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AN ANALYTICAL ESTIMATE OF VALUES OF PRESSURE AND TEMPERATURE AT THE BOUNDARY OF A CONVECTIVE CORE

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Summary: We consider a model of star which is divided in two regions: the „internal" and the „external" one. The model is spherically symmetric, in hydrostatic equilibrium, with an equation of state for a perfect gas with μ = const. The dimensionless integral $I_{\sigma, \nu}$ (Chandrasekhar, 1939) is considered in function of its upper limit, which determines the depth of the region. Analytical estimates are given for $I_{\sigma, \nu}$, $\bar{P}$ and $\bar{T}$, for the „internal" and „external" region and the values for P and T at their boundary.

Particularly, for the models of the zero main-sequence stars, with $4 \leqslant M_0 \leqslant 16$, P and T are estimated at the boundary of the convective core.

Introduction

In the theory of Stellar Structure and Stellar Models, a dimensionless integral (Chandrasekhar, 1939) is considered

$$I_{\sigma, \nu} \equiv \int_0^1 \left(\frac{m}{M}\right)^{\sigma} \left(\frac{r}{R}\right)^{-\nu} d\left(\frac{m}{M}\right) \tag{1}$$

in the whole interior of a star: from centre ($r = 0$, $m = 0$) to surface ($r = R$, $m = M$). For $d\bar{\rho}(r)/dr \leqslant 0$ an estimate is given for $I_{\sigma, \nu}$, with given values of σ and ν, which is:

$$\frac{3}{5} \leqslant I_{2,4} \leqslant \frac{3}{5} (\rho_c / \bar{\rho})^{4/3} \tag{2a}$$

$$\frac{3}{5} \leqslant I_{1,1} \leqslant \frac{3}{5} (\rho_c / \bar{\rho})^{1/3} \tag{2b}$$

$$\frac{3}{2} \leqslant I_{1,4} \leqslant \frac{3}{2} (\rho_c / \bar{\rho})^{4/3} \tag{2c}$$

(see also Cox and Giuli, 1968).Under the assumption that the star is spherically symmetric, with a homogeneous chemical composition, in hydrostatic equilibrium and having an equation of state for a perfect gas with $\mu =$ const, an estimate is further given for $\bar{P}(I_{2,4})$, $\bar{T}(I_{1,1})$ and $P_c(I_{1,4})$.

When the star is considered as a set of concentric layers with given depth, the integral $I_{\sigma,\nu}$ can be defined, for every layer, with discretely variable limits.

We consider, in this paper, a model of star which consists in two regions: an „internal" and an „external" one. In part *I*, $I_{\sigma,\nu}$ is defined and estimated for the two regions. In II, mean values of pressure and temperature are found in those domains, in function of $I_{\sigma,\nu}$ for given σ and ν. Based upon the above, values for P and T are estimated in III, at the boundary of the two domains. Obtained values are applied (part IV) to some models of zero main-sequence stars, having a convective core and an envelope in radiative equilibrium.

I. *The Definition and Estimate of the Integral $I_{\sigma,\nu}$ for the Interior and the Exterior*

Consider a spherically-symmetric model of star with two regions: an interior and an exterior. Let us denote the boundary between the two domains by k and assign to every variable indices k, o and c, related to the interior with the boundary, to the exterior and to the centre of the model respectively. Let R and M be the radius and the mass of the model, r_k and m_k the radius and the mass of its interior.

We define dimensionless quantities q_r and q_m by relations

$$r_k = q_r R, \qquad m_k = q_m M \tag{3}$$

As we have

$$\bar{\rho}(r) = m(r) / (4\pi r^3/3), \tag{4}$$

we obtain, using (3) and (4)

$$(r/R)^3 = (\bar{\rho}_k/\bar{\rho}(r)) (m/M) (q^3_r/q_m) \tag{5}$$

where $m \equiv m(r)$.

Consider, in analogy with $I_{\sigma,\nu}$ from (1), the dimensionless integral for the interior

$$I^k_{\sigma,\nu} = \int_0^k \left(\frac{m}{M}\right)^\sigma \left(\frac{r}{R}\right)^{-\nu} d\left(\frac{m}{M}\right) \tag{6}$$

or, by (5)

$$I^k_{\sigma,\nu} = q_m^{\nu/3} q_r^{-\nu} \int_0^k (\bar{\rho}(r)/\bar{\rho}_k)^{\nu/3} \left(\frac{m}{M}\right)^{\sigma-\nu/3} d\left(\frac{m}{M}\right), (0 \leqslant r \leqslant r_k) \tag{7}$$

