

ON THE GIBBS PHENOMENON FOR A CLASS
OF LINEAR TRANSFORMS¹⁾

by
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I. With a given series $\sum_1^{\infty} a_n$ and a function $\varphi(t)$ we associate the transform

$$\varphi(h) = \sum_1^{\infty} a_n \varphi(nh). \quad (1.1)$$

Let in particular

$$a_n = \frac{\sin nt}{n}, \text{ so that } \sum a_n = \sum \frac{\sin nt}{n} = \frac{1}{2}(\pi - t), \quad 0 < t \leq \pi,$$

and

$$\varphi(h) = \sum_1^{\infty} \frac{\varphi(nh) \sin nt}{n} = T(h, t), \text{ say.} \quad (1.2)$$

The Gibbs ratio corresponding to the transform T is defined by

$$G = \limsup_{\substack{h \rightarrow 0 \\ t \rightarrow 0}} \frac{2}{\pi} T(h, t). \quad (1.3)$$

We assume that the transform (1.1) is regular; it is known that this is the case if and only if

$$\lim_{t \rightarrow +0} \varphi(t) = 1,$$

and

$$\sum_1^{\infty} |\varphi(vh) - \varphi(v+1)h| < M, \quad (1.4)$$

where M is independent of h .

It is clear that in this case $G \geq 1$; we say that the transform (1.2) presents a Gibbs phenomenon when $G > 1$. The condition (1.4) is equi-

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valent to saying that the function $\varphi(t)$ is of bounded variation over the interval $(0, \infty)$, that is

$$\int_0^\infty |d\varphi(t)| < \infty. \quad (1.5)$$

It follows that

$$\lim_{t \rightarrow \infty} \varphi(t) = \chi \text{ exists and is finite.}$$

We assume $\chi = 1$.

The main result of the present paper is:

$$G \equiv \max_{0 < \zeta < \infty} g(\zeta) \equiv \max \frac{2}{\pi} \int_0^\infty \varphi(t) \frac{\sin \zeta t}{t} dt;$$

moreover

$$\frac{2}{\pi} T(h, t) \rightarrow g(\zeta) \text{ as } h^{-1}t \rightarrow \zeta, \quad 0 < \zeta < \infty,$$

$$\frac{2}{\pi} T(h, t) \rightarrow 1, \text{ when } t \rightarrow 0, \quad h \rightarrow 0 \text{ and } h^{-1}t \rightarrow \infty.$$

We then apply this result to some particular transforms. An example of a non-regular transform will be discussed in § 6.

2. Let

$$\rho_n(t) \equiv \sum_{v=n}^{\infty} \frac{\sin vt}{v}, \quad n = 1, 2, 3, \dots;$$

clearly, for $0 < t \leq \pi$

$$\rho_n = \frac{1}{2}(\pi - t) - \sum_1^{n-1} \frac{\sin vt}{v} = \frac{\pi}{2} - \int_0^t \frac{\sin \left(n - \frac{1}{2}\right)x}{2 \sin \frac{x}{2}} dx. \quad (2.1)$$

From (1.2), summing by parts

$$T(h, t) = \rho_1 \varphi(h) - \sum_2^{\infty} \rho_n \{\varphi(\overline{n-1}h) - \varphi(nh)\}; \quad (2.2)$$

using now (2.1)

$$\begin{aligned} T(h, t) &= \frac{1}{2}(\pi - t)\varphi(h) - \frac{\pi}{2}\{\varphi(h) - \chi\} + \\ &+ \sum_2^{\infty} \{\varphi(\overline{n-1}h) - \varphi(nh)\} \int_0^t \frac{\sin \left(n - \frac{1}{2}\right)x}{2 \sin \frac{x}{2}} dx, \end{aligned}$$

so that, for $0 < t \leq \pi$

$$\begin{aligned} T(h, t) = & \frac{\pi}{2} x - \frac{t}{2} \varphi(h) + \\ & + \sum_{n=2}^{\infty} \{ \varphi(n-1)h - \varphi(nh) \} \int_0^t \frac{\sin\left(n-\frac{1}{2}\right)x}{2 \sin \frac{x}{2}} dx. \end{aligned} \quad (2.3)$$

Observe that

$$|\rho_n(t)| < R, \quad R \text{ independent of } n \text{ and } t. \quad (2.4)$$

