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NOTES ON ANALYTIC CONVOLUTED C-SEMIGROUPS

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Abstract. We establish some new structural properties of exponentially bounded, analytic convoluted C-semigroups and state a version of Kato's analyticity criterion for such a class of operator semigroups. Our characterizations completely cover the case of analytic fractionally integrated C-semigroups.

1. Introduction and preliminaries

An important motivational factor for the genesis of this paper presents the fact that several structural properties of exponentially bounded, analytic convoluted C-semigroups have not been fully cleared in the existing literature.

The paper is organized as follows. In Proposition 2.1 and Theorem 2.1, we refine [**4**, Proposition 3.7(a)], [**8**, Theorem 10] and transfer the assertion of [**9**, Theorem 5.2] to analytic convoluted C-semigroups. In Theorem 2.1, we introduce the condition (H_1) which holds in the case of fractionally integrated C-semigroups. In order to better explain the importance of this condition in our investigation, let us recall that the set $\wp(S_K)$ consisted of all subgenerators of a (local) convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}$ need not be finite ([**8**], [**10**], [**13**]) and that, equipped with corresponding algebraic operations, $\wp(S_K)$ becomes a complete lattice whose partially ordering coincides with the usual set inclusion; furthermore, $\wp(S_K)$ is totally ordered iff $\mathrm{card}(\wp(S_K)) \leqslant 2$ ([**10**], [**13**]), and in the case $\mathrm{card}(\wp(S_K)) < \infty$, one can prove that $\wp(S_K)$ is a Boolean, which implies $\mathrm{card}(\wp(S_K)) = 2^n$ for some $n \in \mathbb{N}_0$. In fact, the main objective in Theorem 2.1(i) is to establish the spectral characterizations of the integral generator of an analytic convoluted C-semigroup $(S_K(t))_{t\geqslant 0}$ as well as to show that such characterizations still hold for an arbitrary subgenerator of $(S_K(t))_{t\geqslant 0}$ as long as the condition (H_1) holds. It is an open problem whether the statements (2.6)–(2.9) quoted in the formulation of Theorem 2.1(i) remain true for an arbitrary subgenerator of $(S_K(t))_{t\geqslant 0}$ if the condition (H_1)

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is neglected. Furthermore, the condition (H_1) plays a crucial role in Theorem 2.2 which presents Kato's analyticity criterion for convoluted C-semigroups. Even in the case of regularized semigroups, Theorem 2.2 and Corollary 2.2 improve the corresponding result of Zheng [**14**, Theorem]. It is well known that A generates an (exponentially) bounded, analytic C_0-semigroup of angle $\alpha \in (0, \frac{\pi}{2})$ provided that $e^{\pm i\alpha}A$ are generators of (exponentially) bounded C_0-semigroups $(T_{\pm\alpha}(t))_{t\geqslant 0}$. We transfer this assertion to analytic regularized semigroups by a slight modification of the proof of [**1**, Theorem 3.9.7].

By E and $L(E)$ are denoted a complex Banach space and the Banach algebra of bounded linear operators on E. For a closed linear operator A acting on E, $D(A)$, $\mathrm{Kern}(A)$, $R(A)$ and $\rho(A)$ denote its domain, kernel, range and resolvent set, respectively. By $[D(A)]$ is denoted the Banach space $D(A)$ equipped with the graph norm. Given $\gamma \in (0, \pi]$, put $\Sigma_\gamma := \{\lambda \in \mathbb{C} : \lambda \neq 0,\ \arg(\lambda) \in (-\gamma, \gamma)\}$. In what follows, we assume $L(E) \ni C$ is an injective operator satisfying $CA \subset AC$, $\tau \in (0, \infty]$, K is a complex-valued locally integrable function in $[0, \tau)$ and K is not identical to zero. Put $\Theta(t) := \int_0^t K(s)ds$, $t \in [0, \tau)$; then Θ is an absolutely continuous function in $[0, \tau)$ and $\Theta'(t) = K(t)$ for a.e. $t \in [0, \tau)$. We mainly use the following condition:

(P1): K is Laplace transformable, i.e., it is locally integrable on $[0, \infty)$ and there exists $\beta \in \mathbb{R}$ so that

$$\tilde{K}(\lambda) = \mathcal{L}(K)(\lambda) := \lim_{b\to\infty} \int_0^b e^{-\lambda t} K(t)\, dt := \int_0^\infty e^{-\lambda t} K(t)\, dt$$

exists for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > \beta$. Put $\mathrm{abs}(K) := \inf\{\mathrm{Re}\,\lambda : \tilde{K}(\lambda) \text{ exists}\}$.

Definition 1.1. ([**7**]–[**8**]) Let A be a closed operator and let $0 < \tau \leqslant \infty$. If there exists a strongly continuous family $(S_K(t))_{t\in[0,\tau)}$ in $L(E)$ such that:

(i) $S_K(t)A \subset AS_K(t)$, $t \in [0, \tau)$,
(ii) $S_K(t)C = CS_K(t)$, $t \in [0, \tau)$ and
(iii) for all $x \in E$ and $t \in [0, \tau)$: $\int_0^t S_K(s)x\, ds \in D(A)$ and

$$A\int_0^t S_K(s)x\, ds = S_K(t)x - \Theta(t)Cx, \tag{1.1}$$

then it is said that A is a subgenerator of a (local) K-convoluted C-semigroup $(S_K(t))_{t\in[0,\tau)}$. If $\tau = \infty$, then we say that $(S_K(t))_{t\geqslant 0}$ is an exponentially bounded K-convoluted C-semigroup with a subgenerator A if, additionally, there exist $M > 0$ and $\omega \geqslant 0$ such that $\|S_K(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$.

