CONVEXITY AND NORMED SPACES

by

E. R. LORCH

1. It is common knowledge that in Banach space theory, the triangle inequality, the Hahn-Banach theorem, and the theory of convexity are closely interrelated. The second is a consequence of the first and the existence of support planes in the third has been proved by Mazur with the help of the second. However, the precise nature and the full scope of the interrelations seem not to have been noticed. In order to bring these into strong light, it is necessary to introduce a type of vector space (a space of type **M**), which was studied by Minkowski in his work on convex bodies. An **M**-space differes from a Banach space in that the norm function need not satisfy $|f|=|-f|$. It seems that triangle inequalities, convexity, and the extension of linear functionals not only *may*, but *should* be studied within an **M**-space structure. In this note, the program is carried out. The principal feature of the proofs is the use of the second conjugate of a space. It is also important to note that even though the norm of a space has no properties except „$|f|\geqq 0$, and $|f|=0$ if and only if $f=0$ and $\dfrac{|-f|}{|f|}$ is bounded" then the conjugate space is of type **M**.

2. We consider spaces $\mathfrak{M}$ whose elements are vectors $f, g, \ldots$ over the field of real scalars $\alpha, \beta, \ldots$. That is, $\mathfrak{M}$ is a module which admits of multiplication by real scalars subject to the customary rules. In addition there is defined a real valued function $|f|$ of f (called the norm) subject to the axioms:

1) $|f|\geqq 0$. $|f|=0$ if and only if $f=0$.
2) If $\alpha>0$, $|\alpha f|=\alpha|f|$.
3) $|f+g|\leqq|f|+|g|$.
4) There is a constant $c>0$ such that for all $f\in\mathfrak{M}$, $c^{-1}|f|\leqslant|-f|\leqslant c|f|$.

It is clear that $\mathfrak{M}$ is a metric space if we take as the distance between f and g the number $|f-g|+|g-f|$. We assume that $\mathfrak{M}$ is complete in this metric. Furthermore, $\mathfrak{M}$ is a Banach space if and only if $|f|=|-f|$. A space satisfying all these conditions will be said to be of type **M**.

Consider a real linear functional F over $\mathfrak{M}$. F is said to be bounded if there exists a constant K such that $Ff\leqq K|f|$ for all f; obviously $K\geqq 0$. It is clear that $-Ff=F(-f)\leqq K|-f|$ hence F is bounded if and only if there exists a positive constant K such that $-K|\cdot f| \leqq Ff \leqq K|f|$. The norm of F is defined by $|F| = \text{l.u.b. } Ff/|f|,\ f \neq 0$. It may be shown that the totality of bounded linear functionals is a vector space $\mathfrak{M}^*$ which satisfies axioms 1) to 4) above. If we introduce a metric in $\mathfrak{M}^*$ by defining the distance of F to G to be $|F-G|+|G-F|$, then $\mathfrak{M}^*$ is complete in this metric. We shall see shortly that $\mathfrak{M}^*$ is not empty and that there is a canonical isomorphism (both algebraic and metric) which maps $\mathfrak{M}$ into $\mathfrak{M}^{**}$.

Theorem 1 (Hahn-Banach): *If $\mathfrak{N}$ is a closed linear manifold in a space $\mathfrak{M}$ of type* **M** *and G is a bounded linear functional defined over $\mathfrak{N}$, then there is an extension F of G to the space $\mathfrak{M}$ such that $|F|$ (on $\mathfrak{M}$) equals $|G|$ (on $\mathfrak{N}$).*

The proof proceeds in the usual way by transfinite induction. We set down a few salient facts. Let $g \in \mathfrak{M}-\mathfrak{N}$ and let $f_1, f_2 \in \mathfrak{N}$. Then $Gf_2 - Gf_1 = G(f_2-f_1) \leqq |G||f_2-f_1| \leqq |G||f_2+g|+|G||-f_1-g|$. Hence there exists a real number γ satisfying $-|G||-f_1-g| - Gf_1 \leqq \gamma \leqq |G||f_2+g| - Gf_2$. If $h = f+\beta g$ where $f \in \mathfrak{N}$, define $Fh = Gf + \beta y$. Then it is clear that $F = G$ on N and $|G|$ (on $\mathfrak{N}$) equals $|F|$ (on $\mathfrak{N}+\beta g$).