From (2.1) for $t=\pi$

$$\int_0^\pi \frac{\sin\left(n-\frac{1}{2}\right)x}{2 \sin \frac{x}{2}} dx = \frac{\pi}{2},$$

hence

$$\rho_n = \int_t^\pi \frac{\sin\left(n-\frac{1}{2}\right)x}{2 \sin \frac{x}{2}} dx,$$

and

$$\begin{aligned} T(h, t) = & \frac{1}{2} (\pi - t) \varphi(h) - \\ & - \sum_{n=2}^{\infty} \{ \varphi(n-1)h - \varphi(nh) \} \int_t^\pi \frac{\sin\left(n-\frac{1}{2}\right)x}{2 \sin \frac{x}{2}} dx, \end{aligned} \quad (2.5)$$

for $0 < t \leq \pi$.

Summation by parts yields for $0 < t < \pi$

$$\rho_n(t) = -\frac{\tau_{n-1}}{n} + \sum_{v=1}^{\infty} \frac{\tau_v}{v(v+1)},$$

where

$$\tau_n(t) = \sum_1^n \sin vt = \frac{1}{2} \sin nt + \cot \frac{t}{2} \sin^2 \frac{nt}{2}.$$

It now follows easily that

$$|\rho_n(t)| < \frac{1}{n} \left(1 + \frac{1}{\sin \frac{t}{2}} \right). \quad (2.6)$$

3. We now prove two lemmas.

Lemma 1. If $h \rightarrow 0$ and $t = t(h) \rightarrow 0$ in such a manner that

then $h^{-1} t(h) \rightarrow \zeta < \infty$,

$$T(h, t) - T(h, h\zeta) \rightarrow 0 \text{ as } h \rightarrow 0, \quad 0 < \zeta < \infty, \quad (3.1)$$

$$T(h, t) \rightarrow \frac{\pi}{2} \varphi(\infty), \text{ for } t = o(h), \quad h \rightarrow 0. \quad (3.2)$$

Lemma 2. If $h^{-1} t(h) \rightarrow +\infty$, then

$$T(h, t) \rightarrow \frac{\pi}{2}, \text{ as } h \rightarrow 0 \text{ and } t(h) \rightarrow 0. \quad (3.3)$$

We write

$$\varphi(v-1)h - \varphi(vh) = \Delta_v(h),$$

so that, employing (2.3), for $0 < h\zeta \leq \pi$

$$T(h, t) - T(h, h\zeta) = \frac{1}{2} (h\zeta - t) \varphi(h) + \sum_{n=2}^{\infty} \Delta_n(h) \int_t^{h\zeta} \frac{\sin\left(n - \frac{1}{2}\right)x}{2 \sin \frac{x}{2}} dx.$$

By the assumption of Lemma 1,

$$t = h(\zeta + \varepsilon), \text{ where } \varepsilon = \varepsilon(h) \rightarrow 0,$$

so that

$$t - h\zeta = h\varepsilon = o(h),$$

and, as $\zeta > 0$

$$\int_t^{h\zeta} \frac{\sin\left(n - \frac{1}{2}\right)x}{2 \sin \frac{x}{2}} dx = o\left(\frac{|h\varepsilon|}{h\zeta}\right) = o(|\varepsilon|).$$

It now follows, if $\zeta > 0$, that

$$T(h, t) - T(h, h\zeta) = o(h) + o\left(|\varepsilon| \sum_{n=2}^{\infty} |\Delta_n(h)|\right) = o(1), \text{ as } h \rightarrow 0.$$

In the case $\zeta = 0$ we have, from (1.2), $T(h, 0) = 0$, and $t = h\varepsilon$, $\varepsilon > 0$. Now write (2.3) in the form

$$\begin{aligned} T(h, t) &= \frac{\pi}{2} x - \frac{t}{2} \varphi(h) + \\ &+ \sum_{n=2}^k \Delta_n(h) \int_0^t \frac{\sin\left(n - \frac{1}{2}\right)x}{2 \sin \frac{x}{2}} dx + \sum_{n=k+1}^{\infty} \Delta_n(h) \int_0^t \frac{\sin\left(n - \frac{1}{2}\right)x}{2 \sin \frac{x}{2}} dx, \end{aligned}$$

where $k = [h^{-1} \varepsilon^{-\frac{1}{2}}]$. Then

$$\left| \sum_{n=2}^k \Delta_n(h) \int_0^t \frac{\sin\left(n-\frac{1}{2}\right)x}{2 \sin \frac{x}{2}} dx \right| < t \sum_{n=2}^k n |\Delta_n(h)| < kt \sum_{n=2}^k |\Delta_n(h)| < \frac{\varepsilon h}{h \sqrt{\varepsilon}} = \sqrt{\varepsilon} = O(1), \quad h \rightarrow 0.$$

