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PROPERTIES OF ARMENDARIZ RINGS AND WEAK ARMENDARIZ RINGS

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Abstract. We consider some properties of Armendariz and rigid rings. We prove that the direct product of rigid (weak rigid), weak Armendariz rings is a rigid (weak rigid), weak Armendariz ring. On the assumption that the factor ring R/I is weak Armendariz, where I is nilpotent ideal, we prove that R is a weak Armendariz ring. We also prove that every ring isomorphism preserves weak skew Armendariz structure. Armendariz rings of Laurent power series are also considered.

1. Introduction

Throughout this paper R denotes an associative ring with identity, σ denotes an endomorphism of R and $R[x;\sigma]$ denotes a skew polynomial ring with the ordinary addition and the multiplication subject to the relation $xr = \sigma(r)x$. When σ is an automorphism, $R[x, x^{-1};\sigma]$ denotes a skew Laurent polynomial ring with the multiplication subject to the relation $x^{-1}r = \sigma^{-1}(r)x$.

The notion of Armendariz ring is introduced by Rege and Chhawchharia [**1**]. They defined a ring R to be Armendariz if $f(x)g(x) = 0$ implies $a_ib_j = 0$, for all polynomials $f(x) = \sum_{i=0}^{n} a_ix^i$ and $g(x) = \sum_{j=0}^{m} b_jx^j$ from $R[x]$. The motivation for those rings comes from the fact that Armendariz had shown that reduced rings ($a^2 = 0$ implies $a = 0$) satisfy this condition. The notion of Armendariz ring is natural and useful in understanding the relation between annihilators of rings R and $R[x]$ (see [**4**]). Those rings were also studied by Armendariz himself, Hong and Kim [**5**], Chen and Tong [**3**], Krempa [**6**] and others.

An endomorphism σ is rigid if $a\sigma(a) = 0$ implies $a = 0$, for all $a \in R$ (Krempa [**6**]). Following Hong, a ring is said to be rigid if it has a rigid endomorphism. Hong also generalized the notions of Armendariz and rigid ring to σ-skew Armendariz ring. Ring R is called σ-skew Armendariz if $f(x)g(x) = 0$ implies $a_i\sigma^i(b_j) = 0$, for all $f(x) = \sum_{i=0}^{n} a_ix^i$ and $g(x) = \sum_{j=0}^{m} b_jx^j$ from $R[x;\sigma]$ (see [**5**]). As a generalization of σ-skew Armendariz rings, Ouyang (see [**2**]) introduced a notion of weak σ-skew

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Armendariz ring R as a ring in which $f(x)g(x)=0$ implies $a_i\sigma^i(b_j)$ is the nilpotent element of R for all $f(x)=\sum_{i=0}^n a_ix^i$ and $g(x)=\sum_{j=0}^m b_jx^j$ from $R[x;\sigma]$. Ouyang also introduced a notion of weak σ-rigid ring as a ring with an endomorphism σ that satisfies $a\sigma(a)\in \mathrm{nil}(R)$ if and only if $a\in \mathrm{nil}(R)$ for all $a\in R$ where $\mathrm{nil}(R)$ is the set of all nilpotent elements of R. In [**3**] is shown that R is σ-rigid if and only if R is weak σ-rigid and reduced. Here we show that if A is σ_1-rigid and B is σ_2-rigid, then $A\times B$ is γ-rigid, where endomorphism γ is such that $\gamma(a,b)=(\sigma_1(a),\sigma_2(b))$. In this paper we consider conditions which characterize σ-rigid rings and prove that R is σ-skew Armendariz ring if and only if $R[x,x^{-1};\sigma]$ is σ-skew Armendariz ring. Chen and Tong (see [**3**]) have proved that if R and S are rings and σ is an isomorphism of rings R and S and R is α-skew Armendariz ring, then S is $\sigma\alpha\sigma^{-1}$-skew Armendariz ring. In this paper we prove a variant of this theorem for weak skew Armendariz rings. We also prove that if α is endomorphism of ring R, and the factor ring $R[x]/(x^n)$ is weak $\widetilde{\alpha}$-skew Armendariz, then $V_n(R)$ is weak $\widetilde{\alpha}$-skew Armendariz.

