A NOTE ON QUASI-ANALYTIC FUNCTIONS

by

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Let $C\{M_n\}$ denote the class of functions $f(t)$ infinitely differentiable for $-\infty < t < \infty$ and such that

$$|f^{(n)}(t)| \leqq Ak^n M_n, \quad -\infty < t < \infty, \quad n=0,1,2,\cdots. \tag{1}$$

Here A and k are constants which may depend upon $f(t)$. The class $C\{M_n\}$ is said to be quasi-analytic if and only if for every function $f(t)$ of the class, the conditions

$$f^{(n)}(t_0)=0 \qquad n=0,1,2,\cdots \tag{2}$$

imply that $f(t)$ is identically zero. Let us define a new sequence of numbers M_n^c as follows:

$$M_n^c = \operatorname*{Max}_{r\geqq 0} \frac{r^n}{T(r)}, \quad T(r) = \operatorname*{Max}_{n\geqq 1} \frac{r^n}{M_n}. \tag{3}$$

Clearly

$$M_n^c \leqq M_n. \tag{4}$$

S. Mandelbrojt, [3], has proved that if

$$\lim_{n\to\infty} M_n^{1/n} = \infty, \tag{5}$$

then the class $C\{M_n\}$ is quasi-analytic if and only if

$$\sum_{n=0}^{\infty} \frac{M_n^c}{M_{n+1}^c} = \infty. \tag{6}$$

Compare also [1].

It is our purpose to give here a new proof of the necessity of this condition. In fact we shall prove the Theorem: *If the sequence* M_0, $M_1,\ldots$

satisfies (5), (6), *then the function*

$$f(t)=\frac{M_0^c}{\pi i}\int_{-i\infty}^{i\infty}\frac{e^{st}\,ds}{(s-1)^2\prod_{k=1}^{\infty}\left(1-\frac{s}{a_k}\right)}, \quad a_k=\frac{M_k^c}{M_{k-1}^c}, \tag{7}$$

belongs to the class $C\{M_n\}$, *satisfies* (1), *but is not identically zero.*

The function (7) has appeared in another connection in our work on convolution transforms [2]. By its use we shorten considerably the proof given by Bray and Mandelbrojt, [3] pp. 79—84. However, the function exhibited by them could also be given a form similar to (7) as follows:

$$g(t)=\frac{M_0^c}{\pi i}\int_{-i\infty}^{i\infty}\frac{\sinh^2 s}{s^2}\prod_{k=1}^{\infty}\frac{\sinh(s/a_k)}{s/a_k}\,e^{st}\,ds\,.$$

We turn now to the proof of the theorem. By assumption (6) we see that the infinite product appearing in the integrand (7) converges for all $s=\sigma+i\tau$ and represents an entire function with zeros at the points $a_1, a_2, \cdots$. We have seen in [2] that

$$\left[\prod_{k=1}^{\infty}\left(1-\frac{s}{a_k}\right)\right]^{-1}=O(|\tau|^{-p}), \quad |\tau|\to\infty \tag{8}$$

for any positive number p, uniformly in any finite interval of the σ-axis. This shows ($\sigma=0$, $p=0$) that the integral (7) converges for all real t. Indeed

$$f^{(n)}(t)=\frac{M_0^c}{\pi i}\int_{-i\infty}^{i\infty}\frac{s^n\,e^{st}\,ds}{(1-s)^2\prod_{k=1}^{\infty}\left(1-\frac{s}{a_k}\right)} \qquad n=0,1,2,\cdots.$$

Differentiation under the sign is justified by (8) with $p=n$. Hence

$$|f^{(n)}(t)|\leq\frac{M_0^c}{\pi}\int_{-\infty}^{\infty}\frac{|\tau|^n\,d\tau}{(1+\tau^2)\prod_{k=1}^{\infty}\left(1+\frac{\tau^2}{a_k^2}\right)}\,.$$

Now replace $(1+\tau/a_k)^2$ by $(\tau/a_k)^2$ or by 1 according as $k\leq n$ or $k>n$.

Thus

$$|f^{(n)}(t)| \leqq M_0^c a_1 a_2 \cdots a_n \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\tau}{1+\tau^2} = M_n^c.$$

By (4) we have consequently verified (1) with $A = k = 1$, so that

$$f(t) \in C\{M_n\}.$$

By Cauchy's theorem, using (8) with $p=0$, we may shift the path of integration in (7) as follows:

$$f(t) = \frac{M_0^c}{\pi i} \int_{-R-i\infty}^{-R+i\infty} \frac{e^{st}\, ds}{(s-1)^2 \prod_{k=1}^{\infty}\left(1 - \frac{s}{a_k}\right)}.$$

Here we may choose R as any positive number since the integrand has no singularities for $\tau < 0$. Since the absolute value of the infinite product is certainly less than one on the line $\tau = -R$, we have

$$|f(t)| \leqq \frac{M_0^c e^{-Rt}}{\pi} \int_{-\infty}^{\infty} \frac{d\tau}{(R+1)^2 + \tau^2} \leqq M_0^c e^{-Rt}.$$

Since R may be arbitrarily large, $f(t)$ must be zero for all positive t. Hence (2) is satisfied for any $t_0 > 0$.

Finally, to see that $f(t) \not\equiv 0$ we appeal to the uniqueness theorem for Fourier transforms. By its definition, $f(t)$ is the Fourier transform of a function belonging to $L(-\infty, \infty)$ which is not zero anywhere. This completes the proof of the theorem.

It can be shown in addition that $f(t)$ is analytic in the half-plane where the real part of t is less than zero. Thus $f(t)$ has the property that

$$\overline{\lim_{n \to \infty}} |f^{(n)}(t)|^{1/n} < \infty, \quad -\infty < t < \infty.$$

Needless to say, there exists no uniform bound. Many special properties of $f(t)$ are described in [2] and [4].

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