Next, using (2.4)

$$\begin{aligned} \left| \sum_{n=k+1}^{\infty} \Delta_n(h) \int_0^t \frac{\sin\left(n-\frac{1}{2}\right)x}{2 \sin \frac{x}{2}} dx \right| &< \left(\frac{\pi}{2} + R \right) \sum_{n=k+1}^{\infty} |\Delta_n(h)| = \\ &= O\left(\int_1^{\infty} |d\varphi(t)|\right) = O(1), \quad h \rightarrow 0. \end{aligned}$$

We thus find

$$\lim T(h, t) = \frac{\pi}{2} \leq \frac{\pi}{2}, \quad \text{as } h \rightarrow 0 \text{ and } \frac{t}{h} \rightarrow 0.$$

Proof of Lemma 2. By assumption

$$t = h\omega(h), \quad \text{where } \omega \rightarrow \infty.$$

We get from (2.2) the formula

$$\begin{aligned} T(h, t) &= \frac{1}{2} (\pi - t) \varphi(h) - \sum_{n=2}^{\infty} \rho_n(t) \Delta_n(h) = \\ &= \frac{1}{2} (\pi - t) \varphi(h) - \left(\sum_{nt \leq \omega^{\frac{1}{2}}} \frac{1}{2} + \sum_{nt > \omega^{\frac{1}{2}}} \frac{1}{2} \right) \rho_n(t) \Delta_n(h) = \\ &= \frac{1}{2} (\pi - t) \varphi(h) - S_1 - S_2, \quad \text{say,} \end{aligned}$$

Employing (2.4) yields

$$|S_1| < R \sum_{nh \leq \omega^{-\frac{1}{2}}} |\Delta_n(h)| = O\left(\int_0^{\omega^{-\frac{1}{2}}} |d\varphi(t)|\right) = O(1).$$

Employing (2.6) we get

$$S_2 = O\left(\sum_{nh > \omega^{-\frac{1}{2}}} \frac{1}{nh\omega} |\Delta_n(h)|\right) = O\left(\frac{1}{\omega^{\frac{1}{2}}} \sum |\Delta_n(h)|\right) = O\left(\frac{1}{\omega^{\frac{1}{2}}}\right) = o(1).$$

Letting $h \rightarrow 0$ yields Lemma 2.

We shall also use the lemma [1]¹⁾

Lemma 3. If $\varphi(+0) = \varphi(0) = 1$ and $\int_0^\infty |\varphi(t)| dt < \infty$, $0 < \xi < \infty$, then

$$T(h, h\xi) = \sum_n \frac{\varphi(nh)}{n} \sin nh\xi \rightarrow \int_0^\infty \varphi(t) \frac{\sin \xi t}{t} dt, \text{ as } h \rightarrow 0.$$

The three lemmas combined yield immediately our main result:

Theorem 1. If $\varphi(+0) = \varphi(0) = 1$, $\int_0^\infty |\varphi(t)| dt < \infty$, then

$$T(h, t) = \sum_n \frac{\varphi(nh)}{n} \sin nt \rightarrow \int_0^\infty \varphi(t) \frac{\sin \xi t}{t} dt, \text{ when } h^{-1}t \rightarrow \xi, 0 < \xi < \infty,$$

$$\Gamma(h, t) \rightarrow \frac{\pi}{2} \varphi(\infty) = \frac{\pi}{2} \chi, \text{ for } t = o(h)$$

$$T(h, t) \rightarrow \frac{\pi}{2}, \text{ when } h^{-1}t \rightarrow \infty, t \rightarrow 0, h \rightarrow 0.$$

4. We write

$$g(\xi) = \frac{2}{\pi} \int_0^\infty \varphi(t) \frac{\sin \xi t}{t} dt.$$

Theorem 1 and the definition (1.3) of G now yield the following theorem

Theorem 2. The Gibbs ratio G is determined by

$$G = \max_{0 < \xi < \infty} g(\xi) = \frac{2}{\pi} \max_{0 < \xi < \infty} \int_0^\infty \varphi(t) \frac{\sin \xi t}{t} dt.$$

A related result is due to Kuttner [5, § 2].

