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2-NORMED ALGEBRAS-I

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ABSTRACT. The class of 2-normed real algebras, defined in [**33**], as shown in [**29**] is either void or contains only trivial algebras. In this paper, a new definition of real or complex 2-normed algebras and 2-Banach algebras are proposed. Several examples of such algebras are given.

1. Introduction

Following the introductory notion of Banach algebra by Nagumo in 1936 [**34**], Gelfand et al. put forward a number of fundamental results related to the theory of Banach algebras [**21, 22**]. This has created an impetus to the study of algebras having manifold applications in the theory of dynamical systems, particle physics etc. Later on important contributions were made by Dunford [**12, 13, 14**], Yoshida [**42**], Kaplansky [**26, 27**], Raubenheimer et al. [**35**], Burlando [**6**] and others [**2, 4, 5, 10, 11, 36, 37, 38**] leading to the development of various Banach algebra techniques which have produced many new results and simplified the theories related to matrices, integral equations and operators.

The spectral radius formula established by Gelfand gives an excellent illustration of interplay between the algebraic structure and the metric structure on the class $B(X)$ of all continuous linear transformations from a complex Banach space X into X $(\dim X \geqslant 2)$.

Extensive investigations have been made for the concepts of 2-metric and 2-normed linear spaces with different point of views [**1, 3, 7, 8, 9, 15, 16, 17, 18, 19, 20, 23, 24, 25, 28, 30, 31, 32, 39, 40, 41**].

Considering the concept of 2-normed algebras put forward by Noor Mohammad and Siddiuqe [**33**], Lal et al. have shown that the class of 2-normed algebras with unity as defined in [**33**] is either void or contains only trivial algebras [**29**].

A new definition of 2-normed algebras and an example satisfying this definition is given in Section 5. In a subsequent work we are trying to show that there exist 2-normed algebras (with or without unity) which are not normable and a 2-Banach

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algebra need not be a 2-Banach space. Spectrum will be defined and some results for spectral radius in case of 2-Banach algebras shall be given establishing that the spectral radius formula relates the algebraic and analytical concepts. We also intend to give the 2-norm version of the Gelfand–Mazur theorem.

2.

We recall the definition of 2-normed space [**19, 30**].

DEFINITION 2.1. Let E be a linear space of dimension greater than one over the field $\mathbb{K}$, where $\mathbb{K}$ is the field of real or complex numbers. The real valued function $\|.,.\|$ on $E \times E$ is said to be a 2-norm if it satisfies the following axioms:

(i) $\|x, y\| = 0$, if and only if x and y are linearly dependent in E;
(ii) $\|x, y\| = \|y, x\|$ for all x, y in E;
(iii) $\|\alpha x, y\| = |\alpha| \, \|x, y\|$ for all $\alpha \in \mathbb{K}$ and for all $x, y \in E$;
(iv) $\|x + y, z\| \leqslant \|x, z\| + \|y, z\|$ for all $x, y, z \in E$.

The pair $(E, \|.,.\|)$ is said to be a 2-normed linear space over the field $\mathbb{K}$.

In [**30**] several examples of 2-normed complex linear spaces over $\mathbb{K}$ have been given.

We note here that in a 2-normed linear space $(E, \|.,.\|)$, the 2-norm induces a topology which makes E a locally convex Hausdorff topological vector space. To get this topology, define for each x in E a semi-norm ρ_x on E by $\rho_x(y) = \|x, y\|$, y in E. As dimension $E \geqslant 2$, for each y $(\neq 0)$ in E there exists an x in E such that x, y are linearly independent and hence, $\rho_x(y) = \|x, y\| \neq 0$. Thus the class $\{\rho_x : x \in E\}$ forms a family of semi-norms which separates points in E and the topology formed by this family of semi-norms gives the required topology on E. A typical neighborhood of origin is of the form $\bigcap_{1=i}^{n} \{y \in E : \rho_{x_i}(y) < \varepsilon\}$, where $\varepsilon > 0$ and $\{x_1, x_2, \dots, x_n\}$ is an arbitrary finite subset on E.

We also note here the following elementary but useful proposition [**30**] :

PROPOSITION 2.1. *Let* $(E, \|.,.\|)$ *be a* 2*-normed space over* $\mathbb{K}$*. Then*

(i) $\|x + y, x\| = \|x, y\|$ *for all* x, y *in* E*, and*
(ii) *if for two linearly independent* x *and* y *in* E*,* $\|z, x\| = \|z, y\| = 0$ *for* $z \in E$*, then* $z = 0$.

3.

The following definition of a 2-normed algebra was given [**33**] in 1987.

DEFINITION 3.1. Let E be a real algebra of dim $\geqslant 2$ with the 2-norm $\|.,.\|$. E is said to be a 2-normed algebra if there is some $k > 0$ such that

$$\|xy, z\| \leqslant k\|x, z\| \, \|y, z\| \quad \text{for all } x, y, z \in E.$$

The following theorem [**29**] establishes that such a class of algebras either does not exist or is trivial.

THEOREM 3.1. *Let* $(E, \|.,.\|)$ *be a* 2*-normed algebra according to Definition 3.1. Then*

(a) *the class of such algebra with unity is void and,*
(b) *the class of such algebra without unity is trivial i.e., for all $x, y \in E$, $xy = 0$.*

PROOF. For a proof see [**29**]. □

Here we propose the following definition for 2-normed algebras.

Let E be subalgebra of dimension $\geqslant 2$ of an algebra B, $\|.,.\|$ be a 2-norm in B and $a_1, a_2 \in B$ be linearly independent, non-invertible and be such that for all $x, y \in E$, $\|xy, a_i\| \leqslant \|x, a_i\| \, \|y, a_i\|$, $i = 1, 2$. Then E is called a 2-normed algebra with respect to a_1, a_2.

