ON A TAUBERIAN THEOREM FOR EULER SUMMABILITY

by

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Let Σa_n be an infinite series. Put

$$a_n' = \frac{1}{2^{n+1}} \left(\sum_{k=0}^{n} \binom{n}{k} a_k \right).$$

(1)

$\Sigma a_n'$ is said to be the Euler sum of Σa_n.[1] It is easy to see that $\Sigma a_n'$ converges if Σa_n converges, but the converse is not true. Euler summability was first studied by Knopp.[2]

W. Meyer-König proved[3] that if Σa_n is Euler summable and $a_n = 0$ except if $n = n_i$, $n_{i+1}/n_i > c > 1$, then Σa_n is convergent. He also conjectured that the conclusion of the theorem would follow from the following weaker condition: $a_n = 0$ except if $n = n_i$, where $n_{i+1} - n_i > c n_i^{1/2}$, $c > 0$ any constant. In fact he proved[4] this conjecture under the further assumption that $|a_n| < n^\alpha$ where α is any constant. It is easy to see that this conjecture if true is best possible i. e. if $f(n)$ tends to infinity arbitrarily slowly there exists a series $\Sigma \omega_n$ which is Euler summable but not convergent and for which $a_n = 0$ except if $n = n_i$, $n_{i+1} - n_i > n_i^{1/2} / f(n_i)$.

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[1] (1) gives a series to series transformation method. The corresponding sequence to sequence method would be

$$s_n' = \frac{1}{2^{n+1}} \left(\sum_{k=0}^{n} \binom{n}{k} s_k \right).$$

The two methods are equivalent, but for our present purpose the series to series transformation seems to be more suitable.

[2] Math. Zeitschrift 15 (1922), p. 226—253 and 18 (1923) p. 125—156.

[3] Math. Zeitschrift 49 (1943), p. 151—160.

[4] Math. Zeitschrift 45 (1939), p. 479 - 494.

In the present note we are going to prove the following

Theorem. *There exists a constant $A > 0$ so that if Σa_n is a series which is Euler summable, and for which $a_n = 0$ except if $n = n_i$*

$$n_{i+1} - n_i > A n_i^{1/2}, \tag{2}$$

then Σa_n is convergent.

At present I am unable to decide whether A can be any constant greater than 0, in other words I am unable to prove Meyer-König's conjecture.

Let $\binom{n}{m} a_m / 2^{n+1}$ be the summand of greatest absolute value in (1). If there are several such terms we consider the one with the greatest index m. Put $m = f(n)$, $F(n) = \left| \binom{n}{m} a_m / 2^{n+1} \right|$.

Lemma 1. *$f(n)$ is a non-decreasing function of n.*

To prove lemma 1 it will clearly suffice to show that if

$$\left| \binom{n}{m} a_m \right| \geqslant \left| \binom{n}{l} a_l \right| \quad \text{for} \quad m > l, \quad \text{then} \quad \left| \binom{n+1}{m} a_m \right| > \left| \binom{n+1}{l} a_l \right|.$$

This is true since

i. e.
$$\left| \binom{n+1}{m} a_m \cdot \left(\binom{n}{m} a_m \right)^{-1} \right| > \left| \binom{n+1}{l} a_l \cdot \left(\binom{n}{l} a_l \right)^{-1} \right|$$

$$\frac{n+1}{n-m+1} > \frac{n+1}{n-l+1} \quad \text{for} \quad m > l.$$

Lemma 2. *Assume that $f(n) > n/2$. Then $F(n+1) \geqslant F(n)$.*

Put $f(n) = m > n/2$. We obtain from $F(n+1) \geqslant \dfrac{1}{2^{n+2}} \left| \binom{n+1}{m} a_m \right|$

$$F(n+1)/F(n) \geqslant \binom{n+1}{m} \bigg/ 2 \binom{n}{m} = \frac{n+1}{2(n-m+1)} \geqslant 1,$$

which proves the lemma.

Lemma 3. Let α be arbitrary. Assume that $|a_n| < n^\alpha$ for all n, and $a_n = 0$ except if $n = n_i$, $n_{i+1} - n_i > c n_i^{1/2}$, where $c > 0$ is an arbitrary positive constant. Then if Σa_n is Euler summable it is convergent.

This is a theorem of Meyer-König.[4]

Because of lemma 3 we can now assume that, for infinitely many n, $|a_n| > n$. We shall show that if an infinite series satisfies (2) and

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[4] Math. Zeitschrift 45 (1939), p. 479—494.