The integral generator of $(S_K(t))_{t\in[0,\tau)}$ is defined by

$$\hat{A} := \left\{(x, y) \in E^2 : S_K(t)x - \Theta(t)Cx = \int_0^t S_K(s)y ds,\ t \in [0, \tau)\right\},$$

and it is a closed linear operator which is an extension of any subgenerator of $(S_K(t))_{t\in[0,\tau)}$. Suppose $\{A,B\}\subset \wp(S_K)$. By [**10**, Proposition 1.1], the following holds:

(a) $C^{-1}AC=C^{-1}\hat{A}C=\hat{A}\in\wp(S_K)$,
(b) A and B have the same eigenvalues,
(c) $\rho_C(A)\subseteq\rho_C(B)$ if $A\subseteq B$,
(d) $A=B=\hat{A}$, if $\rho(\hat{A})\neq\emptyset$ or $C=I$.

The proof of the following auxiliary lemma is similar to those of [**7**, Theorem 2.2] and [**9**, Theorem 3.1, Theorem 3.3].

Lemma 1.1. *Suppose K satisfies* (P1) *and A is a closed linear operator.*

(i) *Suppose $M>0$, $\omega\geqslant 0$, A is a subgenerator of an exponentially bounded, K-convoluted C-semigroup $(S_K(t))_{t\geqslant 0}$ satisfying $\|S_K(t)\|\leqslant Me^{\omega t}$, $t\geqslant 0$ and $\omega_1=\max(\omega,\operatorname{abs}(K))$. Then $\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda>\omega_1,\ \tilde{K}(\lambda)\neq 0\}\subset\rho_C(A)$ and $(\lambda-A)^{-1}Cx=\frac{1}{\tilde{K}(\lambda)}\int_0^\infty e^{-\lambda t}S_K(t)x\,dt$ for all $x\in E$ and $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda>\omega_1$ and $\tilde{K}(\lambda)\neq 0$.*

(ii) *Suppose $M>0$, $\omega\geqslant 0$, $(S_K(t))_{t\geqslant 0}$ is a strongly continuous operator family, $\|S_K(t)\|\leqslant Me^{\omega t}$, $t\geqslant 0$ and $\omega_1=\max(\omega,\operatorname{abs}(K))$. If $\{\lambda\in(\omega_1,\infty):\tilde{K}(\lambda)\neq 0\}\subset\rho_C(A)$ and $(\lambda-A)^{-1}Cx=\frac{1}{\tilde{K}(\lambda)}\int_0^\infty e^{-\lambda t}S_K(t)x\,dt$, $x\in E$, $\lambda>\omega_1$, $\tilde{K}(\lambda)\neq 0$, then $(S_K(t))_{t\geqslant 0}$ is an exponentially bounded, K-convoluted C-semigroup with a subgenerator A.*

(iii) *Let A be densely defined. Then A is a subgenerator of an exponentially bounded C-semigroup $(T(t))_{t\geqslant 0}$ satisfying $\|T(t)\|\leqslant Me^{\omega t}$, $t\geqslant 0$ for appropriate constants $M>0$ and $\omega\in\mathbb{R}$ iff $(\omega,\infty)\subset\rho_C(A)$, the mapping $\lambda\mapsto(\lambda-A)^{-1}C$, $\lambda>\omega$ is infinitely differentiable and*

$$\Big\|\frac{d^k}{d\lambda^k}[(\lambda-A)^{-1}C]\Big\|\leqslant\frac{Mk!}{(\lambda-\omega)^{k+1}},\quad k\in\mathbb{N}_0,\ \lambda>\omega.$$

Definition 1.2. [**8**] Let $\alpha\in(0,\frac{\pi}{2}]$ and let $(S_K(t))_{t\geqslant 0}$ be a K-convoluted C-semigroup. Then we say that $(S_K(t))_{t\geqslant 0}$ is an analytic K-convoluted C-semigroup of angle α, if there exists an analytic function $\mathbf{S}_K:\Sigma_\alpha\to L(E)$ which satisfies

(i) $\mathbf{S}_K(t)=S_K(t)$, $t>0$,
(ii) $\lim_{z\to 0,\,z\in\Sigma_\gamma}\mathbf{S}_K(z)x=0$ for all $\gamma\in(0,\alpha)$ and $x\in E$.

It is said that $(S_K(t))_{t\geqslant 0}$ is an exponentially bounded, analytic K-convoluted C-semigroup, resp. bounded analytic K-convoluted C-semigroup, of angle α, if for every $\gamma\in(0,\alpha)$, there exist $M_\gamma>0$ and $\omega_\gamma\geqslant 0$, resp. $\omega_\gamma=0$, such that $\|\mathbf{S}_K(z)\|\leqslant M_\gamma e^{\omega_\gamma\operatorname{Re}z}$, $z\in\Sigma_\gamma$.

Since no confusion seems likely, we will also denote $\mathbf{S}_K$ by S_K. Plugging $K(t)=\frac{t^{r-1}}{\Gamma(r)}$, $t>0$ in Definition 1.1 and Definition 1.2, where $r>0$ and $\Gamma(\cdot)$ denotes the Gamma function, we obtain the well-known classes of (analytic) r-times integrated C-semigroups; an (analytic) 0-times integrated C-semigroup is defined to be an (analytic) C-semigroup (cf. [**3**, Definition 21.3]). The notion of (exponential) boundedness of an analytic r-times integrated C-semigroup, $r\geqslant 0$, is understood in the sense of Definition 1.2.