Let E be a 2-normed algebra with respect to a_1, a_2. If a sequence $\{x_n\}$ in E satisfying $\lim_{m,n\to\infty} \|x_n - x_m, a_i\| = 0$, $i = 1, 2$ is such that there exists an x in E for which $\lim_{n\to\infty} \|x_n - x, a_i\| = 0$, $i = 1, 2$; then E is a 2-Banach algebra with respect to a_1, a_2.

REMARK. Let E be a 2-normed algebra with respect to a_1, a_2. Then $\|xa_i, a_i\| \leqslant \|x, a_i\| \, \|a_i, a_i\| = 0$ for $i = 1, 2$ implying that, $\|xa_i, a_i\| = 0$ for $i = 1, 2$ and so xa_i and a_i are linearly dependent. As a_1, a_2 are linearly independent, $a_i \neq 0$, $i = 1, 2$ and so, for all $x \in E$, there exists $\alpha_i(x) \in \mathbb{K}$, $i = 1, 2$ such that $xa_i = \alpha_i(x)a_i$.

Similarly, for all $x \in E$, there exists $\beta_i(x) \in \mathbb{K}$ such that $a_i x = \beta_i(x)a_i$, $i = 1, 2$.

Now, for $i = 1, 2$ we have for all $x \in E$,

$$a_i x a_i = (a_i x)a_i = \beta_i(x)a_i^2 = a_i(xa_i) = a_i \alpha_i(x)a_i = \alpha_i(x)a_i^2$$

and so, $(\alpha_i(x) - \beta_i(x))a_i^2 = 0$.

Now, if $a_i^2 \neq 0$ for any of $i = 1, 2$, then for this i, $\alpha_i(x) = \beta_i(x)$ and we have $a_i x = xa_i = \alpha_i(x)a_i$ for all $x \in E$, for this i. If for any of $i = 1, 2$ a_i is invertible, then for all $x \in E$, $xa_i = \alpha_i(x)a_i$ and E will be of dimension 1. But E being a 2-normed space, we have the trivial 2-norm on E, that is, $\|x, y\| = 0$ for all $x, y \in E$. To avoid this and the other trivial case $E = \{0\}$, we took a_1 and a_2 non-invertible. The linear independence of a_1 and a_2 is also needed to get a richer structure for 2-normed algebras, as we will see later on. Also note that α_i and β_i, $i = 1, 2$ are homomorphisms. For $(xy)a_i = x(ya_i) = x\alpha_i(y)a_i = \alpha_i(x)\alpha_i(y)a_i = \alpha_i(xy)a_i$ and so, $\alpha_i(xy) = \alpha_i(x)\,\alpha_i(y)$.

Again, for $x, y \in E$, $\alpha, \beta \in \mathbb{K}$

$$(\alpha x + \beta y)a_i = \alpha_i(\alpha x + \beta y)a_i = (\alpha\alpha_i(x) + \beta\alpha_i(x))a_i$$

and so, $\alpha_i(\alpha x + \beta y) = (\alpha\alpha_i(x) + \beta\alpha_i(x))$.

Thus α_i, and similarly β_i, are homomorphisms on E, $i = 1, 2$.

We sum up the above discussions in the following theorem.

THEOREM 3.2. *Let E be a 2-normed algebra with respect to a_1, a_2. Then there exist four homomorphisms α_i, β_i on E, $i = 1, 2$, such that for all $x \in E$, $xa_i = \alpha_i(x)a_i$, $a_i x = \beta_i(x)a_i$ and if any of a_i, $i = 1, 2$ be such that $a_i^2 \neq 0$, then we have $\alpha_i = \beta_i$, and $xa_i = a_i x = \alpha_i(x)a_i$.*

4. Augmentation of unity

Let E be a subalgebra of an algebra $B, a_1, a_2 \in B$, $\|.,.\|$ be a 2-norm on B and $(E, \|.,.\|)$ be a 2-normed algebra (or a 2-Banach algebra) with respect to a_1, a_2 and E be without unity. Our purpose in this section is to augment unity with E (as in the case of 1-normed algebras).

Consider the space $B' = B \times \mathbb{K}$, in which for $(x, \alpha), (y, \beta) \in B'$. Define $(x, \alpha) + (y, \beta) = (x + y, \alpha + \beta)$, $\gamma(x, \alpha) = (\gamma x, \gamma\alpha)$ for $\gamma \in \mathbb{K}$ and $(x, \alpha)(y, \beta) = (xy + \alpha y + \beta x, \alpha\beta)$. Then B' is an algebra with unity $e = (0, 1)$.

Define $\|.,.\|'$ in B' by [for $(x, \alpha), (y, \beta) \in B'$], $\|(x, \alpha), (y, \beta)\|' = \|x, y\| + \|\beta x - \alpha y, a_1\| + \|\beta x - \alpha y, a_2\|$. Then $\|.,.\|$ is a 2-norm on B'.