This means that given any element $f \in \mathfrak{M}$, $f \neq 0$, there exists an $F \in \mathfrak{M}^*$ such that $F \neq 0$ and $Ff = |F|\cdot|f|$. Let $f \in \mathfrak{M}$ and consider the mapping $f \to f^* \in \mathfrak{M}^{**}$ given by $f^*F = Ff$, $F \in \mathfrak{M}^*$. This mapping is an algebraic isomorphism and $|f^*| = |f|$.

We now show that the Hahn-Banach theorem essentially characterizes epaces of type **M**. In particular, the triangle inequality is a consequence of the H-B theorem.

Theorem 2. *Let $\mathfrak{N}$ be a real vector space with a norm satisfying axiom* 1) *and* 4) *and for which the Hahn-Banach theorem holds.* Then $\mathfrak{N}$ *is a linear subspace of a space of type* **M**, *that is, axioms* 2) *and* 3) *also hold.*

Consider the totality $\mathfrak{N}^*$ of bounded linear functionals on $\mathfrak{N}$. Then $\mathfrak{N}^*$ is a space of type **M**. If $F, G \in \mathfrak{N}^*$, then for $f \in \mathfrak{N}$, $(F+G)f \leqq (|F|+|G|)|f|$ hence $F+G \in \mathfrak{N}^*$ and 3) is satisfied. Similarly for any real α $\alpha F \in \mathfrak{N}^*$ and 2) is satisfied. Finally $\mathfrak{N}^*$ is complete.

Consider the space of bounded functionals on $\mathfrak{N}^*$, namely $\mathfrak{N}^{**}$. By the previous discussion, $\mathfrak{N}^{**}$ is of type **M**. If $f \in \mathfrak{N}$ and $F \in \mathfrak{N}^*$ consider the mapping $f \to f^* \in \mathfrak{N}^{**}$ defined by $f^*F = Ff$ for all $F \in \mathfrak{N}^*$. We have $f^*F \leqq |f||F|$ hence $|f^*| \leqq |f|$. By the usual reasoning involving the H-B theorem if $f \neq 0$, there is a linear functional $F \neq 0$ for which $Ff = |F||f|$. Thus $|F||f| = f^*F \leqq |f^*||F|$. Hence $|f| = |f^*|$. Thus $\mathfrak{N}$ is isometrically isomorphic to a subspace of type **M**.

3. We apply these ideas to the theory of convexity in Banach spaces (spaces of type **B**). First we show that any convex set with interior points in a space of type **B** has a hyperplane of support at any point on its boundary. This theorem is due to Mazur (Über konvexen Mengen in linearen normierten Raüme, Studia Math., 4, pp 70—84, (1933)).

Let $\mathfrak{B}$ be a Banach space; let $\mathfrak{K}$ be a closed convex set in $\mathfrak{B}$. Let $\rho\mathfrak{S}$ denote the solid sphere of center 0 and radius ρ in $\mathfrak{B}$, that is, the set of all $f \in \mathfrak{B}$ such that $|f| \leqq \rho$. Suppose that there exist positive constants α and β such that $\alpha\mathfrak{S} \subset \mathfrak{K} \subset \beta\mathfrak{S}$. Suppose f is any vector such that $|f| = 1$. Then there is a positive number γ_f such that $\gamma_f f \in \mathfrak{K}$ but $(\gamma_f + \varepsilon) f \notin \mathfrak{K}$ for any $\varepsilon > 0$. Note that $\alpha \leqq \gamma_f \leqq \beta$.