2. Analytic convoluted C-semigroups

We start this section with the following proposition.

PROPOSITION 2.1. *Suppose K satisfies* (P1), $\alpha \in (0, \frac{\pi}{2}]$ *and A is a subgenerator of an exponentially bounded, analytic K-convoluted C-semigroup $(S_K(t))_{t\geqslant 0}$ of angle α. Suppose, further, that the condition* (H) *holds, where:*

(H) *There exist functions* $c : (-\alpha, \alpha) \to \mathbb{C} \setminus \{0\}$, $\omega_0 : (-\alpha, \alpha) \to [0, \infty)$ *and a family of functions* $(K_\theta)_{\theta \in (-\alpha,\alpha)}$ *satisfying* (P1) *so that:* $\operatorname{abs}(K_\theta) \leqslant \omega_0(\theta)$, $\frac{\operatorname{abs}(K)}{\cos\theta} \leqslant \omega_0(\theta)$,

$$\Phi_\theta =: \{\lambda \in (\omega_0(\theta), \infty) : \tilde{K}(\lambda e^{-i\theta}) = 0\} = \{\lambda \in (\omega_0(\theta), \infty) : \widetilde{K_\theta}(\lambda) = 0\}, \tag{2.1}$$

$$\frac{\widetilde{K_\theta}(\lambda)}{\tilde{K}(\lambda e^{-i\theta})} = c(\theta),\ \ \lambda > \omega_0(\theta),\ \ \lambda \notin \Phi_\theta,\ \ \theta \in (-\alpha, \alpha). \tag{2.2}$$

Then, for every $\theta \in (-\alpha, \alpha)$, the operator $e^{i\theta}A$ is a subgenerator of an exponentially bounded, analytic K_θ-convoluted C-semigroup $(c(\theta)S_K(te^{i\theta}))_{t\geqslant 0}$ of angle $\alpha - |\theta|$. Furthermore,

(i) $S_K(te^{i\theta})A \subset AS_K(te^{i\theta})$, $t \geqslant 0$ *and*

(ii) $A\int_0^{te^{i\theta}} S_K(s)x\,ds = S_K(te^{i\theta})x - \frac{1}{c(\theta)}\int_0^t K_\theta(s)\,ds Cx$, $t \geqslant 0$, $x \in E$, $\theta \in (-\alpha, \alpha)$.

PROOF. Let $\theta \in (-\alpha, \alpha)$ and let $\lambda \in \mathbb{R}$ be sufficiently large with $\widetilde{K_\theta}(\lambda) \neq 0$. Denote $\Gamma_\theta := \{te^{-i\theta} : t \geqslant 0\}$ and notice that $(c(\theta)S_K(te^{i\theta}))_{t\geqslant 0}$ is a strongly continuous, exponentially bounded operator family. Clearly, $\tilde{K}(\lambda e^{-i\theta}) \neq 0$ and Lemma 1.1 yields

$$\begin{aligned} \widetilde{K_\theta}(\lambda)(\lambda - e^{i\theta}A)^{-1}Cx &= \widetilde{K_\theta}(\lambda)e^{-i\theta}(\lambda e^{-i\theta} - A)^{-1}Cx \\ &= e^{-i\theta}\frac{\widetilde{K_\theta}(\lambda)}{\tilde{K}(\lambda e^{-i\theta})}\int_0^\infty e^{-\lambda e^{-i\theta}t}S_K(t)x\,dt = e^{-i\theta}c(\theta)\int_{\Gamma_\theta} e^{-\lambda t}e^{i\theta}S_K(te^{i\theta})x\,dt \\ &= \int_0^\infty e^{-\lambda t}(c(\theta)S_K(te^{i\theta})x)\,dt,\quad x \in E, \end{aligned} \tag{2.3}$$

where (2.3) follows from an elementary application of the Cauchy theorem. Keeping in mind Definition 1.1 and Lemma 1.1(ii), the assertion automatically follows. $\square$

Now we state the following generalization of [**8**, Theorem 10] and [**9**, Theorem 5.2].

THEOREM 2.1. (i) *Suppose K satisfies* (P1), $\omega \geqslant \max(0, \operatorname{abs}(K))$, $\alpha \in (0, \frac{\pi}{2}]$, *and $\tilde{K}(\cdot)$ can be analytically continued to a function $g : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to \mathbb{C}$. Suppose, further, that A is a subgenerator of an analytic K-convoluted C-semigroup $(S_K(t))_{t\geqslant 0}$ of angle α and that*

$$\sup_{z\in\Sigma_\gamma} \|e^{-\omega z}S_K(z)\| < \infty \text{ for all } \gamma \in (0, \alpha). \tag{2.4}$$

Let us denote by $\hat{A}$ the integral generator of $(S_K(t))_{t\geqslant 0}$ and put

$$N := \{\lambda \in \omega + \Sigma_{\frac{\pi}{2}+\alpha} : g(\lambda) \neq 0\}. \tag{2.5}$$

Then:

$$N \subset \rho_C(\hat{A}), \tag{2.6}$$

$$\sup_{\lambda\in N\cap(\omega+\Sigma_{\frac{\pi}{2}+\gamma_1})} \|(\lambda-\omega)g(\lambda)(\lambda-\hat{A})^{-1}C\| < \infty \textit{ for all } \gamma_1 \in (0,\alpha), \tag{2.7}$$

$$\lim_{\lambda\to+\infty,\, \tilde{K}(\lambda)\neq 0} \lambda\tilde{K}(\lambda)(\lambda-A)^{-1}Cx = 0,\ x\in E, \tag{2.8}$$

$$\textit{the mapping } \lambda \mapsto (\lambda-\hat{A})^{-1}C,\ \lambda \in N \textit{ is analytic.} \tag{2.9}$$

Suppose, additionally, that the following condition holds:

(H$_1$) : (H) *holds with* $c(\cdot)$, $\omega_0(\cdot)$, $(K_\theta)_{\theta\in(-\alpha,\alpha)}$, *and additionally,* $\mathrm{abs}(K_\theta) \leqslant \omega\cos\theta$, $\theta\in(-\alpha,\alpha)$.