Note that for $(x, \alpha), (y, \beta) \in B'$, $\|(x, \alpha), (y, \beta)\|' = 0$ iff $\|x, y\| = 0$ and $\|\beta x - \alpha y, a_i\| = 0$, $i = 1, 2$ iff x and y are linearly dependent and $\beta x - \alpha y$ are also linearly dependent to a_1, a_2. But as a_1, a_2 are linearly independent, $\beta x = \alpha y$. If (x, α), (y, β) are linearly dependent in B', then for $(y, \beta) = (0, 0)$, $\|(x, \alpha), (y, \beta)\| = 0$ and if $(y, \beta) \neq 0$, we have for some $\gamma \in \mathbb{K}$ such that $(x, \alpha) = \gamma(y, \beta)$ that is, $x = \gamma y$ and $\alpha = \gamma\beta$ and so, x, y are linearly dependent giving $\alpha y = (\gamma\beta)y = \beta(\gamma y) = \beta x$ and so $\|(x, \alpha), (y, \beta)\|' = 0$. Conversely, if $\|(x, \alpha), (y, \beta)\| = 0$, we have x, y linearly dependent and $\beta x = \alpha y$. Now, if $y = 0$ either $x = 0$ or $\beta = 0$. If $\beta = 0$, then $(y, \beta) = (0, 0)$ and (x, α) and (y, β) are linearly dependent. If $\beta \neq 0$ but $x = 0$, for $\alpha = 0$, (x, α) and (y, β) are linearly dependent and for $\alpha \neq 0$ for some $\gamma \in \mathbb{K}$, $\alpha = \gamma\beta$ and we have, $(x, \alpha) = \gamma(y, \beta)$ and (x, α) and (y, β) are linearly dependent. If $y \neq 0$, then as x, y are linearly dependent there exists $\gamma \in \mathbb{K}$ such that $x = \gamma y$. We assume that $\gamma \neq 0$, that is, $x \neq 0$ as if $x = 0$, the augment above will imply that $(x, \alpha), (y, \beta)$ are linearly dependent. $x = \gamma y$ and $\beta x = \alpha y$ imply that $\beta(\gamma y) = \alpha y$ and as $y \neq 0$, $\alpha = \gamma\beta$ and so $(x, \alpha) = \gamma(y, \beta)$ that is (x, α) and (y, β) are linearly dependent.

It is easy to show that $\|.,.\|'$ satisfies the other 2-norm axioms and consequently $(B', \|.,.\|')$ is a 2-normed space.

We claim that $E' = E \times \mathbb{K}$ is a 2-normed algebra with respect to $b_1 = (ka_1, 0), b_2 = (ka_2, 0) \in B'$ with 2-norm $\|.,.\|'$, for some suitable $k > 0$ if $(E', \|.,.\|)$ is a 2-normed algebra. To see this, let $(x, \alpha), (y, \beta) \in E'$. Then

$$\begin{aligned}
\|(x,\alpha)(y,\beta), b_i\|' &= \|(xy + \alpha y + \beta x, \alpha\beta), (ka_i, 0)\|' \\
&= \|xy + \alpha y + \beta x, ka_i\| + \|0(xy + \alpha y + \beta x) - \alpha\beta k a_i, a_1\| \\
&\qquad + \|0(xy + \alpha y + \beta x) - \alpha\beta k a_i, a_2\| \\
&\leqslant k\|x, a_i\|\,\|y, a_i\| + k|\alpha|\,\|y, a_i\| + k|\beta|\,\|x, a_i\| \\
&\qquad + k|\alpha\beta|\,[\|a_i, a_1\| + \|a_i, a_2\|]
\end{aligned}$$

and

$$\begin{aligned}
\|(x,\alpha), b_i\|' &= \|(x,\alpha), (ka_i, 0)\|' \\
&= k\|x, a_i\| + \|0x - \alpha k a_i, a_1\| + \|0x - \alpha k a_i, a_2\| \\
&= k\|x, a_i\| + k|\alpha|\,[\|a_i, a_1\| + \|a_i, a_2\|], \\
\|(y,\beta), b_i\|' &= k\|y, a_1\| + |\beta| k\,[\|a_i, a_1\| + \|a_i, a_2\|].
\end{aligned}$$

For $i = 1$,

$$\begin{aligned}\|(x,\alpha),b_i\|'\,\|(y,\beta),b_i\|' &= [k\|x,a_1\| + |\alpha|k\|a_1,a_2\|]\,[k\|y,a_1\| + |\beta|k\|a_1,a_2\|] \\ &= k^2\|x,a_1\|\,\|y,a_1\| + k^2|\alpha|\,\|a_1,a_2\|\,\|y,a_1\| \\ &\quad + k^2|\beta|\,\|a_1,a_2\|\,\|x,a_1\| + k^2|\alpha\beta|\,\|a_1,a_2\|^2.\end{aligned}$$

Now, if $\|a_1,a_2\| \geqslant 1$, we take $k = 1$ and if $\|a_1,a_2\| < 1$, take $k = \frac{1}{\|a_1,a_2\|}$ ($\|a_1,a_2\| \neq 0$ as a_1, a_2 are linearly independent) and we see that $\|(x,\alpha)(y,\beta),b_1\|' \leqslant \|(x,\alpha),b_1\|'\,\|(y,\beta),b_1\|'$.

Similarly, $\|(x,\alpha)(y,\beta),b_2\|' \leqslant \|(x,\alpha),b_2\|'\,\|(y,\beta),b_2\|'$ and we see that $(E', \|.,.\|')$ is a 2-normed algebra with unity $e = (0,1)$ with respect to b_i, $i = 1,2$.

That $(E', \|.,.\|')$ is a 2-Banach algebra with respect to b_i, $i = 1,2$ if $(E', \|.,.\|)$ is a 2-Banach algebra as is also easy to see.

Let $\{(x_n,\alpha_n)\}$ be a sequence in E' such that $\lim_{n,m}\|(x_n,\alpha_n)-(x_m,\alpha_m),b_i\|' = 0$ for $i = 1,2$. Then $\lim_{n,m}\{k\|x_n-x_m,a_i\|+k|\alpha_n-\alpha_m|\,[\|a_i,a_1\|+\|a_i,a_2\|]\} = 0$ for $i = 1,2$; which implies that $\lim_{n,m}\|x_n-x_m,a_i\| = 0$, $i = 1,2$ and $\lim_{n,m}|\alpha_n-\alpha_m| = 0$. But then there exists an $x \in E$ such that $\lim_n \|x_n - x, a_i\| = 0$, $i = 1,2$ (as $(E, \|.,.\|)$ is a 2-Banach algebra with respect to a_i, $i = 1,2$) and as $\{\alpha_n\}$ is Cauchy, there exists $\alpha \in \mathbb{K}$ such that $\lim_n |\alpha_n - \alpha| = 0$ and hence $\lim_n\{k\|x_n - x, a_i\| + k|\alpha_n - \alpha|\,[\|a_i,a_1\| + \|a_i,a_2\|]\} = 0$ and so $\lim_n \|(x_n,\alpha_n) - (x,\alpha), b_i\|' = 0$. This shows that $(E', \|.,.\|')$ is a 2-Banach algebra with respect to b_i, $i = 1,2$.