2. Rigid rings and weak rigid rings

In this section we give a simple and straightforward proof that the finite direct product of rigid (weak rigid) rings is a rigid (weak rigid) ring. We also show how the notion of rigidity of a ring can be naturally transferred to the notion of rigidity of the corresponding ring of polynomials.

LEMMA 2.1. *If A is σ_1-rigid ring and B is σ_2-rigid ring, then $A\times B$ is γ-rigid, where $\gamma(a,b)=(\sigma_1(a),\sigma_2(b))$.*

PROOF. Suppose that $(a,b)\gamma(a,b)=(0,0)$; then $(a,b)(\sigma_1(a),\sigma_2(b))=(0,0)$ so that $(a\sigma_1(a),b\sigma_2(b))=(0,0)$. Since $a\sigma_1(a)=0$, $b\sigma_2(b)=0$, from the fact that A,B are rigid rings we have $(a,b)=(0,0)$, which means that $A\times B$ is a γ-rigid ring. □

COROLLARY 2.1. *Finite direct product of σ_i-rigid rings, $1\leqslant i\leqslant n$, is γ-rigid ring, where $\gamma(a_1,a_2,\dots,a_n)=(\sigma_1(a_1),\sigma_2(a_2),\dots,\sigma_n(a_n))$.*

LEMMA 2.2. *If A is a weak σ_1-rigid ring and B is a weak σ_2-rigid ring, then $A\times B$ is a weak γ-rigid ring, where γ is such that $\gamma(a,b)=(\sigma_1(a),\sigma_2(b))$.*

PROOF. Suppose that $(a,b)\gamma(a,b)\in \mathrm{nil}(A\times B)$. From the definition of γ, we have $(a,b)(\sigma_1(a),\sigma_2(b))\in \mathrm{nil}(A\times B)$, so that $(a\sigma_1(a),b\sigma_2(b))\in \mathrm{nil}(A\times B)$ which means that $(a\sigma_1(a),b\sigma_2(b))^n=(0,0)$ for some $n\geqslant 2$. Therefore $(a\sigma_1(a))^n=0$, $(b\sigma_2(b))^n=0$ and $a\sigma_1(a)\in \mathrm{nil}(A)$, $b\sigma_2(b)\in \mathrm{nil}(B)$. From the assumption that A is weak σ_1-rigid and B weak σ_2-rigid we have $a\in \mathrm{nil}(A)$ and $b\in \mathrm{nil}(B)$, so that there exist n_1,n_2 such that $a^{n_1}=0$, $b^{n_2}=0$. Finally we have $(a,b)^{\max(n_1,n_2)}=(0,0)$ which means that $(a,b)\in \mathrm{nil}(A\times B)$.

Conversely, if $(a,b)\in \mathrm{nil}(A\times B)$, using the same arguments we can show that $(a,b)\gamma(a,b)\in \mathrm{nil}(A\times B)$. □

COROLLARY 2.2. *The finite direct product of weak σ_i-rigid rings, $1\leqslant i\leqslant n$, is a weak γ-rigid ring, where $\gamma(a_1,a_2,\dots,a_n)=(\sigma_1(a_1),\sigma_2(a_2),\dots,\sigma_n(a_n))$.*

We now show how the notion of rigidity naturally transferees from the ring R to the ring $R[x]$. If σ is an endomorphism of a ring R, then the map σ can be naturally extended to an endomorphism σ' of the ring $R[x]$ by $\sigma'(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} \sigma(a_i) x^i$.