As a first example consider the Riesz means (R, n^λ, k) of a series Σa_v , defined by

$$\varphi(h) = \sum_{v h \leq 1} a_v (1 - (vh)^\lambda)^k, \quad h \rightarrow 0,$$

so that in (1.1)

$$\varphi(t) = \begin{cases} (1 - t^\lambda)^k & \text{for } 0 \leq t \leq 1 \\ 0 & \text{for } t \geq 1. \end{cases}$$

¹⁾ Numbers in brackets refer to the literature at the end of this paper.

Now

$$g(\zeta) = \frac{2}{\pi} \int_0^1 (1-t^\lambda)^k \frac{\sin \zeta t}{t} dt.$$

Kuttner [5] has shown that if $0 < \lambda < 2$, there is a function $r(\lambda)$ such that we have a Gibbs phenomenon if $k < r(\lambda)$, but not if $k \geq r(\lambda)$. The function $r(\lambda)$ is continuous and increasing, $r(+0)=0$, $r(2)=\infty$. If $\lambda=2$ the Gibbs phenomenon persists for all k .

In the case $\lambda=2$

$$g'(\zeta) = \frac{2}{\pi} \int_0^1 (1-t^2)^k \cos \zeta t dt = \frac{2^{k+\frac{1}{2}}}{\pi} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(k+1)}{\zeta^{k+\frac{1}{2}}} J_{k+\frac{1}{2}}(\zeta),$$

where $J_\alpha(\zeta)$ is the Bessel function of order α . Denote the positive zeros of $J_{k+\frac{1}{2}}(\zeta)$ in ascending order by $0 < \zeta_1 < \zeta_2 < \zeta_3 < \dots$, and use the notation

$$\Delta_\alpha(t) = \left(\frac{2}{t}\right)^\alpha \Gamma(\alpha+1) J_\alpha(t),$$

then

$$g'(\zeta) = \frac{1}{\pi} \Delta_{k+\frac{1}{2}}(\zeta) \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(k+1)}{\Gamma\left(k+\frac{3}{2}\right)},$$

so that

$$g(\zeta) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(k+1)}{\pi \Gamma\left(k+\frac{3}{2}\right)} \int_0^\zeta \Delta_{k+\frac{1}{2}}(t) dt.$$

Min-Teh Cheng [2] has shown that

$$\begin{aligned} G &= \max_{0 < \zeta \leq \infty} g(\zeta) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(k+1)}{\pi \Gamma\left(k+\frac{3}{2}\right)} \int_0^\infty \Delta_{k+\frac{1}{2}}(t) dt \\ &> \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(k+1)}{\pi \Gamma\left(k+\frac{3}{2}\right)} \int_0^\infty \Delta_{k+\frac{1}{2}}(t) dt = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(k+1)}{\pi \Gamma\left(k+\frac{3}{2}\right)} \cdot \frac{\sqrt{\pi} \Gamma\left(k+\frac{3}{2}\right)}{\Gamma(k+1)} = 1, \end{aligned}$$

5. Our next example is

$$\varphi(t) = e^{-t^\lambda}, \quad \lambda > 0,$$

so that

$$g(\zeta) = \frac{2}{\pi} \int_0^\infty e^{-t^\lambda} \frac{\sin \zeta t}{t} dt,$$

and

$$g'(\zeta) = \frac{2}{\pi} \int_0^\infty e^{-t^\lambda} \cos \zeta t dt.$$

It was shown by P. Lévy and B. Kuttner [4] that $g'(\zeta) > 0$ for $0 < \lambda < 2$ and for all ζ . It follows that $g(\zeta)$ is non-decreasing when $0 < \lambda \leq 2$; integration by parts yields

$$g(\zeta) = \int_0^\infty \left(1 + \lambda \frac{u^\lambda}{\zeta^2}\right) \frac{1 - \cos u}{u^2} \exp\left(-\frac{u^\lambda}{\zeta^2}\right) du.$$

Hence

$$g(\zeta) \uparrow \int_0^\infty \frac{1 - \cos u}{u^2} du = \frac{\pi}{2}, \quad \text{as } \zeta \uparrow \infty;$$

so that the transform

$$\sum_1^\infty \frac{\sin vt}{v} \exp(-v^\lambda h^\lambda), \quad h \rightarrow 0, \quad 0 < \lambda \leq 2$$

does not present a Gibbs phenomenon. On the other hand it was shown by Kuttner [5, § 3] that there is a Gibbs phenomenon when $\lambda > 2$. To find G we must consider the zeros of $g'(\zeta)$. It is known [6, vol. 2, pp. 257 and 69] that if $\lambda = 4, 6, 8, \dots$, then $g'(\zeta)$ has infinitely many real zeros and no imaginary zero. For any other $\lambda > 2$, $g'(\zeta)$ has only a finite number of real zeros.

