AN ARITHMETICAL THEOREM CONCERNING LINEAR DIFFERENTIAL — DIFFERENCE EQUATIONS

by

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By combining analytic methods with ideas from the theory of numbers one is often led to theorems of mixed arithmetical and analytical character. I give here a new result of this type.

Theorem[1]. *Let all constants* $A_{\mu\nu}$, ω_ν *in the linear differential difference equation*

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} A_{\mu\nu}\, y^{(\mu)}(z+\omega_\nu)=0 \tag{1}$$

be algebraic, the $A_{\mu\nu}$ *not vanishing simultaneously and* $\omega_0, \omega_1, \ldots, \omega_n$ *being different. Let the* $n+1$ *equations*

$$\sum_{\mu=0}^{m} A_{\mu\nu}\, t^{\mu}=0 \qquad (\nu=0, 1, \ldots, n) \tag{2}$$

have at most $t=0$ *as a common root.*

Then there exists no integral transcendental function of exponential type

$$y(z)=\sum_{h=0}^{\infty} c_h \frac{z^h}{h!}, \quad c_h = O(q^h), \tag{3}$$

with algebraic coefficients $c_0, c_1, c_2, \ldots$, *satisfying the equation* (1).

Here q denotes an arbitrary positive number.

As a simple example take $y(z)=e^{\alpha z}=\sum_{h=0}^{\infty} \alpha^h \frac{z^h}{h!}$; this function satisfies the difference equation

$$e^{\alpha}\, y(z)-y(z+1)=0.$$

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[1]) This theorem was communicated without proof on September 1, 1950, at the International Congress of Mathematicians held in Cambridge (Mass.).

It follows from our theorem, that α and e^{α} cannot be algebraic simultaneously, except in the trivial case $\alpha = 0$.

The condition in this theorem, that the $n+1$ equations (2) must have no common root except perhaps $t=0$, is necessary. For let all other conditions of the theorem be fulfilled and let $\alpha \neq 0$ be a common root of the equations (2). Now all numbers $A_{\mu\nu}$ are algebraic and do not vanish simultaneously; hence α is algebraic also. If we take

$$y(z) = e^{\alpha z} = \sum_{h=0}^{\infty} \alpha^h \frac{z^h}{h!},$$

then

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} A^{\mu\nu} y^{(\mu)}(z+\omega_\nu) \equiv \sum_{\nu=0}^{n} \left(\sum_{\mu=0}^{m} A_{\mu\nu} \alpha^{\mu} \right) e^{\alpha(z+\omega_\nu)} \equiv 0.$$

Hence in this case there certainly exists an integral transcendental function of the form (3) with algebraic coefficients $c_0, c_1, c_2, \ldots$, satisfying the linear differential — difference equation (1).

To prove the theorem we need the following known results:

a) The Lindemann-Weierstrass theorem[2]: *Let* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *denote different algebraic numbers, let* $\beta_1, \beta_2, \ldots, \beta_n$ *denote arbitrary algebraic numbers. If*

$$\beta_1 e^{\alpha_1} + \beta_2 e^{\alpha_2} + \cdots + \beta_n e^{\alpha_n} = 0,$$

then necessarily

$$\beta_1 = \beta_2 = \cdots = \beta_n = 0.$$

b) The analogous but elementary theorem: *Let* $\rho_1, \rho_2, \ldots, \rho_n$ *denote different numbers, let* $P_1(z), P_2(z), \ldots, P_n(z)$ *denote arbitrary polynomials. If*

$$P_1(z) e^{\varrho_1 z} + P_2(z) e^{\varrho_2 z} + \cdots + P_n(z) e^{\varrho_n z} \equiv 0,$$

then necessarily

$$P_1(z) \equiv P_2(z) \equiv \cdots \equiv P_n(z) \equiv 0.$$

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[2] Lindemann, F., Ueber die Zahl π, Math. Ann. 20, 213—225 (1882).
Lindemann, F., Über die Ludolph'sche Zahl. S. B. preuss. Akad. Wiss. 1882, 679--682.
Weierstrass, K., Zu Lindemann's Abhandlung „Über die Ludolph'sche Zahl", Math. Werke II, 341—362.

c) A theorem essentially due to Schürer[3]): *Let the integral function*

$$y(z)=\sum_{h=0}^{\infty} c_h \frac{z^h}{h!}, \quad \limsup_{h\to\infty} \sqrt[h]{|c_h|} \leqq q,$$

satisfy a linear differential equation of infinite order

$$a_0 y(z)+a_1 y'(z)+a_2 y''(z)+\cdots=0,$$

with constant coefficients not vanishing simultaneously, and such that the characteristic function

$$A(t)=a_0+a_1 t+a_2 t^2+\cdots$$

is regular for $|t|\leqq q$.

