\text{(b)}\quad B=-\frac{1}{4}\,g^{ik}\,g^{jh}\,g_{st}\,\mathbf{R}^s{}_{ij}\,\mathbf{R}^t{}_{kh},\\&\text{(c)}\quad C=-2g^{jh}\,\frac{\partial C_h}{\partial y^j}\,.\end{aligned}\tag{2.24}$$

By the homogeneity properties of the functions on the right side of (2.24), we conclude that A,B and C are positively homogeneous functions of degrees 0, 1 and -2, respectively. Then, from (2.23) and (2.24), we deduce that $\widetilde{\rho}$ is positively homogeneous of degree 0 if and only if we have

$$\text{(a)}\quad g^{ik}\,g^{jh}\,g_{st}\,\mathbf{R}^s{}_{ij}\,\mathbf{R}^t{}_{kh}=0,\qquad\text{(b)}\quad g^{jh}\,\frac{\partial C_h}{\partial y^j}=0.\tag{2.25}$$

Clearly, (2.25a) holds if and only if $\mathbf{R}^k{}_{ij} = 0$, for each $i, j, k \in \{1, \dots, m\}$, that is, $\mathbb{F}^m$ is a flat Finsler manifold. On the other hand, by using (2.21) and (2.19), we deduce that

$$\begin{aligned} g^{jh} \frac{\partial C_h}{\partial y^j} &= g^{jh} \left\{ \frac{\partial g_{hk}}{\partial y^j} C^k + g_{hk} \frac{\partial C^k}{\partial y^j} \right\} \\ &= \frac{\partial C^j}{\partial y^j} + 2 g^{jh} C_{hkj} C^k = \frac{\partial C^j}{\partial y^j} + 2 g_{kh} C^k C^h. \end{aligned}$$

Thus, (2.25b) is equivalent to (2.22). This completes the proof of the theorem. □

Next, we recall the following results on the geometry of $\mathbb{F}^m$.

THEOREM 2.3 (Akbar–Zadeh [**1**]). *Let $\mathbb{F}^m = (M, F)$ be a compact connected boundaryless flat Finsler manifold. Then, $\mathbb{F}^m$ is locally Minkowskian.*

THEOREM 2.4 (Deicke [**5**]). *Let $\mathbb{F}^m = (M, F)$ be a Finsler manifold such that F is positive and C^4-differentiable for any nonzero (y^i). If the torsion vector field vanishes on M, then $\mathbb{F}^m$ must be Riemannian.*

Now, by combining Theorems 2.2 and 2.3, we obtain the following:

COROLLARY 2.1. *Let $\mathbb{F}^m = (M, F)$ be a be a compact connected boundaryless Finsler manifold. If the scalar curvature of (TM^0, G) is a positively homogeneous function of degree zero with respect to (y^i), then $\mathbb{F}^m$ is locally Minkowskian.*

Also, we can prove the following.

COROLLARY 2.2. *Let $\mathbb{F}^m = (M, F)$ be a Finsler manifold such that F is positive and C^4-differentiable for any nonzero (y^i). Suppose the torsion vector field of $\mathbb{F}^m$ satisfies*

$$\operatorname{trace}\left[\frac{\partial C^i}{\partial y^j}\right] \geqslant 0. \tag{2.26}$$

Then the scalar curvature of (TM^0, G) is positively homogeneous of degree 0 with respect to (y^i) if and only if $\mathbb{F}^m$ is locally Euclidean.

PROOF. If $\mathbb{F}^m$ is locally Euclidean, then both the flag curvature and the torsion vector field of $\mathbb{F}^m$ vanish on M, and by Theorem 2.2 we conclude that $\widetilde{\rho}$ is positively homogeneous of degree 0. Conversely, suppose that $\widetilde{\rho}$ is positively homogeneous of degree 0. Then, from (2.26) and (2.22), we deduce that the torsion vector field of $\mathbb{F}^m$ vanishes on M. Thus, by Theorem 2.4, we conclude that $\mathbb{F}^m$ must be Riemannian. Finally, from Theorem 2.2 we see that $\mathbb{F}^m$ is a flat Finsler manifold. Hence, F^m is locally Euclidean. □

COROLLARY 2.3. *Let $\mathbb{F}^m = (M, F)$ be a Landsberg manifold. If the scalar curvature of (TM^0, G) is a positively homogeneous function of degree 0 with respect to (y^i), then it vanishes on TM.*

Proof. Since ${B_j}^i{}_k = 0$ for all $i, j, k \in \{1, \ldots, m\}$, by (2.23) and (2.24) we deduce that $\widetilde{\rho} = g^{ik}\, g^{jh}\, K_{ijkh}$. Also, by assertion (i) of Theorem 2.2, we conclude that $\mathbb{F}^m$ is of flag curvature $\lambda = 0$. On the other hand, the hh-curvature tensor of the Chern–Rund connection for a Landsberg manifold of constant curvature λ is given by (cf. Bao–Chern–Shen [**2**, p. 314])

$$K_{ijkh} = \lambda(g_{jk}\, g_{ih} - g_{ik}\, g_{jh}).$$

Thus, $\widetilde{\rho}$ vanishes on TM. □

Corollary 2.4. *Let $\mathbb{F}^m = (M, F)$ be a Berwald manifold. Suppose that the scalar curvature $\widetilde{\rho}$ of (TM^0, G) is a positively homogeneous function of degree 0 with respect to (y^i). Then $\widetilde{\rho} = 0$, and $\mathbb{F}^m$ is locally Minkowskian.*

Proof. As $\mathbb{F}^m$ is a Landsberg manifold, we apply Corollary 2.3 and obtain $\widetilde{\rho} = 0$. Then, by assertion (iv) of Theorem 8.5 of Bejancu–Farran [**4**, p. 65], we deduce that the hv-curvature tensor field F_{ijkh} of the Chern–Rund connection vanishes on M. Finally, from the proof of Corollary 2.3, we deduce that $K_{ijkh} = 0$. Hence, by assertion (iv) of Theorem 8.6 of Bejancu–Farran [**4**, p. 66], we conclude that $\mathbb{F}^m$ is locally Minkowskian. □

Corollary 2.5. *Let $\mathbb{F}^m = (M, F)$ be a Riemannian manifold and TM be equipped with the Sasaki metric G. Suppose that the scalar curvature $\widetilde{\rho}$ of (TM^0, G) is a positively homogeneous function of degree 0 with respect to (y^i). Then $\widetilde{\rho} = 0$, and $\mathbb{F}^m$ is locally Euclidean.*

Proof. By Corollary 2.4 we have $\widetilde{\rho} = 0$ and $K_{ijkh} = 0$. As in this case K_{ijkh} are the local components of the curvature tensor field of the Levi–Civita connection on (M, g), we conclude that (M, g) is locally Euclidean. □

Finally, we note that Corollary 2.5 improves some well-known results of Musso–Tricerri [**9**] and Yano–Okubo [**11**] which state the following:

If the scalar curvature of (TM, G) is constant, then (M, g) is locally Euclidean.

and

If the scalar curvature of (TM, G) vanishes, then (M, g) is locally Euclidean.

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