6. We now consider the example

$$\varphi(t) = \frac{2}{\pi} \int_t^\infty \frac{\sin u}{u} du = \frac{2}{\pi} \operatorname{Si}(t);$$

so that

$$\varphi'(t) = -\frac{2}{\pi} \frac{\sin t}{t};$$

clearly the condition (1.5) is not satisfied, and the transform (1.1) is not

regular in this case. But (1.2) exists, in fact

$$T(h, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Si}(nh) \sin nt,$$

and, in view of $\operatorname{Si}(x)=O(1)$ the series is absolutely convergent for $h > 0$. Furthermore $\varphi(\infty)$ exists and is $=0$.

We now have

$$\operatorname{Si} nh = \int_{nh}^{\infty} \frac{\sin u}{u} du = \int_h^{\infty} \frac{\sin ny}{y} dy = \frac{\pi}{2} - \int_0^h \frac{\sin ny}{y} dy$$

so that

$$\begin{aligned} T(h, t) &= \frac{2}{\pi} \sum_{n=1}^{\infty} \int_h^{\infty} \frac{\sin ny}{ny} \sin nt dy = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin nt}{n} \left(\frac{\pi}{2} - \int_0^h \frac{\sin ny}{y} dy \right) = \\ &= \frac{1}{2} (\pi - t) - \frac{2}{\pi} \sum_{n=1}^{\infty} \int_0^h \frac{\sin nt \sin ny}{ny} dy = \\ &= \frac{1}{2} (\pi - t) - \frac{2}{\pi} \int_0^h \frac{1}{y} \sum_{n=1}^{\infty} \frac{\sin nt \sin ny}{ny} dy; \end{aligned}$$

termwise integration being permissible as in a similar case in Hardy and Rogosinski [3, p. 178]. Now

$$\sum_{n=1}^{\infty} \frac{\sin nt \sin ny}{n} = \frac{1}{2} \log \frac{\sin \frac{1}{2}(t+y)}{\sin \frac{1}{2}|t-y|} < 0 \text{ for } 0 < t < \pi, \quad 0 < y < \pi,$$

hence $T(h, t) \rightarrow \frac{1}{2}(\pi - t)$ as $h \rightarrow 0$, and $T(h, t) < \frac{1}{2}(\pi - t)$,

so that $T(h, t)$ presents no Gibbs phenomenon.

7. The function

$$\varphi(t) = \frac{te^{-t}}{1-e^{-t}}, \quad \varphi(0) = 1,$$

defines Lambert summability. In this case

$$\begin{aligned} g(\zeta) &= \frac{2}{\pi} \int_0^{\infty} \frac{te^{-t}}{1-e^{-t}} \frac{\sin \zeta t}{t} dt = \frac{2}{\pi} \int_0^{\infty} \sin \zeta t \sum_{n=1}^{\infty} e^{-nt} dt = \\ &= \frac{2}{\pi} \zeta \sum_{n=1}^{\infty} \frac{1}{n^2 + \zeta^2} = 2 \left\{ \frac{1}{1-e^{-2\pi\zeta}} - \frac{1}{2\pi\zeta} - \frac{1}{12} \right\}, \end{aligned}$$

hence

$$g\left(\frac{x}{2\pi}\right) = 2\left\{\frac{1}{1-e^{-x}} - \frac{1}{x} - \frac{1}{2}\right\} = \frac{(x-2)e^x + x+2}{x(e^x - 1)} = g_1(x), \text{ say.}$$

Now

$$g_1'(x) = 2\left\{\frac{1}{x^2} - \frac{e^{-x}}{(1-e^{-x})^2}\right\} > 0 \text{ for } x > 0,$$

hence $g(\xi)$ is monotone increasing, $g(\infty) = 1$, so that Lambert summability presents no Gibbs phenomenon.

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