If $A(t)$ *has no zeros in the circle* $|t|\leqq q$, *then necessarily* $y(z)\equiv 0$.

In all other cases there exists a polynomial $b_0+b_1 t+\cdots+b_k t^k$ *with zeros* (also with respect to their multiplicities) *identical with those of* $A(t)$ *in the circle* $|t|\leqq q$. *Then* $y(z)$ *satisfies the linear differential equation of finite order*

$$b_0 y(z)+b_1 y'(z)+\cdots+b_k y^{(k)}(z)=0.$$

First we deduce from these theorems the following

Lemma: *Let the integral function of exponential type*

$$y(z)=\sum_{h=0}^{\infty} c_h \frac{z^h}{h!} \not\equiv 0, \quad c_h=O(q^h),$$

with algebraic coefficients $c_0, c_1, c_2, \ldots$ *satisfy a linear differential equation of infinite order*

$$a_0 y(z)+a_1 y'(z)+a_2 y''(z)+\cdots=0$$

with constant coefficients not vanishing simultaneously and such, that the characteristic function

$$a_0+a_1 t+a_2 t^2+\cdots$$

is regular for $|t|\leqq q$.

Then $y(z)$ *can be written*

$$y(z)=\sum_{i=1}^{j} P_i(z)\, e^{Q_i z},$$

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[3]) Schürer, F., Eine gemeinsame Methode zur Behandlung gewisser Funktionalgleichungsprobleme. Leipziger Ber. 70, 185—240 (1918); Satz VI.

See also: Perron, O., Über Summengleichungen und Poincarésche Differenzengleichungen. Math. Ann. 84, 1—15 (1921); Satz I, and: Scheffer, I. M., Systems of differential equations of infinite order with constant coefficients. Ann. of Math. 30, 250—264 (1929); theorem 1, 2.

where $P_1(z),\ P_2(z),\ldots,P_j(z)$ *represent polynomials and where* $\rho_1,\ \rho_2,\ldots,\rho_j$ *are different algebraic numbers and also zeros of the characteristic function* $a_0+a_1t+a_2t^2+\cdots$.

Proof: The function $y(z)$ considered in this lemma satisfies all the conditions of Schürer's theorem c); moreover $y(z)\not\equiv 0$, hence the characteristic function $a_0+a_1t+a_2t^2+\cdots$ necessarily has zeros in the circle $|t|\leqq q$. Let $\rho_1,\ \rho_2,\ldots,\ \rho_s$ represent these zeros and let $\nu_1,\nu_2,\ldots,\nu_s$ denote their multiplicities. Let $b_0+b_1t+\cdots+b_kt^k$ be a polynomial with zeros $\rho_1,\rho_2,\ldots,\rho_s$ of multiplicities $\nu_1,\nu_2,\ldots,\nu_s$. Then by Schürer's theorem the function $y(z)$ satisfies the equation

$$b_0\,y(z)+b_1\,y'(z)+\cdots+b_k\,y^{(k)}(z)=0, \tag{4}$$

with coefficients $b_0,b_1,\ldots,b_k$ not vanishing simultaneously. Hence $y(z)$ can be written

$$y(z)=\sum_{\sigma=1}^{s}P_\sigma(z)\,e^{\varrho_\sigma z}, \tag{5}$$

where every $P_\sigma(z)$ represents a polynomial of degree $\nu_\sigma-1$ at most $(\sigma=1,2,\ldots,s)$.

Now we shall use the condition, that all coefficients $c_0,\ c_1,\ c_2$ of $y(z)$ are algebraic. In stead of (4) we may write

$$L[y(z)]\equiv 0, \tag{6}$$

if we introduce the linear differential operator

$$L=b_0+b_1,\ D+\cdots+b_kD^k.$$

The $k+1$ number $b_0,b_1,\ldots,b_k$ have a linear independent basis $\tau_1,\tau_2,\ldots,\tau_r$ with respect to the field of algebraic numbers; hence

$$b_\varkappa=b_{\varkappa 1},\tau_1+b_{\varkappa 2}\tau_2+\ldots+b_{\varkappa r}\tau_r \qquad (\varkappa=0,1,\ldots,k),$$

with algebraic $b_{\varkappa 1},b_{\varkappa 2},\ldots,b_{\varkappa r}$. It follows

$$L=\tau_1L_1+\tau_2L_2+\ldots+\tau_rL_r, \tag{7}$$

if we put

$$L_\rho=b_{0\,\rho}+b_{1\,\rho}D+\ldots+b_{k\,\rho}D^k \qquad (\rho=1,2,\ldots,r).$$