For

$$d\bar{\rho}(r)/dr \leqslant 0, \quad 0 \leqslant r \leqslant R, \tag{8}$$

we have

$$\bar{\rho}_k \leqslant \bar{\rho}(r) \leqslant \rho_c, \quad 0 \leqslant r \leqslant r_k \tag{9}$$

and for the integral (7) one obtain

$$\frac{3}{3(\sigma+1)-\nu} q_m^{(\sigma+1)} q_r^{-\nu} \leqslant I^k_{\sigma,\nu} \leqslant \frac{3}{3(\sigma+1)-\nu} q_m^{(\sigma+1)} q_r^{-\nu} \left(\frac{\rho_c}{\bar{\rho}_k} \right)^{\nu/3} \tag{10}$$

Let us consider now, the dimensionless integral for the exterior

$$I^0_{\sigma,\nu} = \int_k^1 \left(\frac{m}{M} \right)^{\sigma} \left(\frac{r}{R} \right)^{-\nu} d\left(\frac{m}{M} \right) \tag{11}$$

We have from (8)

$$\bar{\rho}(R) \equiv \bar{\rho} \leqslant \bar{\rho}(r) \leqslant \bar{\rho}_k, \quad r_k \leqslant r \leqslant R \tag{12}$$

and with (5) and (12) one obtains for integral (11)

$$\frac{3}{3(\sigma+1)-\nu} \left[1 - q_m^{(\sigma+1)-\nu/3} \right] \leqslant I^0_{\sigma,\nu} \leqslant \frac{3}{3(\sigma+1)-\nu} \left[1 - q_m^{(\sigma+1)-\nu/3} \right] q_m^{\nu/3} q_r^{-\nu} \tag{13}$$

For

$$\nu < 3(\sigma + 1) \tag{14}$$

the left and right hand sides in (10) and (13) are positive.

II. $\bar{P}$ *and* $\bar{T}$ *for the Interior and the Exterior*

The model is in hydrostatic equilibrium (the mass conservation law holds) so that

$$dP(r) = -(Gm / 4\pi r^4)\, dm, \tag{15}$$

with an equation of state

$$P = (k/\beta\, \mu\, m_H)\rho T, \quad \beta = P_g/P \tag{16}$$

The mean value of the structure function F, for the internal region is

$$\bar{F}_k = \frac{1}{m_k} \int_0^{m_k} F\, dm \tag{17}$$

and for the external one it is

$$\bar{F}_0 = \frac{1}{M - m_k} \int_{m_k}^{M} F\, dm \tag{18}$$

a) *The Mean Value of The Pressure*

Let us perform partial integration of (17), with $F \equiv P(r)$, using condition (15). The integral $I^k_{\sigma,\nu}$ from (6) gives, for $\sigma = 2$, $\nu = 4$,

$$\int_0^{m_k} \frac{m^2}{r^4}\, dm = \frac{M^3}{R^4} I^k_{2,4} \tag{19}$$

so that (17) becomes finally

$$m_k \bar{P}_k = m_k P_k + \frac{GM^3}{4\pi R^4} I^k_{2,4} \tag{20}$$

Making use of the result for $\sigma = 2$, $\nu = 4$ from (11), i.e.

$$\int_{m_k}^{M} \frac{m^2}{r^4}\, dm = \frac{M^3}{R^4} I^0_{2,4} \tag{21}$$

and the hypothesis that $P \equiv P(R) \ll P_k$, relation (18) becomes similarly, for $F \equiv P(r)$

$$M(1 - q_m)\bar{P}_0 = -m_k P_k + \frac{GM^3}{4\pi R^4} I^0_{2,4} \tag{22}$$

Adding (20) and (22) one obtains the well-known result

$$\bar{P}(R) \equiv \bar{P} = \frac{GM^2}{4\pi R^4} I_{2,4} \tag{23}$$

b) *The Mean Value of The Temperature*

Let it be, in the first approximation

$$\beta(r) \equiv P_g(r) / P(r) = 1 \quad \text{and} \quad \mu(r) = \text{const}, \quad 0 \leqslant r \leqslant R.$$

We get then, from (16)

$$\int T(r)\, dm = \frac{1}{C} \int P(r)\, dV \tag{24}$$

where $C \equiv \mu m_H / k = \text{const.}$

Using the result from (6) for $\sigma = 1$, $\nu = 1$, i.e.

$$\int_0^{m_k} (m/r)\, dm = \frac{M^2}{R} I^k_{1,1} \tag{25}$$

and relations (24) and (15), the result of partial integration (17) for $F \equiv T(r)$ is

$$Cm_k \bar{T}_k = P_k V_k + \frac{GM^2}{3R} I^k_{1,1} \tag{26}$$

Similarly, with the assumption that $P/P_k \ll V_k/V$ $(V \equiv V(R))$, equation (18) with $F \equiv T(r)$ gives