$|a_n| > n$ for infinitely many n then it cannot be Euler summable. This together with lemma 3 will complete the proof of our theorem. First we prove

Lemma 4. *Let $c_1 > 0$ be suitable constant. Then there exist infinitely many integers n satisfying*

$$n/2 \leqslant f(n) = f(n+1) = \cdots = f(n+t), \qquad t \geqslant \frac{A}{3} n^{1/2} \qquad (3)$$

and

$$F(n) > c_1 n^{1/2}, \quad F(n+1) > c_1 n^{1/2}, \ldots, F(n+t) > c_1 n^{1/2}. \qquad (4)$$

First of all it is easy to see that there exist infinitely many integers n_i satisfying

$$|a_{n_i}| > n_i, \quad |a_k| < |a_{n_i}| \quad \text{for } 1 \leqslant k < n_i. \qquad (5)$$

To prove (5) it suffices to choose $|a_n| > n$ and define a_{n_i} as the a_k of largest absolute value for $1 \leqslant k \leqslant n$.

Put

$$a'_{2n_i} = \frac{1}{2^{2n_i+1}} \sum_{k=0}^{2n_i} \binom{2n_i}{k} a_k.$$

By the second inequality of (5) we have $f(2n_i) \geqslant n_i$, and by the first inequality of (5) for sufficiently large n_i

$$F(2n_i) \geqslant \left| \binom{2n_i}{n_i} a_{n_i} / 2^{2n_i+1} \right| > n_i \binom{2n_i}{n_i} / 2^{2n_i+1} > c_2 n_i^{1/2}. \qquad (6)$$

Assume first that for infinitely many n satisfying (5) we have $f(2n_i) = n_i$. From lemma 1 we have for $x \geqslant 0$

$$f(2n_i - x) \leqslant f(2n_i) = n_i. \qquad (7)$$

Further, a simple argument shows that for $t \leqslant n_i - n_{i-1}$ and $j \geqslant 1$

$$\binom{2n_i - t}{n_i} \geqslant \binom{2n_i - t}{n_{i-j}}. \qquad (8)$$

Therefore by the second inequality of (5)

$$\left| \binom{2n_i - t}{n_i} a_{n_i} \right| > \left| \binom{2n_i - t}{n_{i-j}} a_{n_{i-j}} \right|. \qquad (9)$$

(7) and (9) imply that for $0 \leqslant t \leqslant n_i - n_{i-1}$

$$f(2n_i - t) = f(2n_i) = n_i. \qquad (10)$$

A simple computation gives that for $t < A n_i^{1/2}$

$$\binom{2n_i - t}{n_i} > c_3 \binom{2n_i}{n_i} / 2^t . \tag{11}$$

Therefore from (2)[5] (6) and (11) we have for $t < A n_i^{1/2}$

$$F(2n_i - t) \geqslant \frac{1}{2^{2n_i - t + 1}} \binom{2n_i - t}{n_i} |a_{n_i}| > c_3 / 2^{2n_i + 1} \binom{2n_i}{n_i} |a_{n_i}| >$$

$$> c_2 c_3 n_i^{1/2} > 1 \tag{12}$$

(10) and (12) prove our lemma.

Assume next that for all sufficiently large n_i satisfying (5) we have $f(2n_i) > n_i$. Put $n_i = n_{i_0}$, $f(2n_{i_0}) = n_{i_1}$, $f(2n_{i_1}) = n_{i_2} \ldots$ There thus exists an infinite sequence $n_{i_0}, n_{i_1}, \ldots$ satisfying

$$n_{i_0} < n_{i_1} < \ldots, \quad 2n_{i_r} \geqslant n_{i_{r+1}}, \quad f(2n_{i_r}) = n_{i_{r+1}}. \tag{13}$$

To simplify the notation we shall write n_r instead of n_{i_r} whenever there is no danger of confusion. First of all we show that all the n_r satisfy (5). We use induction. By assumption n_0 satisfies (5). Assume that n_r satisfied (5). A simple computation gives for sufficiently large A

$$\binom{2n_r}{n_{r+1}} < \binom{2n_r}{n_r + A[n_r]^{1/2}} < \frac{1}{2} \binom{2n_r}{n_r} .$$

Thus

$$\left| \binom{2n_r}{n_{r+1}} a_{n_{r+1}} \right| \geqslant \left| \binom{2n_r}{n_r} a_{n_r} \right|$$

implies

$$|a_{n_{r+1}}| > 2 |a_{n_r}| > 2 n_r \geqslant n_{r+1}$$

which is the first inequality of (5). Further since the binomial coefficients $\binom{2n_r}{n_r + l}$ decrease as l increases, it follows from $f(2n_r) = n_{r+1}$ that

$$|a_{n_{r+1}}| > |a_n| \quad \text{for} \quad n_r \leqslant n < n_{r+1} .$$

But then since n_r satisfied the first inequality of (5) it clearly follows that n_{r+1} also satisfied it, which completes our proof.