Now all coefficients in $y(z)=c_0+c_1\dfrac{z}{1!}+c_2\dfrac{z^2}{2!}+\ldots$ are algebraic; hence

$$L_\rho[y(z)]=c_{0\,\rho}+c_{1\,\rho}\frac{z}{1!}+c_{2\,\rho}\frac{z^2}{2!}+\ldots,$$

with algebraic coefficients $c_{0\rho}, c_{1\rho}, c_{2\rho}, \ldots$. From (6) and (7) it follows therefore

$$c_{01}\tau_1 + c_{02}\tau_2 + \cdots + c_{0r}\tau_r = 0,$$
$$c_{11}\tau_1 + c_{12}\tau_2 + \cdots + c_{1r}\tau_r = 0,$$
$$\cdots\cdots\cdots\cdots\cdots,$$

hence, $\tau_1, \tau_2, \ldots, \tau_r$ being linearly independent,

$$c_{01} = c_{02} = \cdots = c_{0r} = 0,$$
$$c_{11} = c_{12} = \cdots = c_{1r} = 0,$$
$$\cdots\cdots\cdots\cdots,$$

or

$$L_\rho[y(z)] \equiv 0 \; for \; \rho = 1, 2, \ldots, r.$$

Every linear differential operator $L_\rho = b_{0\rho} + b_{1\rho} D + \ldots + b_{k\rho} D^k$ has algebraic coefficients; these $r(k+1)$ coefficients $b_{\kappa\rho}$ do not vanish simultaneously on account of (7): hence $y(z)$ certainly satisfies a linear differential equation

$$\bar{b}_0 y(z) + \bar{b}_1 y'(z) + \cdots + \bar{b}_k y^{(k)}(z) = 0,$$

with *algebraic* coefficients $\bar{b}_0, \bar{b}_1, \ldots, \bar{b}_k$, not vanishing simultaneously. Its auxiliary equation

$$\bar{b}_0 + \bar{b}_1 t + \cdots + \bar{b}_k t^k = 0$$

clearly has algebraic roots $\bar{\rho}_1, \bar{\rho}_2, \ldots, \bar{\rho}_l$. Hence $y(z)$ can be written

$$y(z) = \sum_{\lambda=1}^{l} \bar{P}_\lambda(z) e^{\bar{\rho}_\lambda z}, \tag{8}$$

where $\bar{P}_1(z), \bar{P}_2(z), \ldots, \bar{P}_l(z)$ are polynomials and $\bar{\rho}_1, \bar{\rho}_2, \ldots, \bar{\rho}_l$ are different *algebraic* numbers.

We observe that the righthand-sides of (5) and (8) are identical functions of z. Using the elementary theorem b) we may suppose, without loss of generality,

$$\rho_i = \bar{\rho}_i, \quad \bar{P}_i(z) \equiv \bar{P}_i(z) \not\equiv 0 \qquad (i = 1, 2, \ldots, j) \tag{9}$$

and

$$P_\sigma(z) \equiv 0 \quad (\sigma = j+1, j+2, \ldots s), \quad \bar{P}_\lambda(z) \equiv 0 \quad (\lambda = j+1, j+2, \ldots, l)$$

for a positive integer $j \leq \mathrm{Min}(s, l)$. Hence

$$y(z) = \sum_{i=1}^{j} P_i(z) e^{\rho_i z}.$$

On account of (9) the numbers $\rho_1, \rho_2, \ldots, \rho_j$ are algebraic and on the other hand they represent zeros of the characteristic function $a_0 + a_1 t + a_2 t^2 + \ldots$ by definition.

This proves our lemma.

Proof of the theorem: Suppose that, contrary to the assertion of our theorem, the transcendental function

$$y(z) = \sum_{h=0}^{\infty} c_h \frac{z^h}{h!}, \quad c_h = O(q^h), \tag{3}$$

with algebraic coefficients $c_0, c_1, c_2, \ldots$ is a solution of

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} A_{\mu\nu} y^{(\mu)}(z + \omega_\nu) = 0, \tag{1}$$

with algebraic $A_{\mu\nu}$, ω_ν; the constants $A_{\mu\nu}$ not vanishing simultaneously and $\omega_0, \omega_1, \ldots, \omega_n$ being different.