5. Examples

In this section we give examples of 2-normed algebras and 2-Banach algebras. It is well known that a finite dimensional algebra over $\mathbb{K}$ is an 1-normed (and hence 1-Banach) algebra for a suitably defined 1-norm. Our first example shows that a finite dimensional algebra with dim $\geqslant 2$ also becomes a 2-normed (and 2-Banach) algebra with respect to a_1, a_2 (suitably chosen) with suitably defined 2-norm.

EXAMPLE 5.1. Let E be a finite dimensional ($\dim E = n \geqslant 2$) algebra over $\mathbb{K}$ and let $\{e_1, \ldots, e_n\}$ be a basis for E. Let a, b be two symbols (not in E) and define $B = \{x + \alpha a + \beta b : x \in E,\ \alpha, \beta \in \mathbb{K}\}$, with the agreement that $x + \alpha a + \beta b = 0$, if and only if $x = 0$, $\alpha = \beta = 0$, for $x \in E$, $\alpha, \beta \in \mathbb{K}$.

For $y_i = x_i + \alpha_i a + \beta_i b \in B$, $i = 1,2$, $\alpha \in \mathbb{K}$, define $y_1 + y_2 = (x_1 + x_2) + (\alpha_1 + \alpha_2)a + (\beta_1 + \beta_2)b$ and $\alpha y_1 = \alpha x_1 + (\alpha\alpha_1)a + (\alpha\beta_1)b$, $y_1 y_2 = x_1 x_2 + \alpha_1\alpha_2 a + \beta_1\beta_2 b$. Then B is an algebra over $\mathbb{K}$ and if E has unit e, then $\tilde{e} = e + a + b$ is the unit of B.

For $x = \sum_{i=1}^n \alpha_i e_i + s_1 a + s_2 b$, $y = \sum_{i=1}^n \beta_i e_i + t_1 a + t_2 b \in B$ define $\|x,y\|$ by

$$\|x,y\|^2 = \left(\sum_{i=1}^n |\alpha_i|^2 + |s_1|^2 + |s_2|^2\right)\left(\sum_{i=1}^n |\beta_i|^2 + |t_1|^2 + |t_2|^2\right) - \left|\sum_{i=1}^n \alpha_i\overline{\beta_i} + s_1\overline{t_1} + s_2\overline{t_2}\right|^2$$

Then $\|.,.\|$ defines a 2-norm in B (See [**30**]).

On E, define $\|.\|_1$ by $\|x\|_1 = \|x,a\|$ for $x \in E$. Note that $\|.\|_1$ is a 1-norm on E. Let $\|.\|$ be a 1-norm on E so that $(E, \|.\|)$ is a 1-normed algebra. Then E being finite dimensional, both 1-norms, $\|.\|_1$ and $\|.\|$ on E are equivalent and hence there exist $k_1, k_2 > 0$ such that for every $x \in E$, $\|x\|_1 \leqslant k_1\|x\|$ and $\|x\| \leqslant k_2\|x\|_1$.

For $x, y \in E$, we have

$$\|xy, a\| = \|xy\|_1 \leqslant k_1\|xy\| \leqslant k_1\|x\|\,\|y\| \leqslant k_1k_2\|x\|_1 k_2\|y\|_1 = k_1k_2^2\|x, a\|\,\|y, a\|$$

and so for $a_1 = k_1k_2^2 a$, we have for all $x, y \in E$, $\|xy, a_1\| \leqslant \|x, a_1\|\,\|y, a_1\|$.

Similarly, for suitably chosen $k_3 > 0$, we have for $a_2 = k_3 b$, $\|xy, a_2\| \leqslant \|x, a_2\|\,\|y, a_2\|$ for all $x, y \in E$ and $(E, \|., .\|)$ becomes a 2-normed algebra over $\mathbb{K}$ with respect to a_1, a_2.

To see that $(E, \|., .\|)$ is a 2-Banach algebra with respect to a_1, a_2, let $\{x_n\}$ be a sequence in E so that $\lim_{n,m\to\infty} \|x_n - x_m, a_i\| = 0$, $i = 1, 2$. Then $\lim_{n,m} \|x_n - x_m\|_1 = 0$, and E being finite dimensional $(E, \|.\|_1)$ is a Banach space and hence there is an $x \in E$, such that $\lim_n \|x_n - x\|_1 = 0$ or equivalently $\lim_n \|x_n - x, a_1\| = 0$. Now if we define for $x \in E$, $\|x\|_2 = \|x, a_2\|$, then $(E, \|.\|_2)$ also becomes a 1-normed space and the norms $\|.\|_1$ and $\|.\|_2$ on E being equivalent $\lim_n \|x_n - x\|_2 = 0$ or equivalently $\lim_n \|x_n - x, a_2\| = 0$, and the proof that $(E, \|., .\|)$ is a 2-Banach algebra with respect to a_1, a_2 is complete.

Before we proceed further, we note the following: for $x, y \in E$, we say $x \perp_B y$ (that is x is orthogonal to y in Birkhoff's sense) if $\|x\| \leqslant \|x + \alpha y\|$ for all $\alpha \in \mathbb{K}$.

LEMMA 5.1. *There exists, for an* $0 \neq x \in E$*, a* $\phi \in E^*$ *such that* $\|\phi\| = 1$ *and* $\phi(x) = \|x\|$.