Next we prove that for all $n \geqslant 2n_0$

$$F(n) > c_4 n^{1/2} . \tag{14}$$

From (13) and lemma 1 it follows that for $n \geqslant 2 n_0^*$, $f(n) > n/2$. Hence we have from lemma 2 that for $n \geqslant 2 n_0$ $F(n)$ is an increasing function of n. Let

$$2 n_r \leqslant n < 2 n_{r+1} \leqslant 4 n_r.$$

Since n_r satisfies (5) we have

$$F(n) \geqslant F(2 n_r) \geqslant \left| \binom{2 n_r}{n_r} a_{n_r} \right| > c_5 n_r^{1/2} > c_7 n^{1/2} \quad \text{q. e. d.}$$

Consider now the interval $2 n_{i_0} \leqslant n \leqslant 4 n_{i_0}$. Clearly $n_{i_0} < f(n) \leqslant 4 n_{i_0}$. Also $f(n)$ must be one of the $n_j's$. But by (2) the difference of two consecutive $n_j's$ is greater than $A n_{i_0}^{1/2}$, $(n_j > n_{i_0})$. Thus the number of $n_j's$ in the interval $(n_{i_0}, 4 n_{i_0})$ is less than $3 n_{i_0}^{1/2}/A$. Hence there must be at least

$$2 n_{i_0} | (3 n_{i_0}/A) | = \frac{2 A}{3} n_{i_0}^{1/2}$$

integers in the interval $(n_{i_0}, 4 n_{i_0})$ with the same $f(n)$ and by Lemma 1 they must be consecutive integers say $n, n+1, \ldots n+t$ $t > A/3 \, n^{1/2}$. Thus (14) completes the proof of Lemma 4.

Now we can prove our theorem. Let n satisfy lemma 4 and choose

$$t = \left[\frac{A}{3} n^{1/2} \right] + 1. \quad \text{Put} \quad \left[\frac{2n+t}{2} \right] = M. \quad \text{We have} \quad a'_M = \frac{1}{2^{M+1}} \sum_{k=0}^{M} \binom{M}{k} a_k.$$

We shall show that $|a'_M| > c_6 M^{1/2}$ where c_6 is an absolute constant independent of n. This will of course show that $\Sigma a'_n$ can not converge, hence Σa_n was not Euler summable and the proof of our theorem will be complete.

Put $f(M) = n_j$. We have by (4)

$$F(M) = \frac{1}{2^{M+1}} \left| \binom{M}{n_j} a_{n_j} \right| > c_1 n^{1/2} > c_1/2 \, M^{1/2}. \tag{15}$$

We have

$$|a'_M| \geqslant \frac{1}{2^{M+1}} \left[\binom{M}{n_j} | a_{n_j} | - \sum_{n_r > n_j} \binom{M}{n_r} | a_{n_r} | - \sum_{n_r < n_j} \binom{M}{n_r} | a_{n_r} | \right] = $$
$$= \frac{1}{2^{M+1}} \left[\binom{M}{n_j} | a_{n_j} | - \Sigma_1 - \Sigma_2 \right]. \tag{16}$$

For an estimate of Σ_1 put $r - j = k$, then $n_r - n_j > A k n_j^{1/2}$. Put

$$n+t = M+x, \quad \frac{A}{6} n^{1/2} \leqslant x \leqslant \frac{A}{6} n^{1/2} + 1.$$

We have by

$$f(n) = f(n+t) = n_j \leqslant x > \frac{A}{12} M^{1/2}$$

$$\left| \binom{M+x}{n_r} a_{n_r} \right| \leqslant \left| \binom{M+x}{n_j} a_{n_j} \right|$$

Hence

$$\left| \binom{M}{n_r} a_{n_r} \right| \leqslant \left| \binom{M}{n_j} a_{n_j} \right| \frac{M-n_r+1}{M-n_j+1} \cdot \frac{M-n_r+2}{M-n_j+2} \cdots \frac{M-n_r+x}{M-n_j+x} <$$

$$< \left| \binom{M}{n_j} a_{n_j} \right| \left(1 - \frac{kA\, n_j^{1/2}}{M} \right)^x < \left| \binom{M}{n_j} a_{n_j} \right| \left(1 - \frac{kA\, (M/2)^{1/2}}{M} \right)^{\frac{A}{12} M^{1/2}} <$$

$$< \left| \binom{M}{n_j} a_{n_j} \right| \left(1 - \frac{kA}{2 M^{1/2}} \right)^{\frac{A}{12} M^{1/2}} < \left| \binom{M}{n_j} a_{n_j} \right| e^{-k \frac{A^2}{24}}$$

since from $f(M) = n_j$, $n_j \geqslant M/2$ (lemma 4). Thus for sufficiently large A [5)]

$$\Sigma_1 < \left| \binom{M}{n_j} a_{n_j} \right| \sum_{k=1}^{\infty} e^{-k \frac{A^2}{24}} < \frac{1}{4} \left| \binom{M}{n_j} a_{n_j} \right|. \qquad (17)$$

In the same way we can show

$$\Sigma_2 < \frac{1}{4} \left| \binom{M}{n_j} a_{n_j} \right|. \qquad (18)$$

Thus by (15), (16), (17) and (18)

$$\left| a'_M \right| > \frac{1}{2} \left| \binom{M}{n_j} a_{n_j} \right| / 2^{M+1} = \frac{1}{2} F(M) > \frac{c_1}{2} M^{1/2}$$

which completes the proof of the theorem.

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