THEOREM 2.1. *If R is σ-rigid, then $R[x]$ is σ'-rigid ring.*

PROOF. Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ and $f(x)\sigma'(f(x)) = 0$. We have to prove that $f(x) = 0$. From the relation

$$(a_0 + a_1 x + \cdots + a_n x^n)(\sigma(a_0) + \sigma(a_1)x + \cdots + \sigma(a_n)x^n) = 0,$$

we have that $a_0\sigma(a_0) = 0$, which means $a_0 = 0$. Since the coefficient of x^2 has to be zero, we have $a_0\sigma(a_2) + a_1\sigma(a_1) + a_2\sigma(a_0) = 0$, so that $a_1\sigma(a_1) = 0$, and since R is σ-rigid, we have $a_1 = 0$. Continuing in this way, since the coefficient of x^{2n-2} has to be zero, and since $a_{n-2} = 0$, from the previous step, we have

$$a_{n-2}\sigma(a_n) + a_{n-1}\sigma(a_{n-1}) + a_n\sigma(a_{n-2}) = 0,$$

which means that $a_{n-1}\sigma(a_{n-1}) = 0$, so that from the rigidity of the ring R we have $a_{n-1} = 0$. Finally, from the fact that the coefficient of x^{2n} has to be zero, we obtain $a_n\sigma(a_n) = 0$, which means that $a_n = 0$ and so $f(x) = 0$. □

3. Skew Polynomial Laurent series Rings

In this section we introduce Laurent σ-Armendariz rings and Laurent σ-skew power series rings and we give their useful characterization in terms of σ-skew Armendariz rings. Throughout this section σ is a ring automorphism.

A ring R is a σ-skew Armendariz ring of Laurent type if for every two polynomials $f(x) = \sum_{i=-p}^{q} a_i x^i$, and $g(x) = \sum_{j=-t}^{s} b_j x^j$ from $R\left[x, x^{-1}; \sigma\right]$,

$$f(x)g(x) = 0 \text{ implies } a_i\sigma^i(b_j) = 0, -p \leqslant i \leqslant q, -t \leqslant j \leqslant s.$$

We say that R is a σ-skew power series Armendariz ring of Laurent type if for every $f(x) = \sum_{i=-p}^{\infty} a_i x^i$, and $g(x) = \sum_{j=-t}^{\infty} b_j x^j$ from the power series ring $R[[x, x^{-1}; \sigma]]$,

$$f(x)g(x) = 0 \text{ implies } a_i\sigma^i(b_j) = 0, -p \leqslant i \leqslant \infty, -t \leqslant j \leqslant \infty.$$

In the following two theorems we give a useful characterization of Laurent σ-skew Armendariz rings and Laurent σ-skew power series rings.

THEOREM 3.1. *The following conditions are equivalent:*

(1) *R is a σ-skew Armendariz ring,*
(2) *R is a σ-skew Armendariz ring of Laurent type.*

PROOF. Suppose that $f(x) = \sum_{i=-p}^{q} a_i x^i$ and $g(x) = \sum_{j=-t}^{s} b_j x^j$ are polynomials from the ring $R[x, x^{-1}; \sigma]$ such that $f(x)g(x) = 0$. Since $x^p f(x)$ and $x^t g(x)$ are polynomials from the ring $R[x; \sigma]$ we have that $x^p f(x) g(x) x^t = 0$ which gives $\sigma^p(a_i)\sigma^{i+p}(b_j) = 0, -p \leqslant i \leqslant q, -t \leqslant j \leqslant s$. Since σ is an automorphism,

$$\sigma^p(a_i\sigma^i(b_j)) = 0,$$

so that we have $a_i\sigma^i(b_j) = 0$. The converse is evident since $R[x; \sigma] \subset R\left[x, x^{-1}; \sigma\right]$. □

THEOREM 3.2. *The following conditions are equivalent:*

(1) *R is a σ-skew power series Armendariz ring,*
(2) *R is a σ-skew power series Armendariz ring of Laurent type.*

PROOF. The same as the proof of the previous theorem. □

We close this section with an interesting remark which gives a sufficient condition for the power series ring $R[[x;\sigma]]$ to be reduced.