LEMMA 5.2. [**25**] *Let* $x, y \in E$*,* $x \neq 0$*. Then* $x \perp_B y$ *if and only if there exists a* $\phi \in E^*$ *such that* $\|\phi\| = 1$*,* $\phi(x) = \|x\|$ *and* $\phi(y) = 0$.

LEMMA 5.3. *For* $x, y \in E$*, both nonzero, there exists an* $\alpha \in \mathbb{K}$ *such that* $x \perp_B (\alpha x + y)$.

PROOF. From Lemma 5.1, there exists a $\phi \in E^*$ such that $\|\phi\| = 1$ and $\phi(x) = \|x\|$. Now, let $\alpha = -\phi(y)/\phi(x)$. Then $\phi(\alpha x + y) = 0$ and the lemma follows by Lemma 5.2. □

LEMMA 5.4. [**28**] *Let* $(E, \|.\|)$ *be a* 1*-normed linear space over* $\mathbb{K}$*. For* $x, y \in E$*, define* $\|x, y\|$ *by*

$$\|x, y\| = \sup_{\phi, \Psi \in E^*\, \|\phi\| = \|\Psi\| = 1} |\phi(x)\Psi(y) - \phi(y)\Psi(x)|.$$

Then $(E, \|., .\|)$ *is a* 2*-normed linear space over* $\mathbb{K}$.

The above lemma leads us to the following: let $(E, \|., .\|)$ be a 1-normed linear space over $\mathbb{K}$. The 2-norm defined above is called the induced 2-norm.

We also note the following:

LEMMA 5.5. [**28**] *Let* $(E, \|.\|)$ *be a* 1*-normed linear space over* $\mathbb{K}$ *and let* $(E, \|., .\|)$ *be the corresponding induced* 2*-normed linear space. Then for all* $x, y \in E$*,* $\|x, y\| \leqslant 2\|x\|\,\|y\|$*. If further* $x \perp_B y$ *or* $y \perp_B x$*, then* $\|x\|\,\|y\| \leqslant \|x, y\|$.

Let E be an algebra over $\mathbb{K}$, $\phi : E \to \mathbb{K}$ be such that ϕ is linear and for all $x, y \in E$, $\phi(xy) = \phi(x)\phi(y)$, we say ϕ is a $\mathbb{K}$-homomorphism on E.

Before we give our next example as the following theorem, we note that from Theorem 3.2 it is obvious that not all 1-normed algebras (or 1-Banach algebras) can be made into a 2-normed algebra (or 2-Banach algebra) respectively as the existence of $\mathbb{K}$-homomorphism on a 1-normed algebra is not guaranteed though Theorem 3.2 states that $\mathbb{K}$-homomorphism on a 2-normed algebra (or 2-Banach algebra) $(E, \|.,.\|)$ with respect to a_1, a_2 is guaranteed.

THEOREM 5.1. *Let* $(E, \|.\|)$ $(\dim E \geqslant 2)$ *be a* 1*-normed algebra* (*or* 1*-Banach algebra*) *over* $\mathbb{K}$ *with or without unity, over which a nontrivial* $\mathbb{K}$*-homomorphism* θ *exists. Then there exist an algebra* A *over* $\mathbb{K}$ *of which* E *is a subalgebra and a* 2*-norm* $\|.,.\|$ *on* A*,* $a_1, a_2 \in A$ *such that* $(E, \|.,.\|)$ *becomes a* 2*-normed algebra* (*or a* 2*-Banach algebra respectively*) *with respect to* a_1, a_2*.*

PROOF. We prove the theorem for the case of 1-normed algebras. In the course of the proof, we will see that the case of 1-Banach algebras is also proved.

Let $(E, \| \,.\, \|)$ be a 1-normed algebra over $\mathbb{K}$. If E is without unity, we augment the unity in the usual way and extend θ over the extended 1-normed algebra $E \times \mathbb{K}$. For $(x, \alpha), (y, \beta) \in E \times \mathbb{K}$, we define $(x, \alpha) + (y, \beta) = (x + y, \alpha + \beta)$, $(x, \alpha)(y, \beta) = xy + \alpha y + \beta$ and $\|(x, \alpha)\| = \|x\| + |\alpha|$.

Unit of $E \times \mathbb{K}$ is given by $e = (0, 1)$ and $\theta(x, \alpha) = \theta(x) + \alpha$ over $E \times \mathbb{K}$ becomes a $\mathbb{K}$-homomorphism. We can now without any loss of generality assume that 1-normed algebra $(E, \|.\|)$ is with unity. Let $\tilde{E}$ be the completion of E. Now, let $a \in E$ be such that $\|a\| < 1$ and $|\theta(a)| \geqslant 1$. Then $\theta(e - \alpha^{-1}a) = 0$ where $\alpha = \theta(a)$. Write $b = e - \alpha^{-1}a$. Then $\theta(b) = 0$ and as $\|\alpha^{-1}a\| < 1$, b is invertible in $\tilde{E}$ and hence is invertible in E itself. But θ being a nontrivial $\mathbb{K}$-homomorphism, $\theta(b) \neq 0$. Hence $|\theta(a)| < 1$ and we conclude that θ is continuous over E and hence θ can be extended continuously to $\tilde{\theta}$ over $\tilde{E}$ and $\tilde{\theta}$ is a nontrivial $\mathbb{K}$-homomorphism over $\tilde{E}$ with $\|\tilde{\theta}\| = \|\theta\| = 1$. Therefore $(\tilde{E}, \|.\|)$ is a 1-Banach algebra, with unity e and $\tilde{\theta}$ is a nontrivial $\mathbb{K}$-homomorphism over $\tilde{E}$. Then $\tilde{E}$ is isometrically isomorphic to a closed subalgebra M of $B(X)$, the 1-Banach algebra of all bounded linear operators on a 1-Banach-space $(X, \|.\|_x)$ over $\mathbb{K}$. Now, let a, b be two symbols (not in X) and consider the space $X_1 = \{x + \alpha a + \beta b : x \in X,\ \alpha, \beta \in \mathbb{K}\}$, with the agreement that $x + \alpha a + \beta b = 0$ if and only if $x = 0$, $\alpha = \beta = 0$. For, $y_i = x_i + \alpha_i a + \beta_i b \in X_1$, $i = 1, 2$ define $y_1 + y_2 = (x_1 + x_2) + (\alpha_1 + \alpha_2)a + (\beta_1 + \beta_2)b \in X_1$ and $\alpha y_1 = (\alpha x_1) + (\alpha\alpha_1)a + (\alpha\beta_1)b \in X_1$, for $\alpha \in \mathbb{K}$.

