NOTE ON THE LOCATION OF ZEROS OF EXTREMAL POLYNOMIALS IN THE NON-EUCLIDEAN PLANE

by

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The non-euclidean (hyperbolic) plane H: $|z| < 1$ is a subregion of the plane of the complex variable z. In H we define a NE (i. e., non-euclidean) *polynomial* as a function of the form

$$\lambda \prod_{1}^{n} \frac{z - \alpha_k}{1 - \bar{\alpha}_k z}, \qquad |\lambda| = 1, \quad |\alpha_k| < 1, \tag{1}$$

and define the degree of (1) as n. The function (1) determines an n- to-one map of the NE plane H onto itself, has precisely n zeros in H, and has the modulus unity on C: $|z| = 1$. The object of the present note is to prove

Theorem 1. *Let the closed set E lie interior to some circle $|z| < r (< 1)$. Let n be given. Then there exists at least one extremal polynomial* (1), *namely a NE polynomial* (1) *whose maximum modulus on E is not greater than the maximum modulus on E of any other NE polynomial* (1). *If E contains at least n distinct points, all zeros of this extremal polynomial lie in the smallest NE convex set K containing E.*

A NE line is defined as the arc in H of a euclidean circle orthogonal to C, and a NE half-plane is defined as either of the two subregions into which a NE line separates H. A NE half-plane is *convex* in the sense that if it contains two points it also contains the unique NE line segment joining them. The set K may be defined as the totality of points common to all NE half-planes containing E, and is obviously convex and closed.

If z_1 and z_2 are points of H, their *pseudo-distance* is defined as $\frac{|z_1 - z_2|}{|1 - \bar{z}_1 z_2|}$, which is less than unity. As is well known and immediately verifiable, pseudo-distance is invariant under the most general analytic transformation $w = \frac{\mu(z-\alpha)}{(1-\bar{\alpha}z)}$, $|\mu| = 1$, $|\alpha| < 1$, which transforms H into itself in a one-to-one manner. It is to be noted that the modulus of (1) is precisely the product of the pseudo-distances from z to the n points α_k.

If the set E contains precisely n distinct points, it is clear that (1) can be chosen to vanish in each point of E; the extremal polynomial (1) is uniquely determined except for the factor λ. If E contains fewer than n distinct points, the function (1) can still be chosen to vanish in each point of E, but the zeros of (1) are no longer uniquely determined and need not all lie in K. These trivial cases are henceforth excluded.

Our chief tool in the proof of Theorem 1 is the

Lemma. *Let the point P lie exterior to the NE half-plane H_0. Then motion of P toward H_0 along the NE line through P orthogonal to the boundary of H_0 decreases the pseudo-distance from P to each point of H_0.*

Let us choose P as the center O of C, an choose H_0 as a NE half-plane to the left of P and symmetric in the axis of reals. By virtue of the invariance of pseudo-distance under transformation of H, the pseudo-distances of the Lemma may be studied either by moving P to the left along the axis of reals and keeping the points of H_0 fixed, or by keeping P fixed at O and suitably moving the points of H_0; with the latter interpretation, each point z of H_0 may be moved toward the right along the circle through $-1, z$, and $+1$. If z remains in the left half of H, the latter motion clearly decreases the euclidean distance from z to O, which in this case is equal to the pseudo-distance; the Lemma is established.

It is a consequence of the Lemma that if E containing more than n points is given, together with a polynomial (1) not all of whose zeros lie in K, then a new polynomial of the same degree can be found by moving toward K (in the NE sense) those zeros of (1) which do not lie in K; the modulus of this new polynomial is in each point of E *less* than the modulus (if not zero) of the original polynomial; consequently, the maximum modulus on E of the new polynomial is less than the maximum modulus on E of the original polynomial.

The first part of Theorem 1 now follows in any non-trivial case by the use of normal families of functions; in studying for fixed n a sequence of functions of type (1) whose respective maximum moduli on E approach the least upper bound of all such maximum moduli, it is sufficient to consider only functions (1) whose zeros lie in K; all such functions are uniformly bounded not merely in H but in a region of the z- plane containing $C+H$ in its interior;[1] any limit function of the sequence is of modulus unity on C and has precisely n zeros in H, hence is of type (1) and is extremal. The latter part of Theorem 1 is likewise a consequence of the Lemma.

We recall the classical theorem of Lucas, that the smallest euclidean convex set containing the zeros of a (euclidean) polynomial in z also contains the zeros of the derived polynomial. Fejér proved the related result[2] that if a closed bounded set E of the euclidean plane is given, and if K is the smallest euclidean convex set containing E, then (in any non-trivial case) K contains all zeros of any extremal (euclidean) polynomial of given degree n, namely a polynomial of the form

$$\prod_{1}^{n} (z-\alpha_k) \tag{2}$$

whose maximum modulus on E is not greater than the maximum modulus on E of any other polynomial (2).

Lucas's Theorem has a precise NE analogue, namely that the smallest NE convex set in H containing the zeros of the NE polynomial (1) also contains all zeros in H of the derivative of (1)[3]. The latter part of Theorem 1 bears a relation to this result similar to the relation that Fejér's Theorem bears to Lucas's Theorem. The proof of the latter part of Theorem 1 is the precise NE analogue of Fejér's proof of his Theorem.

Theorem 1 depends only on the property expressed in the Lemma, so Theorem 1 clearly admits of extensions: (i) to functions of form (1) where the norm is not the maximum modulus of (1) on E but is the line

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[1] For the algebraic details here, the reader may compare Walsh, Interpolation and Approximation (New York 1935), p. 229.

[2] L. Fejér, Mathematische Annalen, vol. 85 (1922), pp. 41-48; compare also M. Fekete and J. L. v. Neumann, Jahresbericht d. d. Math-Vereinigung. vol. 31 (1922), pp. 125-138.

[3] See Walsh, The Location of Critical Points (New York 1950), p. 157.

or surface integral over E of the p-th power $(p>0)$ of the modulus of (1) with norm function positive and continuous on E; (ii) to functions no longer of form (1) but of form

$$\lambda \prod_{1}^{n} \Phi\left(\frac{z-\alpha_k}{1-\bar{\alpha}_k z}\right), \quad |\lambda|=1, \quad |\alpha_k|<1,$$

where $|\Phi(w)|$ increases strongly monotonically with $|w|$, for $|w|<1$.