Then (2.6)–(2.7) *and* (2.9) *hold with* $\hat{A}$ *replaced by* A *therein.*

(ii) *Assume* $\alpha\in(0,\frac{\pi}{2}]$, K *satisfies* (P1) *and* $\omega \geqslant \max(0,\mathrm{abs}(K))$. *Suppose that* A *is a closed linear operator with* $\{\lambda\in\mathbb{C} : Re\lambda>\omega,\ \tilde{K}(\lambda)\neq 0\}\subset\rho_C(A)$ *and that the function* $\lambda\mapsto\tilde{K}(\lambda)(\lambda-A)^{-1}C$, $\mathrm{Re}\,\lambda>\omega$, $\tilde{K}(\lambda)\neq 0$, *can be analytically extended to a function* $\tilde{q} : \omega+\Sigma_{\frac{\pi}{2}+\alpha}\to L(E)$ *satisfying*

$$\sup_{\lambda\in\omega+\Sigma_{\frac{\pi}{2}+\gamma}} \|(\lambda-\omega)\tilde{q}(\lambda)\| < \infty \textit{ for all } \gamma\in(0,\alpha), \tag{2.10}$$

$$\lim_{\lambda\to+\infty} \lambda\tilde{q}(\lambda)x = 0,\ \ x\in E,\ \textit{if}\ \overline{D(A)}\neq E. \tag{2.11}$$

Then the operator A is a subgenerator of an exponentially bounded, analytic K-convoluted C-semigroup of angle α.

Proof. The proof of (i) can be obtained as follows. By Lemma 1.1(i), we have $\{\lambda\in\mathbb{C} :\ \mathrm{Re}\,\lambda>\omega,\ \tilde{K}(\lambda)\neq 0\}\subset\rho_C(A)$ and

$$\tilde{K}(\lambda)(\lambda-A)^{-1}Cx = \int_0^\infty e^{-\lambda t}S_K(t)x\,dt,\ \mathrm{Re}\,\lambda>\omega,\ \tilde{K}(\lambda)\neq 0,\ x\in E.$$

Put $q(\lambda) := \int_0^\infty e^{-\lambda t}S_K(t)\,dt$, $\mathrm{Re}\,\lambda>\omega$. An application of [**1**, Theorem 2.6.1] gives that the function $q(\cdot)$ can be extended to an analytic function $\tilde{q} : \omega+\Sigma_{\frac{\pi}{2}+\alpha}\to L(E)$ satisfying $\sup_{\lambda\in\omega+\Sigma_{\frac{\pi}{2}+\gamma}}\|(\lambda-\omega)\tilde{q}(\lambda)\|<\infty$ for all $\gamma\in(0,\alpha)$. Further on, N is an open subset of $\mathbb{C}$ and it can be easily seen that every two point belonging to N can be connected with a C^∞ curve lying in N; in particular, N is a connected open subset of $\mathbb{C}$. The function $F : N\to L(E)$ given by $F(\lambda) := \frac{\tilde{q}(\lambda)}{g(\lambda)}$, $\lambda\in N$ is analytic and

$$\{\lambda\in\mathbb{C} : \mathrm{Re}\,\lambda>\omega,\ \tilde{K}(\lambda)\neq 0\}\subset\{\lambda\in N\cap\rho_C(A) : F(\lambda)=(\lambda-A)^{-1}C\}. \tag{2.12}$$

Let us denote $V = \{\lambda \in N \cap \rho_C(A) : F(\lambda) = (\lambda - A)^{-1}C\}$ and suppose $\mu \in \rho_C(A)$, $x \in D(A)$ and $y \in E$. Since

$$F(\lambda)(\lambda - A)x = (\lambda - A)^{-1}C(\lambda - A)x = Cx,\ \lambda \in V, \tag{2.13}$$

$$F(\lambda)Cy = CF(\lambda)y,\ \lambda \in V, \tag{2.14}$$

$$F(\lambda)Cy = (\lambda - A)^{-1}C^2y = (\mu - A)^{-1}C^2y - (\lambda - \mu)(\mu - A)^{-1}CF(\lambda)y,\ \lambda \in V, \tag{2.15}$$

the uniqueness theorem for analytic functions (cf. [**1**, Proposition A2, Proposition B.5]) implies that (2.13)–(2.15) remain true for all $\lambda \in N$. Suppose now that $(\lambda - A)x = 0$ for some $\lambda \in N$ and $x \in D(A)$. Owing to (2.13), one gets $Cx = 0$, $x = 0$ and $\lambda - A$ is injective. By the assertion (b), we obtain that $\lambda - \hat{A}$ is injective. Furthermore,

$$\begin{aligned}(\lambda - A)CF(\lambda)y &= (\lambda - A)F(\lambda)Cy\\ &= (\lambda - A)[(\mu - A)^{-1}C^2y - (\lambda - \mu)(\mu - A)^{-1}CF(\lambda)y]\\ &= C^2y + (\lambda - \mu)[(\mu - A)^{-1}C^2y - CF(\lambda)y - (\lambda - \mu)(\mu - A)^{-1}CF(\lambda)y],\end{aligned}$$