X_1 is a linear space over $\mathbb{K}$. In X_1 define $\|.\|_{x_1}$ by $\|x + \alpha a + \beta b\|_{x_1} = \|x\|_x + |\alpha| + |\beta|$ for $x + \alpha a + \beta b \in X_1$. $\|.\|_{x_1}$ defines a 1-norm in X_1 with $\|a\|_{x_1} = \|b\|_{x_1} = 1$, and $\|x\|_{x_1} = \|x\|_x$ for $x \in X$.

Let $\{x_n + \alpha_n a + \beta_n b\}$ be a Cauchy sequence in X_1. Then $\{x_n\}$ is Cauchy in X and $\{\alpha_n\}, \{\beta_n\}$ are also Cauchy in $\mathbb{K}$ and so $\{x_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ converge to $x \in X$ and $\alpha, \beta \in \mathbb{K}$ respectively and $\{x_n + \alpha_n a + \beta_n b\}$ to $x + \alpha a + \beta b$ in X_1 and so, $(X_1, \|x\|_{x_1})$ is a 1-Banach space.

For $T \in M$, let $\tilde{T}$ be defined on X_1 by

$$\tilde{T}(x + \alpha a + \beta b) = T(x) + \tilde{\theta}_1(T)[\alpha a + \beta b] \quad \text{for all } x + \alpha a + \beta b \text{ in } X_1,$$

where $\tilde{\theta}_1$ is the $\mathbb{K}$-homomorphism over M corresponding to $\tilde{\theta}$ over $\tilde{E}$. For $x+\alpha a+\beta b \in X_1$, $T \in M$, we have

$$\begin{aligned}\|\tilde{T}(x+\alpha a+\beta b)\|_{x_1} &= \|T(x)\|_x + |\tilde{\theta}_1(T)|\,(|\alpha|+|\beta|) \\ &\leqslant \|T\|_x\|x\|_x + \|T\|_x(|\alpha|+|\beta|) \\ &= \|T\|_x\|(x+\alpha a+\beta b)\|_{x_1} \qquad \text{for all } x+\alpha a+\beta b \in X_1\end{aligned}$$

and so, $\tilde{T} \in B(X_1)$ and as $\|\tilde{T}\|_{x_1} \leqslant \|T\|_x$ we have $\|\tilde{T}\|_{x_1} = \|T\|_x$.

Let $\tilde{M} = \{\tilde{T} \in B(X_1) : \tilde{T}$ as defined above for $T \in M\}$. Then $\tilde{M}$ is a 1-Banach algebra with unity, a closed subalgebra of $B(X_1)$.

Let $T_a, T_b \in B(X_1)$ be defined as $T_a(x+\alpha a+\beta b) = \alpha a$ and $T_b(x+\alpha a+\beta b) = \beta b$, for $x+\alpha a+\beta b \in X_1$. Then, $\|T_a\|_{x_1} = \|T_b\|_{x_1} = 1$. For $\tilde{T} \in \tilde{M}$, and $x+\alpha a+\beta b \in X_1$, we have

$$\tilde{T}T_a(x+\alpha a+\beta b) = \tilde{T}(\alpha a) = \tilde{\theta}_1(T)\alpha a = \tilde{\theta}_1(T)T_a(x+\alpha a+\beta b) = T_a\tilde{T}(x+\alpha a+\beta b)$$

and so $\tilde{T}T_a = T_a\tilde{T} = \tilde{\theta}_1(T)T_a$. Similarly $\tilde{T}T_b = T_b\tilde{T} = \tilde{\theta}_1(T)T_b$ and $T_a^2 = T_a$, $T_b^2 = T_b$.

Now, if we identify $B(X_1)$ with the 1-Banach algebra B of which $\tilde{E}$ is a closed 1-normed subalgebra, (writing $\|.\|_B$ for the norm in B) and T_a, T_b be identified with a, b respectively in B, then we have, for each $x \in \tilde{E}$, $xa = ax = \tilde{\theta}(x)a$ and $xb = bx = \tilde{\theta}(x)b$, and $a^2 = a$, $b^2 = b$, $\|a\|_B = \|b\|_B = 1$.

Let us equip B with the induced 2-norm. We claim that $(\tilde{E}, \|.,.\|)$, $\|.,.\|$ is the induced 2-norm on B, is a 2-Banach algebra with respect to $2a, 2b$. To prove the claim, we first prove that $(\tilde{E}, \|.,.\|)$ is a 2-normed algebra with respect to $2a, 2b$. We begin by showing that for every $x, y \in \tilde{E}$, $\|xy, a\| \leqslant 2\|x, a\|\,\|y, a\|$.