Consider now a vector space $\mathfrak{M}$ of type **M** defined as follows: The elements of $\mathfrak{M}$ and their algebraic structure are identical with those of $\mathfrak{B}$. Thus the symbol f refers to an element in $\mathfrak{B}$ and also in $\mathfrak{M}$. A norm is introduced in $\mathfrak{M}$ — it will be denoted by $|f|_1$ — as follows: If $|f| = 1$, then for $\delta > 0$, $\gamma_f |\delta f|_1 = |\delta f|$. Thus conditions 1) and 2) on the norm are satisfied. To prove 3) note that if $f' = f/|f|_1$ and $g' = g/|g_1|$, then $|f'|_1 = |g'|_1 = 1$, hence $f', g' \in \mathfrak{K}$. Also for, $0 \leqq \theta \leqq 1$, $\theta f' + (1-\theta) g' \in \mathfrak{K}$ since $\mathfrak{K}$ is convex. Setting $\theta = |f|_1 (|f|_1 + |g|_1)^{-1}$ we obtain 3). The proof of 4) is simple.

It is clear that the canonical mapping of $\mathfrak{M}$ onto $\mathfrak{B}$ is a homeomorphism. For we have for $|f| = 1$, $\alpha |f|_1 \leqq \gamma_f |f|_1 = |f| \leqq \beta |f|_1$. Thus $\mathfrak{M}$ is complete and of type **M**.

Let F be a linear functional defined on $\mathfrak{B}$, and hence also on $\mathfrak{M}$. Then F is bounded on $\mathfrak{B}$ if and only if it is bounded on $\mathfrak{M}$ since $\alpha |f|_1 \leqq |f| \leqq \beta |f|_1$. Thus $\mathfrak{M}^*$ and $\mathfrak{B}^*$ are algebraically isomorphic but differ metrically. The norm in $\mathfrak{M}^*$ will be denoted by $|F|_1$. We may now prove.

Theorem 3 (Mazur): *Let $\mathfrak{K}$ be a closed bounded convex set containing the origin as an interior point in a Banach space $\mathfrak{B}$ and let f be a vector on the boundary of $\mathfrak{K}$. Then there exists a bounded linear functional $F \in \mathfrak{B}^*$ such that for each $g \in \mathfrak{K}$, $Fg \leqq Ff$.*

Proof: Introduce into $\mathfrak{B}$ a new norm as above for which the boundary of $\mathfrak{K}$ is the „unit sphere", thus changing $\mathfrak{B}$ into a space $\mathfrak{M}$ of type **M**. By the Hahn-Banach theorem (theorem 1) there is a bounded linear functional F in $\mathfrak{M}^*$ such that $|F|_1 = 1$, $Ff = |F|_1 |f|_1 = 1$. Thus for every $g \in \mathfrak{K}$ we have $Fg \leqq |F|_1 |g|_1 = |g|_1 \leqq 1 = Ff$. Now F is also in $\mathfrak{B}^*$ and since $Fg \leqq Ff$, $g \in \mathfrak{K}$, we have a supporting hyperplane at f.

We turn to the following

Problem: *Given a Banach space $\mathfrak{B}$ and a set $\mathfrak{L}$ in $\mathfrak{B}$ which is bounded and contains the origin as interior point. To construct the convex cover of $\mathfrak{L}$ (that is, the smallest closed convex set which contains $\mathfrak{L}$).*

We supose as before that $\rho\mathfrak{S}$ is the solid sphere of radius ρ in $\mathfrak{B}$, $f| \leqq \rho$, and that for suitable α, β, $0 < \alpha \leqq \beta < \infty$, $\alpha\mathfrak{S} \subset \mathfrak{L} \subset \beta\mathfrak{S}$. For

f such that $|f|=1$, find $\gamma_f=$ l.u.b. ρ such that $\rho f \in \mathfrak{L}$. Introduce a norm function into $\mathfrak{B}$ considered as an algebraic (but not a metric) structure by defining for arbitrary $\delta > 0$ and f such that $|f|=1: \gamma_f |\delta f|_1 = |\delta f|$. The norm $|g|_1$ has properties 1) and 2) as well as 4). Denote the space with the new norm by $\mathfrak{N}$.