and thanks to the validity of (2.15) for all $\lambda \in N$, one obtains that

$$(\lambda - A)CF(\lambda)y = C^2y,\ \lambda \in N. \tag{2.16}$$

The last equality, injectiveness of C and assertion (a) taken together imply:

$$\lambda F(\lambda)y = C^{-1}AC[F(\lambda)y] + Cy = \hat{A}F(\lambda)y + Cy,\ \lambda \in N, \text{i.e.,} \tag{2.17}$$

$$(\lambda - \hat{A})F(\lambda)y = Cy,\ \ \lambda \in N. \tag{2.18}$$

This implies $R(C) \subset R(\lambda - \hat{A})$, $\lambda \in N$, $N \subset \rho_C(\hat{A})$, $F(\lambda) = (\lambda - \hat{A})^{-1}C$, $\lambda \in N$, (2.6) and (2.9). The estimate (2.7) is an immediate consequence of [**1**, Theorem 2.6.1]. Let $x \in E$ be fixed. Then $z \mapsto S_K(z)x$, $z \in \Sigma_\alpha$ is an analytic function which satisfies the condition (i) quoted in the formulation of [**1**, Theorem 2.6.1]. Since $\lim_{t\downarrow 0} S_K(t)x = 0$, an application of [**1**, Theorem 2.6.4] implies that $\lim_{\lambda\to+\infty} \lambda q(\lambda) = 0$. This gives $\lim_{\lambda\to+\infty,\ \tilde{K}(\lambda)\neq 0} \lambda\tilde{K}(\lambda)(\lambda - A)^{-1}Cx = 0$, i.e., (2.8) and the first part of the proof is completed. Suppose now that (H_1) holds. Then $\operatorname{abs}(K_\theta) \leqslant \omega\cos\theta$, $\theta \in (-\alpha,\alpha)$, and by Lemma 1.1(i), we have that, for every $\theta \in (-\alpha,\alpha)$, $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\cos\theta,\ \widetilde{K_\theta}(\lambda) \neq 0\} \subset \rho_C(e^{i\theta}A)$ and that:

$$\widetilde{K_\theta}(\lambda)e^{-i\theta}(\lambda e^{-i\theta} - A)^{-1}Cx = \int_0^\infty e^{-\lambda t}(c(\theta)S_K(te^{i\theta}))x\,dt, \tag{2.19}$$

for all $x \in E$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega\cos\theta$ and $\widetilde{K_\theta}(\lambda) \neq 0$. Fix a number $\theta \in (-\alpha,\alpha)$ and define $G_\theta : \{\omega + te^{i\varphi} : t > 0,\ \varphi \in (-(\frac{\pi}{2}+\theta),\ \frac{\pi}{2}-\theta)\} \cap N \to \mathbb{C}$ by $G_\theta(\lambda) := \frac{\widetilde{K_\theta}(\lambda e^{i\theta})}{g(\lambda)}$, $\lambda \in D(G_\theta(\cdot))$. Then it is clear that $D(G_\theta(\cdot))$ is an open, connected subset of $\mathbb{C}$ and that, owing to (2.1)–(2.2), there exists $a > 0$ such that $\Phi_{\theta,a} := \{te^{-i\theta} \cap N : t \geqslant a\} \subset D(G_\theta(\cdot))$ and that $G_\theta(\lambda) = c(\theta)$, $\lambda \in \Phi_{\theta,a}$. By the uniqueness theorem for analytic functions, one obtains that $G_\theta(\lambda) = c(\theta)$,

$\lambda \in D(G_\theta(\cdot))$. Hence, (2.19) implies $\{\omega + te^{i\varphi} : t > 0,\ \varphi \in (-(\frac{\pi}{2}+\theta),\ \frac{\pi}{2}-\theta)\} \cap N \subset \rho_C(A)$,

$$(z - A)^{-1}Cx = \frac{e^{i\theta}}{g(z)} \int_0^\infty e^{-ze^{i\theta}t} S_K(te^{i\theta})x\,dt, \tag{2.20}$$