Let $x, y \in \tilde{E}$, $x \neq 0 \neq y$. Then by Lemma 5.3 there exist $\alpha_1, \alpha_2 \in \mathbb{K}$ such that $a \perp_B (\alpha_1 a + x)$ and $a \perp_B (\alpha_2 a + y)$ and let $x_1 = \alpha_1 a + x, x_2 = \alpha_2 a + y$, then $x_1x_2 = \alpha_1\alpha_2 a^2 + \alpha_1 ay + \alpha_2 xa + xy = \alpha_1\alpha_2 a + \alpha_1\tilde{\theta}(y)a + \alpha_2\tilde{\theta}(x)a + xy$ and we have, by Proposition 2.1, $\|x_1x_2, a\| = \|xy, a\|$. Now

$$\begin{aligned}\|xy, a\| &= \|x_1x_2, a\| \\ &\leqslant 2\|x_1x_2\|_B\|a\|_B && \text{(by Lemma 5.5)} \\ &\leqslant 2\|x_1\|_B\|x_2\|_B.1 && \text{(as } (B, \|.\|_B) \text{ is a 1-normed algebra)} \\ &\leqslant 2\|x_1, a\|\,\|x_2, a\| && \text{(by Lemma 5.5 as } a \perp_B x_1, a \perp_B x_2) \\ &= 2\|x, a\|\,\|y, a\| && \text{(again by Proposition 2.1).}\end{aligned}$$

If either of x or y in $\tilde{E}$ is zero, then $\|xy, a\| = 0 = \|x, a\|\,\|y, a\|$, and we have, for every $x, y \in \tilde{E}$, $\|xy, a\| \leqslant 2\|x, a\|\,\|y, a\|$. This proves that $\|xy, 2a\| \leqslant \|x, 2a\|\,\|y, 2a\|$, and similarly it can be shown that $\|xy, 2b\| \leqslant \|x, 2b\|\,\|y, 2b\|$.

Now to complete the proof of the claim, which proves the theorem also, it remains to be shown that $(\tilde{E}, \|.,.\|)$ is a 2-Banach algebra with respect to $2a, 2b$. To prove this let $\{x_n\}$ be a sequence in $\tilde{E}$ satisfying $\lim_{m,n\to\infty}\|x_m - x_n, 2a\| = \lim_{m,n\to\infty}\|x_m - x_n, 2b\| = 0$. By Lemma 5.1, there exists a $\phi \in B^*$ such that $\|\phi\| = 1$ and $\phi(a) = \|a\|_B = 1$. We define a sequence $\{x^n\}$ in B by, $x^n = x_n - \phi(x_n)a$. Then $\phi(x^n) = \phi(x_n) - \phi(x_n)\phi(a) = 0$, for each $n \in \mathbb{N}$, and so, for each $m, n \in \mathbb{N}$, $\phi(x^m - x^n) = 0$ and so $a \perp_B (x^m - x^n)$ using Lemma 5.2.

Now, by Lemma 5.5, and Proposition 2.1, we have for each $m, n \in \mathbb{N}$,

$$\|x^m - x^n\|_B = \|x^m - x^n\|_B \|a\|_B \leqslant \|x_m - x_n, a\| \to 0 \text{ as } m, n \to \infty$$

which implies that $\{x^n\}$ is a Cauchy sequence in $(B, \|.\|_B)$. □

Now we consider the following cases:

CASE 1. Let the sequence $\{\phi(x_n)\}$ be bounded. There is a subsequence of $\{\phi(x_n)\}$ which is convergent in $\mathbb{K}$, to say s. Then the corresponding subsequence of $\{x^n\}$ in B converges to some $x_0 - sa$ in B. But $\{x^n\}$ being Cauchy in B, it converges to $x_0 - sa$ in B, and as the subsequence of $\{\phi(x_n)\}$ converges to s, it follows that $\{x_n\}$ converges to x_0 in B. Since $\{x_n\}$ is in $\tilde{E}$ and $\tilde{E}$ closed in B, we conclude that $\{x_n\}$ converges to x_0 in $\tilde{E}$ in $\|.\|_B$ norm. Using Lemma 5.5, we now have $\|x_n - x_0, 2a\| \leqslant 2\|x_n - x_0\|_B \|2a\|_B \to 0$ as $n \to \infty$ and we have an $x_0 \in \tilde{E}$ such that $\lim_{n\to\infty} \|x_n - x_0, 2a\| = 0$.

Similarly it can be shown that $\lim_{n\to\infty} \|x_n - x_0, 2b\| = 0$, and our claim is proved in this case.

CASE 2. Let the sequence $\{\phi(x_n)\}$ be unbounded and for infinitely many n's say for $n_1 < n_2 < \ldots, \phi(x_{n_i}) = 0$. In this case $x^{n_i} = x_{n_i}$ and as we have observed in the Case 1, $\{x^{n_i}\} = \{x_{n_i}\}$ is a Cauchy sequence in B and hence converges to some $x_0 \in B$. But $\tilde{E}$ is closed in B and $\{x_{n_i}\}$ is in $\tilde{E}$ and hence $\{x_{n_i}\}$ converges to x_0 in $\tilde{E}$. Again as observed above, $\{x_n\}$ is Cauchy in B. Hence, $\{x_n\}$ converges to $x_0 \in \tilde{E}$. Then we conclude that, as in the Case 1, $\lim_{n\to\infty} \|x_n - x_0, 2a\| = \lim_{n\to\infty} \|x_n - x_0, 2b\| = 0$ and the claim is proved in this case also.

CASE 3. Let the sequence $\{\phi(x_n)\}$ be unbounded and $\phi(x_n) = 0$ for finitely many n's only. We shed x_n's for which $\phi(x_n) = 0$ and writing the remaining sequence by $\{x_n\}$ again, we have $\{x_n\}$ in $\tilde{E}$ such that $\phi(x_n) \neq 0$ for all $n \in \mathbb{N}$.