We have

$$y^{(\mu)}(z + \omega_\nu) = y^{(\mu)}(z) + \frac{\omega_\nu}{1!} y^{(\mu+1)}(z) + \frac{\omega_\nu^2}{2!} y^{(\mu+2)}(z) + \cdots$$

Substitution in (1) gives a linear differential equation of infinite order

$$a_0 y(z) + a_1 y'(z) + a_2 y''(z) + \cdots = 0,$$

with characteristic function

$$a_0 + a_1 t + a_2 t^2 + \cdots = \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} A_{\mu\nu} \left(t^\mu + \frac{\omega_\nu}{1!} t^{\mu+1} + \frac{\omega_\nu^2}{2!} t^{\mu+2} + \cdots \right)$$

$$= \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} A_{\mu\nu} t^\mu e^{\omega_\nu t}.$$

The function $y(z)$ considered here obviously fulfills all conditions of the foregoing lemma. Hence

$$y(z) = \sum_{i=1}^{j} P_i(z) e^{\rho_i z}, \tag{10}$$

where $P_1(z), P_2(z), \ldots, P_j(z)$ represent polynomials and $\rho_1, \rho_2, \ldots, \rho_j$ different algebraic roots of the equation

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} A_{\mu\nu} t^\mu e^{\omega_\nu t} = 0. \tag{11}$$

However, we will show, that *all non-zero roots of this equation are transcendental.* For suppose that $\rho \neq 0$ represents an algebraic root of (11), then $\omega_0 \rho, \omega_1 \rho, \ldots, \omega_n \rho$ are algebraic and different. From

$$\sum_{\nu=0}^{n} \left(\sum_{\mu=0}^{m} A_{\mu\nu} \rho^{\mu} \right) e^{\omega_\nu \rho} = 0$$

it follows by the Lindemann — Weierstrass theorem, that the $n+1$ algebraic numbers

$$\sum_{\mu=0}^{m} A_{\mu\nu} \rho^{\mu} \qquad (\nu = 0, 1, \ldots, n)$$

must vanish. But this conclusion contradicts one of the conditions of our theorem, the $n+1$ equations (2) having no common root, except perhaps $t=0$.

Hence the set of algebraic numbers $(\rho_1, \rho_2, \ldots, \rho_j)$ must consist of only one number $\rho_1 = 0$; it follows from (10)

$$y(z) = P_1(z),$$

but this gives a contradiction, for $y(z)$ is a transcendental function by hypothesis.

This proves our theorem.

By analogous reasoning I found the following similar theorem: *Let the integral transcendental function of exponential type*

$$y(z) = \sum_{h=0}^{\infty} c_h \frac{z^h}{h!}, \quad c_h = O(q^h),$$

with algebraic coefficients $c_0, c_1, c_2, \ldots$ *satisfy an equation*

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} A_{\mu\nu} y^{(\mu)}(z + \omega_\nu) = 0$$

where the constants $A_{\mu\nu}$ *do not vanish simultaneously and where* $\omega_0, \omega_1, \ldots, \omega_n$ *are different.*

Then $y(z)$ *is a transcendental number for every algebraic value of* z *with exception of a finite number of values for* z. (Clearly $z=0$ always is such an exceptional value.)

If moreover $\sum_{\nu=0}^{n} A_{0\nu} \neq 0$, *then these exceptional values for* z *which differ from 0 necessarily are zeros of* $y(z)$

This theorem, however, is a very special case of the following theorem concerning differential equations of infinite order:

Let the integral transcendental function

$$y(z) = \sum_{h=0}^{\infty} c_h \frac{z^h}{h!}, \quad \limsup_{h \to \infty} \sqrt[h]{|c_h|} \leqq q,$$

with algebraic coefficients $c_0, c_1, c_2, \ldots,$ *satisfy a linear differential equation of infinite order*

$$a_0 y(z) + a_1 y'(z) + a_2 y''(z) + \cdots = 0,$$

with constant coefficients $a_0, a_1, a_2, \ldots,$ *not vanishing simultaneously. Let the corresponding characteristic functian*

$$a_0 + a_1 t + a_2 t^2 + \ldots$$

be regular in the circle $|t| \leqq q$ *and let* v *denote the maximum of the multiplicities of its zeros in the region* $0 < |t| \leqq q$.

Then $y(z)$ *is a transcendental number for every algebraic value of* z *with exception of* v *values for* z *at most.*

If moreover $a_0 \neq 0$, *then these exceptional values for* z *which differ from 0 necessarily are zeros of* $y(z)$.

The proofs of these theorems will appear elsewhere.