Let $\mathfrak{N}^*$ denote the set of all bounded linear functionals over $\mathfrak{N}$. In the first place, $\mathfrak{N}^*$ is clearly a space of type **M**. Next, $F \in \mathfrak{N}^*$ if and only if $F \in \mathfrak{B}^*$. Let the norm of F in $\mathfrak{N}^*$ be denoted by $|F|_1$. If $f \in \mathfrak{N}$, there is an element $f^* \in \mathfrak{N}^{**}$ such that $f^*F = Ff$ for all $F \in \mathfrak{N}^*$. The mapping $f \to f^*$ is an (algebraic) isomorphism of $\mathfrak{N}$ into $\mathfrak{N}^{**}$: The mapping is clearly homomorphic and if $f^*=0$ then $Ff=0$ for all F, and thus $f=0$; for if $f \neq 0$ then there is an $F \in \mathfrak{B}^*$, $F \neq 0$, such that $Ff \neq 0$. It is easily seen that $|f^*|_1 \leqq |f|_1$.

Let $\delta_f = |f|_1 (|f^*|_1)^{-1}$. Let $\mathfrak{K}$ be the set of all vectors εf where $0 \leqq \varepsilon \leqq \gamma_f \delta_f = (|f^*|_1)^{-1}$, $|f|=1$. Then $\mathfrak{K}$ is the convex cover of $\mathfrak{L}$.

First $\mathfrak{K} \supset \mathfrak{L}$ since $\delta_f \geqq 1$. Note in passing that if $\mathfrak{L}$ is closed and convex, $\delta_f = 1$ and $\mathfrak{K} = \mathfrak{L}$ since $\mathfrak{N}$ is of type M. Next, $\mathfrak{K}$ is closed and convex. In order to show this it is sufficient to show that $\mathfrak{K}$ is the intersection of closed convex sets. It will be shown that there exist linear functionals F_i (i ranging over a suitable set Ω) and constants α_i such that $\mathfrak{K}$ is precisely the set of vectors h such that $F_i h \leqq \alpha_i$ for all $i \in \Omega$. Consider any vector on the boundary of $\mathfrak{K}$ and write it in the form $\gamma_f \delta_f f$, $|f|=1$. Suppose that there is a functional $F \in \mathfrak{N}^*$ such that $F \neq 0$ and $f^*F = Ff = |f^*|_1 |F|_1$. Then, for any g with $|g|=1$ we have $F\gamma_g \delta_g g \leqq \leqslant \gamma_g \delta_g |F|_1 |g^*|_1 = |F|_1 = F\gamma_f \delta_f f$ and if we set $|F|_1 = \alpha$, the vectors h of $\mathfrak{K}$ satisfy $Fh \leqq \alpha$. Suppose that for a given f with $|f|=1$, it is not possible to find $F \neq 0$ such that $f^*F = |f^*|_1 |F|_1$. Let $\varepsilon > 0$ be arbitrary and find F such that $f^*F \geqq |f^*|_1 |F|_1 - \varepsilon$ and $|F|_1 = 1$. Then if we set $\alpha = (f^*F + \varepsilon)(|f^*|_1)^{-1}$, $h \in \mathfrak{K}$ implies $Fh \leqq \alpha$. If $\beta f \notin \mathfrak{K}$, then it is clear that by taking ε sufficiently small, there are an α and an F such that $F\beta f \geqq \alpha$. This proves that $\mathfrak{K}$ is closed and convex.

We complete the solution of the problem by showing that $\mathfrak{K}$ is contained in the closed convex cover of $\mathfrak{L}$. For $F \in \mathfrak{B}^*$ and a real number α, the set of $h \in \mathfrak{B}$ such that $Fh \leqq \alpha$ is called a half space. It will be sufficient to show that $\mathfrak{K}$ lies in any half space contaning $\mathfrak{L}$, since by theorem 3 the closed convex cover of $\mathfrak{L}$ has a supporting hyperplane at each point of its surface. If F is arbitrary in $\mathfrak{B}^*$, then the l.u.b. Fh, $h \in \mathfrak{L}$, equals the l.u.b. $F\gamma_g g$, $|g|=1$, that is, $Fh \leqq$ l.u.b. $Fh = |F|_1$. Then $F\gamma_g \delta_g g = = \gamma_g \delta_g Fg = (|g^*|_1)^{-1} g^*F \leqq |F|_1$. This means that $\mathfrak{K}$ lies in the half-space $Fg \leqq |F|_1$ Therefore $\mathfrak{K}$ is the convex cover of $\mathfrak{L}$.

Barnard College, Columbia University
New York.