for all $z \in \{\omega + te^{i\varphi} : t > 0,\ \varphi \in (-(\frac{\pi}{2}+\theta),\ \frac{\pi}{2}-\theta)\} \cap N$ and $x \in E$, and the mapping $z \mapsto (z-A)^{-1}C$, $z \in N$, $\arg(z-\omega) \in (-(\frac{\pi}{2}+\theta), \frac{\pi}{2}-\theta)$ is analytic. One can apply the same argument to $e^{-i\theta}A$ in order to see that $\{z \in N : \arg(z-\omega) \in (\theta-\frac{\pi}{2}, \frac{\pi}{2}+\theta)\} \subset \rho_C(A)$ and that the mapping $z \mapsto (z-A)^{-1}C$, $z \in N$, $\arg(z-\omega) \in (\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2})$ is analytic. Thereby, $\{z \in N : |\arg(z-\omega)| < \theta + \frac{\pi}{2}\} \subset \rho_C(A)$ and the mapping $z \mapsto (z-A)^{-1}C$, $z \in N$, $|\arg(z-\omega)| < \theta + \frac{\pi}{2}$ is analytic. This completes the proof of (i). The proof of (ii) in the case $\overline{D(A)} \neq E$ is given in [**8**]. Suppose now that $\overline{D(A)} = E$. We will prove that (2.11) automatically holds for every $x \in E$. Arguing as in the proof of [**8**, Theorem 10], one obtains that there exists an analytic function $S_K : \Sigma_\alpha \to L(E)$ such that $\sup_{z \in \omega + \Sigma_{\frac{\pi}{2}+\beta}} \|e^{-\omega z} S_K(z)\| < \infty$ for all $\beta \in (0, \alpha)$. By [**1**, Proposition 2.6.3(b)] and the proof of [**8**, Theorem 10], it suffices to show that $\lim_{t \downarrow 0} S_K(t)x = 0$. Suppose, for the time being, $x \in D(A)$. Since $\tilde{q}(\lambda)x = \tilde{K}(\lambda)(\lambda - A)^{-1}Cx$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > \omega$, $\tilde{K}(\lambda) \neq 0$ we have that $\mathcal{L}(\int_0^t S_K(s)Ax\,ds)(\lambda) = \frac{\tilde{q}(\lambda)}{\lambda}Ax = \tilde{q}(\lambda)x - \frac{\tilde{K}(\lambda)}{\lambda}Cx = \mathcal{L}(S_K(t)x - \Theta(t)Cx)(\lambda)$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > \omega$, $\tilde{K}(\lambda) \neq 0$ and the uniqueness theorem for Laplace transforms implies $\int_0^t S_K(s)Ax\,ds = S_K(t)x - \Theta(t)Cx$, $t \geqslant 0$. Therefore $\|S_K(t)x\| \leqslant |\Theta(t)|Cx + te^{\omega t}\|Ax\|$, $t \geqslant 0$ and $\lim_{t\downarrow 0} S_K(t)x = 0$. Combined with the exponential boundedness of $S_K(\cdot)$, this indicates that $\lim_{t \downarrow 0} S_K(t)x = 0$ for every $x \in E$. □

Let $\emptyset \neq \Omega \subset \rho_C(A)$ be open. By [**5**, Remark 2.7], we have that the continuity of mapping $\lambda \mapsto (\lambda - A)^{-1}C$, $\lambda \in \Omega$ implies its analyticity. Furthermore, it can be simply verified that the function $K(t) = \frac{t^{r-1}}{\Gamma(r)}$, $t > 0$, $r > 0$ satisfies the condition (H_1) with $c(\theta) = e^{-ir\theta}$, $\omega_0(\theta) = 0$ and $K_\theta(t) = K(t)$, $\theta \in (-\alpha, \alpha)$, $t > 0$. Keeping in mind Proposition 1.1, Theorem 2.1 and these remarks, one immediately obtains the proof of the following corollary; notice only that, in the case $r = 0$, the equality (2.24) follows from [**1**, Theorem 2.6.4] and elementary definitions.

Corollary 2.1. (i) *Suppose $r \geqslant 0$ and $\alpha \in (0, \frac{\pi}{2}]$. Then the operator A is a subgenerator of an exponentially bounded, analytic r-times integrated C-semigroup $(S_r(t))_{t \geqslant 0}$ of angle α iff for every $\gamma \in (0, \alpha)$, there exist $M_\gamma > 0$ and $\omega_\gamma \geqslant 0$ such that:*

$$\omega_\gamma + \Sigma_{\frac{\pi}{2}+\gamma} \subset \rho_C(A), \tag{2.21}$$

$$\|(\lambda - A)^{-1}C\| \leqslant M_\gamma(1 + |\lambda|)^{r-1}, \quad \lambda \in \omega_\gamma + \Sigma_{\frac{\pi}{2}+\gamma}, \tag{2.22}$$

(2.23)

the mapping $\lambda \mapsto (\lambda - A)^{-1}C,\ \lambda \in \omega_\gamma + \Sigma_{\frac{\pi}{2}+\gamma}$ is analytic (continuous) and

$$\lim_{\lambda \to +\infty} \frac{(\lambda - A)^{-1}Cx}{\lambda^{r-1}} = \chi_{\{0\}}(r)Cx, \ \ x \in E, \ \ \textit{if } \overline{D(A)} \neq E. \tag{2.24}$$

(ii) *Let* $\theta \in (-\alpha, \alpha)$ *and let* A *be a subgenerator of an exponentially bounded, analytic* r*-times integrated* C*-semigroup* $(S_r(t))_{t\geqslant 0}$ *of angle* α*. Then* $e^{i\theta}A$ *is a subgenerator of an exponentially bounded, analytic* r*-times integrated* C*-semigroup* $(e^{-i\theta r}S_r(te^{i\theta}))_{t\geqslant 0}$ *of angle* $\alpha - |\theta|$*,* $S_r(z)A \subset AS_r(z)$ *and* $A\int_0^z S_r(s)x\,ds = S_r(z)x - \frac{z^r}{\Gamma(r+1)}Cx$*,* $z \in \Sigma_\alpha$*,* $x \in E$*.*

Now we state Kato's analyticity criterion for convoluted C-semigroups.