THEOREM 3.3. *If an endomorphism σ of a reduced ring R satisfies the so-called compatibility condition: $a\sigma(b)=0 \Leftrightarrow ab=0$, then the power series ring $R[[x;\sigma]]$ is reduced.*

PROOF. Let $f(x)=\sum_{i=0}^{\infty} a_i x^i$ and $(f(x))^2=0$. We have to prove that $f(x)=0$. It is clear that $a_0^2=0$, so that $a_0=0$. Now, since the coefficient of x^2 has to be zero, we have $a_0a_2+a_1\sigma(a_1)+a_2\sigma^2(a_0)=0$, so that we obtain $a_1\sigma(a_1)=0$. From the compatibility condition we obtain $a_1^2=0$ and since R is reduced, we have $a_1=0$. Continuing in this way, since the coefficient of x^{2n} is zero, we have $a_n\sigma^n(a_n)=0$ and, using compatibility condition once again, we have $a_n\sigma^{n-1}(a_n)=0$ and in the same way $a_n\sigma(a_n)=0$, so that $a_n=0$. By induction, we have $a_i=0$, for all i. This means that $f(x)=0$ and so the ring $R[[x;\sigma]]$ is reduced. □

Without compatibility condition the previous theorem is not true. Since if the ring $R=Z_2\oplus Z_2$ and σ is defined by $\sigma(a,b)=(b,a)$, it is easy to check that $R[[x;\sigma]]$ is not reduced. Observe that $(1,0)(0,1)=(0,0)$ but $(1,0)\sigma(0,1)\neq(0,0)$.

4. Weak Armendariz rings

In this section we generalize some results from [**3**], which are related to σ-skew Armendariz rings, to the weak σ-skew Armendariz case.

A ring R is weak Armendariz if $f(x)g(x)=0$ implies $a_ib_j\in \text{nil}(R)$ for every two polynomials $f(x)=a_0+a_1x+\cdots+a_nx^n$, $g(x)=b_0+b_1x+\cdots+b_mx^m$ from the ring $R[x]$. This definition is equivalent with the fact that ideal 0 is weak Armendariz ideal. We will prove that the class of weak Armendariz rings is closed for direct products. Also, if the factor ring R/I is a weak Armendariz ring, for some nilpotent ideal I, then the ring R is weak Armendariz.

THEOREM 4.1. *The finite direct product of weak Armendariz rings is a weak Armendariz ring.*

PROOF. Suppose that $R_1,R_2,\dots,R_n$ are weak Armendariz rings and $R=\prod_{i=1}^{n} R_i$. If $f(x)g(x)=0$ for some polynomials

$$f(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n,\ \ g(x)=b_0+b_1x+\cdots+b_mx^m\in R[x],$$

where $a_i=(a_{i1},a_{i2},\dots,a_{in})$, $b_i=(b_{i1},b_{i2},\dots,b_{in})$ are elements of the product ring R, define

$$f_k(x)=a_{0k}+a_{1k}x+\cdots+a_{nk}x^n,\ \ g_k(x)=b_{0k}+b_{1k}x+\cdots+b_{mk}x^m.$$

From $f(x)g(x) = 0$, we have $a_0b_0 = 0$, $a_0b_1 + a_1b_0 = 0, \dots, a_nb_m = 0$, and this implies

$$a_{01}b_{01} = a_{02}b_{02} = \cdots = a_{0n}b_{0n} = 0$$
$$a_{01}b_{11} + a_{11}b_{01} = \cdots = a_{0n}b_{1n} + a_{1n}b_{0n} = 0$$
$$a_{n1}b_{m1} = a_{n2}b_{m2} = \cdots = a_{nn}b_{mn} = 0.$$