Now, as $\{\phi(x_n)\}$ is unbounded, $\{1/\phi(x_n)\}$ is bounded. Let $s_n = 1/\phi(x_n)$, $y^n = s_n x_n - a$, for all $n \in \mathbb{N}$. Then $\phi(y^n) = s_n\phi(x_n) - \phi(a) = 1 - 1 = 0$ and so, for all $m, n \in \mathbb{N}$, $\phi(y^m - y^n) = 0$, and we have by Lemmas 5.2, 5.5 and Proposition 2.1,

$$\begin{aligned}\|y^m - y^n\|_B = \|a\|_B \|y^m - y^n\|_B &\leqslant \|y^m - y^n, a\| \\ &\leqslant \|s_n x_n - s_n x_m, a\| + \|(s_n - s_m)x_m, a\| \\ &\leqslant |s_n|\, \|x_n - x_m, a\| + |s_n - s_m|\, \|x_m, a\|\end{aligned}$$

Note that $\lim_{m,n\to\infty} \|x_m - x_n, a\| = 0$ implies that $\{\|x_n, a\|\}$ is bounded, for there exists $\mathbb{N}$ such that $\|x_m - x_n, a\| < 1$, for all $m, n \geqslant N$ and this implies that $\|x_m - x_N, a\| < 1$, for all $m \geqslant N$. Consequently $\|x_m, a\| - \|x_N, a\| \leqslant \|x_m - x_N, a\| < 1$ for all $m \geqslant N$ and we get $\|x_m, a\| < 1 + \|x_N, a\|$ for all $m \geqslant N$ and this proves that $\|x_m, a\|$ is bounded.

Again $\{s_n\}$ being bounded, there is a subsequence of s_n, say $\{s_{n_i}\}$, converging to some $s \in \mathbb{K}$. Then, as $\lim_{m,n\to\infty} \|x_m - x_n, a\| = 0$, the above inequality shows that the sequence $\{y^{n_i}\}$ is a Cauchy sequence in B and hence, converges in $(B, \|.\|_B)$. But as for all $m, n \in \mathbb{N}$, $\|s_m x_m - s_n x_n\|_B = \|y^m - y^n\|_B$ and $\{y^{n_i}\}$ is Cauchy in B, $\{s_{n_i} x_{n_i}\}$ converges in B and as $\{s_{n_i} x_{n_i}\}$ is in $\tilde{E}$ and $\tilde{E}$ is closed, $\{s_{n_i} x_{n_i}\}$ converges to say sy in $\tilde{E}$.

Now, for $x \in \tilde{E}$, define $\|x\|_1 = \max\{\|x,a\|, \|x,b\|\}$. Observe that for $x \in \tilde{E}$, $\|x\|_1 = 0$ if and only if x is linearly dependent on both a and b, that is if and only if $x = 0$ as both $a, b \notin \tilde{E}$. Thus $\|.\|_1$ defines a 1-norm in $\tilde{E}$.

Let $(\tilde{\tilde{E}}, \|.\|_1)$ be the completion of $(\tilde{E}, \|.\|)$. Since for $j = 1, 2$, $a_1 = 2a$, $a_2 = 2b$, we have $\lim_{m,n\to\infty} \|x_m - x_n, a_j\| = 0$, and therefore, $\lim_{m,n\to\infty} \|x_m - x_n\|_1 = 0$, that is, $\{x_n\}$ is Cauchy in $(\tilde{\tilde{E}}, \|.\|_1)$. Hence, there is an $x_0 \in \tilde{\tilde{E}}$ such that $\lim_{n\to\infty} \|x_n - x_0\|_1 = 0$, and as $\{s_{n_i}\}$ converges to s in $\mathbb{K}$, $\lim_{n\to\infty} \|s_{n_i} x_{n_i} - s x_0\|_1 = 0$. But by Lemma 5.5, we have for $j = 1, 2$ $\|s_{n_i} x_{n_i} - sy, a_j\| \leqslant 2\|s_{n_i} x_{n_i} - sy\|_B \|a_j\|_B \to 0$ as $n_i \to \infty$. As $\{s_{n_i} x_{x_i}\}$ converges to sy in B, it follows that $\lim_{n_i\to\infty} \|s_{n_i} x_{n_i} - sy\|_1 = 0$, which implies that $x_0 = y \in \tilde{E}$, and we conclude that, there exists an $x_0 \in \tilde{E}$, $s \in K$, such that for $j = 1, 2$ $\lim_{n_i\to\infty} \|s_{n_i} x_{n_i} - s x_0, a_j\| = 0$.

Now observe that $\|s_{n_i} x_{n_i} - s x_0, a_j\| \geqslant |s_{n_i}| \, \|x_{n_i} - x_0, a_j\| - |s_{n_i} - s| \, \|x_0, a_j\|$ and therefore as $n_i \to \infty$ we have $|s| \lim_{n_i\to\infty} \|x_{n_i} - x_0, a_j\| \leqslant 0$ which implies that $\lim_{n_i\to\infty} \|x_{n_i} - x_0, a_j\| = 0$ for $j = 1, 2$. Now $\lim_{m,n} \|x_m - x_n, a_j\| = 0$ and $\lim_{n_i\to\infty} \|x_{n_i} - x_0, a_j\| = 0$ imply that for $\varepsilon > 0$ there exists N such that $\|x_{n+p} - x_n, a_j\| < \varepsilon$ for all $n \geqslant N$, $p \geqslant 0$, and $\|x_{n_i} - x_0, a_j\| < \varepsilon$ for all $n_i \geqslant N$.

For all $n > N$, let $n_i > N$ and then $\|x_n - x_0, a_j\| \leqslant \|x_n - x_{n_i}, a_j\| + \|x_{n_i} - x_0, a_j\|$ $< \varepsilon + \varepsilon$, and so, $\lim_{n\to\infty} \|x_n - x_0, a_j\| = 0$ for $j = 1, 2$. This establishes our claim and the theorem is completely established.

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