THEOREM 2.2. *Suppose* $\alpha \in (0, \frac{\pi}{2}]$*,* K *satisfies* (P1)*,* $\omega \geqslant \max(0, \operatorname{abs}(K))$*, there exists an analytic function* $g : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to \mathbb{C}$ *such that* $g(\lambda) = \tilde{K}(\lambda)$*,* $\lambda \in \mathbb{C}$*,* $\operatorname{Re}\lambda > \omega$ *and* (H_1) *holds. Then* A *is a subgenerator of an analytic* K*-convoluted* C*-semigroup* $(S_K(t))_{t\geqslant 0}$ *satisfying* (2.4) *iff:*

(i.1) *For every* $\theta \in (-\alpha, \alpha)$*,* $e^{i\theta}A$ *is a subgenerator of a* K_θ*-convoluted* C*-semigroup* $(S_\theta(t))_{t\geqslant 0}$*, and*

(i.2) *for every* $\beta \in (0, \alpha)$*, there exists* $M_\beta > 0$ *such that*

$$\left\|\frac{1}{c(\theta)}S_\theta(t)\right\| \leqslant M_\beta e^{\omega t\cos\theta}, \;\; t \geqslant 0, \;\theta \in (-\beta, \beta). \tag{2.25}$$

PROOF. Suppose A is a subgenerator of an analytic K-convoluted C-semigroup $(S_K(t))_{t\geqslant 0}$ satisfying (2.4). By Proposition 1.1, we have that (i.1) and (i.2) hold with $S_\theta(t) = c(\theta)S_K(te^{i\theta})$, $t \geqslant 0$, $\theta \in (-\alpha, \alpha)$. To prove the converse statement, notice that the argumentation given in the final part of the proof of Theorem 2.1 implies that $(\omega + \Sigma_{\frac{\pi}{2}+\alpha}) \cap N \subset \rho_C(A)$ and that there exists an analytic mapping $G : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(E)$ such that $G(\lambda) = g(\lambda)(\lambda - A)^{-1}C$, $\lambda \in (\omega + \Sigma_{\frac{\pi}{2}+\alpha}) \cap N$, where N is defined by (2.5). Furthermore, for every $\theta \in (-\alpha, \alpha)$:

$$G(\lambda) = e^{i\theta}\int_0^\infty e^{-\lambda te^{i\theta}}\Big(\frac{1}{c(\theta)}S_\theta(t)\Big)\,dt \text{ if } \arg(\lambda - \omega) \in \Big(-\Big(\frac{\pi}{2} + \theta\Big), \frac{\pi}{2} - \theta\Big), \tag{2.26}$$

$$G(\lambda) = e^{-i\theta}\int_0^\infty e^{-\lambda te^{-i\theta}}\Big(\frac{1}{c(-\theta)}S_{-\theta}(t)\Big)\,dt \text{ if } \arg(\lambda - \omega) \in \Big(\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2}\Big). \tag{2.27}$$

Keeping in mind (i.2) as well as (2.26)–(2.27), we have that, for every $\beta \in (0, \alpha)$, $\sup_{\lambda \in \omega + \Sigma_{\frac{\pi}{2}+\beta}} \|(\lambda - \omega)G(\lambda)\| < \infty$. By [**1**, Theorem 2.6.1], one gets the existence of an analytic mapping $S_K : \Sigma_\alpha \to L(E)$ such that $\sup_{z \in \Sigma_\beta} \|e^{-\omega z}S_K(z)\| < \infty$ for all $\beta \in (0, \alpha)$ and that $G(\lambda) = \widetilde{S_K}(\lambda)$ for all $\lambda \in (\omega, \infty)$. Furthermore, the uniqueness theorem for Laplace transforms implies $S_K(z) = \frac{1}{c(\arg(z))}S_{\arg(z)}(|z|)$, $z \in \Sigma_\alpha$, and since $c(0) = 1$ and $K_0 = K$, it suffices to show that, for every fixed $x \in E$ and $\beta \in (0, \alpha)$, one has $\lim_{z \in \Sigma_{-\beta},\, z \to 0} S_K(z)x = 0$ (cf. also Lemma 1.1(ii)). To this end, notice that $\lim_{t\downarrow 0} S_K(t)x = \lim_{t\downarrow 0} S_0(t)x = 0$ and that [**1**, Proposition 2.6.3(b)] implies $\lim_{z \in \Sigma_\beta,\, z \to 0} e^{-\omega z}S_K(z)x = \lim_{z \in \Sigma_\beta,\, z \to 0} S_K(z)x = 0$, $z \in \Sigma_\alpha$. $\square$

In the following corollary, we remove any density assumption from [**14**, Theorem]:

COROLLARY 2.2. *Suppose* $r \geqslant 0$, $\alpha \in (0, \frac{\pi}{2}]$ *and* $\omega \in [0, \infty)$ *if* $r > 0$, *resp.* $\omega \in \mathbb{R}$ *if* $r = 0$. *Then* A *is a subgenerator of an analytic* r*-times integrated* C*-semigroup* $(S_r(t))_{t \geqslant 0}$ *of angle* α *satisfying* $\sup_{\lambda \in \Sigma_\beta} \|e^{-\omega z} S_r(z)\| < \infty$ *for all* $\beta \in (0, \alpha)$ *iff the following conditions hold:*

(i.1) *For every* $\theta \in (-\alpha, \alpha)$, $e^{i\theta}A$ *is a subgenerator of an* r*-times integrated* C*-semigroup* $(S_\theta(t))_{t \geqslant 0}$, *and*

(i.2) *for every* $\beta \in (0, \alpha)$, *there exists* $M_\beta > 0$ *such that* $\|S_\theta(t)\| \leqslant M_\beta e^{\omega t \cos\theta}$, $t \geqslant 0$, $\theta \in (-\beta, \beta)$.

Now we state the following extension of [**1**, Theorem 3.9.7] and [**1**, Corollary 3.9.9]:

THEOREM 2.3. *Suppose* $\alpha \in (0, \frac{\pi}{2})$, A *is densely defined and* $e^{\pm i\alpha}A$ *are subgenerators of* (*exponentially*) *bounded* C*-semigroups* $(T_{\pm\alpha}(t))_{t \geqslant 0}$. *Then* A *is a subgenerator of an* (*exponentially*) *bounded, analytic* C*-semigroup of angle* α.