This means that $f_k(x)g_k(x) = 0$ in $R_k[x]$, $1 \leqslant k \leqslant n$, and since R_k are weak Armendariz rings, we have $a_{ik}b_{jk} \in \mathrm{nil}(R_k)$. Now, for each i, j, there exists positive integers m_{ijk} such that $(a_{ik}b_{jk})^{m_{ijk}} = 0$ in the ring R_k, $1 \leqslant k \leqslant n$. If we take $m_{ij} = \max\{m_{ijk} : 1 \leqslant k \leqslant n\}$, then it is clear that $(a_ib_j)^{m_{ij}} = 0$ and this means that R is a weak Armendariz ring. □

THEOREM 4.2. *If I is a nilpotent ideal of ring R such that R/I is a weak Armendariz ring, then R is a weak Armendariz ring.*

PROOF. Let $f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m$ are polynomials from $R[x]$ such that $f(x)g(x) = 0$. This implies

$$(\overline{a_0} + \overline{a_1}x + \cdots + \overline{a_n}x^n)(\overline{b_0} + \overline{b_1}x + \cdots + \overline{b_m}x^m) = 0,$$

and since R/I is weak Armendariz, we have that $\overline{a_i}\overline{b_j} \in \mathrm{nil}(R|I)$. From the fact that the ideal I is nilpotent, we obtain that $a_ib_j \in \mathrm{nil}(R)$. □

Recall that a ring R is weak σ-rigid if $a\sigma(a) \in \mathrm{nil}(R) \Leftrightarrow a \in \mathrm{nil}(R)$. It is easy to see that the notion of weak σ-rigid ring generalizes the notion of a σ-rigid ring. Every homomorphism σ of rings R and S can be extended to the homomorphism of rings $R[x]$ and $S[x]$ by $\sum_{i=0}^{m} a_ix^i \mapsto \sum_{i=0}^{m} \sigma(a_i)x^i$, which we also denote by σ. Chen and Tong in [**3**] prove that if σ is a ring isomorphism of rings R and S and R is α-skew Armendariz, then S is a $\sigma\alpha\sigma^{-1}$ skew Armendariz ring. We prove the weak skew Armendariz variant of this theorem.

THEOREM 4.3. *Let R and S be rings with a ring isomorphism $\sigma : R \to S$. If R is weak α-skew Armendariz, then S is weak $\sigma\alpha\sigma^{-1}$-skew Armendariz.*

PROOF. Let $f(x) = \sum_{i=0}^{m} a_ix^i$ and $g(x) = \sum_{j=0}^{m} b_jx^j$ are polynomials from the ring $S[x; \sigma\alpha\sigma^{-1}]$. We have to prove that $f(x)g(x) = 0$ implies $a_i(\sigma\alpha\sigma^{-1})^i(b_j) \in \mathrm{nil}(S)$, for all i and j.

As we noted, σ extends to the isomorphism of the corresponding polynomial rings, so that there exist polynomials $f_1(x) = \sum_{i=0}^{m} a'_ix^i$ and $g_1(x) = \sum_{j=0}^{m} b'_jx^j$ from $R[x]$ such that

$$f(x) = \sigma(f_1(x)) = \sum_{i=0}^{m} \sigma(a'_i)x^i \quad \text{and} \quad g(x) = \sigma(g_1(x)) = \sum_{j=0}^{m} \sigma(b'_j)x^j.$$

First, we shall show that $f(x)g(x) = 0$ implies $f_1(x)g_1(x) = 0$. If $f(x)g(x) = 0$, we have

$$a_0b_k + a_1(\sigma\alpha\sigma^{-1})(b_{k-1}) + \cdots + a_k(\sigma\alpha\sigma^{-1})^k(b_0) = 0,$$

for any $0 \leqslant k \leqslant m$. From the definition of $f_1(x)$ and $g_1(x)$, we have,

$$\sigma(a_0')\sigma(b_k') + \sigma(a_1')(\sigma\alpha\sigma^{-1})\sigma(b_{k-1}') + \cdots + \sigma(a_k')(\sigma\alpha\sigma^{-1})^k\sigma(b_0') = 0,$$

so that $(\sigma\alpha\sigma^{-1})^t = \sigma\alpha^t\sigma^{-1}$ we obtain

$$a_0'b_k' + a_1'\alpha(b_{k-1}') + \cdots + a_k'\alpha^k(b_0') = 0,$$

which means that $f_1(x)g_1(x) = 0$ in the ring $R[x;\alpha]$.

It remains to prove that $f_1(x)g_1(x) = 0$ implies $a_i(\sigma\alpha\sigma^{-1})^i(b_j) \in \mathrm{nil}(S)$. From the fact that R is weak α-skew Armendariz we have $a_i'\alpha^i(b_j') \in \mathrm{nil}(R)$, and since $a_i' = \sigma^{-1}(a_i), b_j' = \sigma^{-1}(b_j)$, we have $\sigma^{-1}(a_i)\alpha^i\sigma^{-1}(b_j) \in \mathrm{nil}(R)$. This implies

$$\sigma^{-1}(a_i)\sigma^{-1}\sigma\alpha^i\sigma^{-1}(b_j) = \sigma^{-1}(a_i(\sigma\alpha\sigma^{-1})^i(b_j)) \in \mathrm{nil}(R)$$

and finally we obtain

$$a_i(\sigma\alpha\sigma^{-1})^i(b_j) \in \mathrm{nil}(S),\ 0 \leqslant i, j \leqslant m.$$

Hence S is weak $\sigma\alpha\sigma^{-1}$-skew Armendariz. □

In our closing result, we shall show that, under certain condition, the subring of upper triangular skew matrices over a ring R has a weak skew Armendariz structure.

Let $E_{ij} = (e_{st} : 1 \leqslant s, t \leqslant n)$ denotes $n \times n$ unit matrices over ring R, in which $e_{ij} = 1$ and $e_{st} = 0$ when $s \neq i$ or $t \neq j$, $0 \leqslant i, j \leqslant n$, for all $n \geqslant 2$. If $V = \sum_{i=1}^{n-1} E_{i,i+1}$, then $V_n(R) = RI_n + RV + \cdots + RV^{n-1}$ is the subring of upper triangular skew matrices.

Corollary 4.1. *Suppose that α is an endomorphism of ring R. If the factor ring $R[x]/(x^n)$ is weak $\widetilde{\alpha}$-skew Armendariz, then $V_n(R)$ is weak $\widetilde{\alpha}$-skew Armendariz.*

Proof. Suppose that $R[x]/(x^n)$ is weak $\widetilde{\alpha}$-skew Armendariz and define the ring isomorphism $\theta : V_n(R) \to R[x]/(x^n)$ by

$$\theta(r_0I_n + r_1V + \cdots + r_{n-1}V^{n-1}) = r_0 + r_1x + \cdots + r_{n-1}x^{n-1} + (x^n).$$

Now we have that $V_n(R)$ is weak $\theta^{-1}\widetilde{\alpha}\theta$-skew Armendariz and

$$\begin{aligned}\theta^{-1}\widetilde{\alpha}\theta(r_0I_n + r_1V + \cdots + r_{n-1}V^{n-1})&\\ &= \theta^{-1}\widetilde{\alpha}(r_0 + r_1x + \cdots + r_{n-1}x^{n-1} + (x^n))\\ &= \theta^{-1}(\alpha(r_0) + \alpha(r_1)x + \cdots + \alpha(r_{n-1})x^{n-1} + (x^n))\\ &= \alpha(r_0)I_n + \alpha(r_1)V + \cdots + \alpha(r_{n-1})V^{n-1}\\ &= \widetilde{\alpha}(r_0I_n + r_1V + \cdots + r_{n-1}V^{n-1}),\end{aligned}$$

which means that $V_n(R)$ is a weak $\widetilde{\alpha}$-skew Armendariz ring. □

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