PROOF. Suppose $\|T_{\pm\alpha}(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$ for appropriate constants $M \geqslant 0$ and $\omega \geqslant 0$. Put $\mu := \frac{\omega}{\cos\alpha}$ and $A_\mu := A - \mu$. Then $e^{\pm i\alpha}A_\mu$ are subgenerators of bounded C-semigroups $(S_{\pm\alpha}(t) := e^{-e^{\pm i\alpha}\mu t} T_{\pm\alpha}(t))_{t \geqslant 0}$ and $\|S_{\pm\alpha}(t)\| \leqslant M,\ t \geqslant 0$. Proceeding as in the proof of [**1**, Theorem 3.9.7], one gets that $\Sigma_{\frac{\pi}{2}+\alpha} \subset \rho_C(A_\mu)$ and that the mapping $\lambda \mapsto (\lambda - A_\mu)^{-1}C$, $\lambda \in \Sigma_{\frac{\pi}{2}+\alpha}$ is analytic. Then the proof of [**5**, Corollary 2.8] implies that, for every $n \in \mathbb{N}_0$ and $\lambda \in \Sigma_{\frac{\pi}{2}+\alpha}$:

$$R(C) \subset R\big((\lambda - A_\mu)^{n+1}\big) \text{ and } \frac{d^n}{d\lambda^n}(\lambda - A_\mu)^{-1}C = (-1)^n n!(\lambda - A_\mu)^{-(n+1)}C. \tag{2.28}$$

Put now $T_{n,k}(z) := (I - \frac{z}{n}A_\mu)^{-k}C$, $z \in \overline{\Sigma_\alpha}$, $k \in \mathbb{N}$, $n \in \mathbb{N}$. By (2.28), we obtain that, for every $r \geqslant 0$:

$$\begin{aligned} \|T_{n,k}(re^{\pm i\alpha})\| &= \Big\|\Big(I - \frac{re^{\pm i\alpha}}{n}A_\mu\Big)^{-k}C\Big\| = \Big\|\frac{n^k}{r^k}\Big(\frac{n}{r}I - e^{\pm i\alpha}A_\mu\Big)^{-k}C\Big\| \\ &= \Bigg\|\frac{n^k}{r^k}\frac{(\frac{d^{k-1}}{d\lambda^{k-1}}(\lambda - e^{\pm i\alpha}A_\mu)^{-1}C)_{|\lambda=\frac{n}{r}}}{(-1)^{k-1}(k-1)!}\Bigg\| \\ &= \Bigg\|\frac{n^k}{r^k}\frac{(-1)^{k-1}\int_0^\infty e^{-\frac{n}{r}t}t^{k-1}S_{\pm\alpha}(t)\,dt}{(-1)^{k-1}(k-1)!}\Bigg\| \leqslant M. \end{aligned} \tag{2.29}$$

Arguing similarly, we get:

$$\|T_{n,k}(z)\| \leqslant \frac{M}{\cos^k\alpha},\ z \in \Sigma_\alpha,\ k \in \mathbb{N},\ n \in \mathbb{N}. \tag{2.30}$$

Taking into account the Phragmén–Lindelöf principle (cf. for instance [**1**, Theorem 3.9.8]) and (2.29)–(2.30), one obtains that $\|T_{n,k}(z)\| \leqslant M$, $z \in \overline{\Sigma_\alpha}$, $k \in \mathbb{N}$, $n \in \mathbb{N}$. In particular, $\|\frac{d^n}{d\lambda^n}(\lambda I - A_\mu)^{-1}C\| \leqslant \frac{Mn!}{\lambda^n}$, $\lambda > 0$, $n \in \mathbb{N}_0$ and Lemma 1.1(iii) implies that A_μ is a subgenerator of a bounded C-semigroup $(T(t))_{t \geqslant 0}$ such that $(\lambda - A_\mu)^{-1}Cx = \int_0^\infty e^{-\lambda t}T(t)x\,dt$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > 0$, $x \in E$. By the Post–Widder inversion formula [**1**, Theorem 1.7.7], one obtains $T(t)x = \lim_{n\to\infty} T_{n,n+1}(\frac{t}{n})x$, $x \in E$, $t \geqslant 0$ and Vitali's theorem [**1**, Theorem A.5, p. 458] implies that there exists an analytic mapping $\tilde{T} : \Sigma_\alpha \to L(E)$ such that $\tilde{T}(t) = T(t)$, $t > 0$ and that

$\|\tilde{T}(z)\| \leqslant M$, $z \in \Sigma_\alpha$. By [**1**, Proposition 2.6.3(b)], one yields that the mapping $z \mapsto \tilde{T}(z)x$, $z \in \overline{\Sigma_\beta}$ is continuous for every fixed $x \in E$ and $\beta \in (0, \alpha)$ and the proof of theorem completes a routine argument. □

The preceding theorem has been recently generalized in [**11**]:

THEOREM 2.4. *Suppose $\alpha \in (0, \frac{\pi}{2})$, $r \geqslant 0$, and $e^{\pm i\alpha}A$ are subgenerators of exponentially bounded r-times integrated C semigroups $(S_r^{\pm\alpha}(t))_{t \geqslant 0}$. Then, for every $\zeta > 0$, A is a subgenerator of an exponentially bounded, analytic $(r+\zeta)$-times integrated C semigroup $(S_{r+\zeta}(t))_{t \geqslant 0}$ of angle α; if A is densely defined, then A is a subgenerator of an exponentially bounded, analytic r-times integrated C semigroup $(S_r(t))_{t \geqslant 0}$ of angle α.*

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