$$CM(1-q_m)\bar{T}_0 = -P_k V_k + \frac{GM^2}{3R} I^0_{1,1} \tag{27}$$

where use was made of the result from (13) for $\sigma = 1$, $\nu = 1$, that is

$$\int_{m_k}^{M} (m/r)\, dm = \frac{M^2}{R} I^0_{1,1} \tag{28}$$

By adding (26) and (27) one obtains the well-known result

$$\bar{T}(R) \equiv \bar{T} = \frac{GM}{3CR} I_{1,1} \tag{29}$$

III. *The Estimate of P and T on the Boundary of the Two Regions*

Relation (10) becomes, for $\sigma = 2$, $\nu = 4$

$$\frac{3}{5} q_m^3 q_r^{-4} \leqslant I^k_{2,4} \leqslant \frac{3}{5} q_m^3 q^{-4} (\rho_c \mid \bar{\rho}_k)^{4/3} \tag{30}$$

and for $\sigma = \nu = 1$,

$$\frac{3}{5} q_m^2 q_r^{-1} \leqslant I^k_{1,1} \leqslant \frac{3}{5} q_m^2 q_r^{-1} (\rho_c \mid \bar{\rho}_k)^{1/3} \tag{31}$$

Relation (13) becomes, for $\sigma = 2$, $\nu = 4$

$$\frac{3}{5}(1 - q_m^{5/3}) \leqslant I^0_{2,4} \leqslant \frac{3}{5}(1 - q_m^{5/3})\, q_m^{4/3}\, q_r^{-4} \tag{32}$$

and for $\sigma = \nu = 1$,

$$\frac{3}{5}(1 - q_m^{5/3}) \leqslant I^0_{1,1} \leqslant \frac{3}{5}(1 - q_m^{5/3})\, q_m^{1/3}\, q_r^{-1} \tag{33}$$

By substituting in (20) the minimal value of $I^k_{2,4}$ from (30), one obtains a result some more rigorous than Theorem 5 (Chandrasekhar, 1939, page 70), for $r = r_k\,(m = m_k)$. Moreover, when putting $\bar{P}_k < P_c$, one obtains

$$P_k < P_c - \frac{3\,G}{20\pi}\left(\frac{m_k}{r_k^2}\right)^2 \tag{34}$$

i.e.

$$P_k < P_c - 0.5369 \times 10^{-2}\,(q_m/q^2{}_r)^2\,(M_0/R^2{}_0)^2 \tag{35}$$

Here $M_0 = M/M_\odot$, $R_0 = R/R_\odot$, P_c and P_k are in units 10^{17} dyn/cm². Equation (22) with $\bar{P}_0 < P_k$, and $I^{0\,(min)}_{2.4}$ from (32) yields

$$P_k > 0.5369 \times 10^{-2}\,(1 - q_m^{5/3})\,(M_0/R^2{}_0)^2 \tag{36}$$

Equation (27) with $\bar{T}_0 < T_k$, $\rho_k\,\mathbf{V}_k < m_k$ and $I^{0\,(min)}_{1,1}$ from (33) yields

$$T_k > 4.585\,\mu\,(1 - q_m^{5/3})\,(M_0/R_0) \tag{37}$$

for T_k in units $10^6 K$ and $C = 1.202743 \times 10^{-8}\,\mu$, or using (29) and (2)

$$T_k > T^{min}\,(1 - q_m^{5/3}) \tag{38}$$

IV. *P and T at the Boundary of the Convective Core for some Models of the Zero Main-Sequence Stars*

Consider the models of the zero main-sequence stars with convective core. When equating the integration limit k for integrals $I^k_{\sigma,\nu}$ and $I^0_{\sigma,\nu}$ with the limit of the convective core, relations (35), (36) and (37) allow an analytic estimate for P and T at the boundary of the core, when P_c, q_m and q_r are known in function of M and R. One obtains correlations for P_c, q_m and q_r from the models of the zero main-sequence stars (Angelov, 1973, 1974) in the form:

$$LgP_c = + \underset{34}{0.357} - \underset{35}{0.843}\, LgM_0 \tag{39}$$

(P_c in 10^{17} dyn/cm²),

$$Lgq_m = -1.11588931 + 0.59698577\, LgM_0 \tag{40}$$

$$q_r = \underset{32}{+\,0.0744} + \underset{59}{0.5151}\, q_m \tag{41}$$

Results (39), (40) and (41) hold only for interval $4 \leqslant M_0 \leqslant 16$ and for chemical composition ($X = 0.650$, $Z = 0.040$) with $\mu = 0.643$. For instance, for $q_m(M_0)$ one can draw the conclusion that coefficients in (40) are mutually distinct for $4 \leqslant M_0 \leqslant 16$ (Angelov, 1974) and for $M_0 > 16$ (Kotok, 1966).

One can show that for these models $\beta \lesssim 1$, so that the estimate for P and T, in the first approximation, can be performed making use of (35), (36) and (37).

Values for P_k^{max}, P_k^{min} and T_k^{min}

(P_k in units 10^{17} dyn/cm², T_k in units $10^6 K$)

M_0	R_0	P_k^{max}	P_k^{min} (10^2)	T_k^{min}	P_k(mod)	T_k(mod)
4	2.35	0.59	0.27	4.7	0.36	18.6
5	2.69	0.48	0.24	5.1	0.27	19.2
8	3.56	0.32	0.19	5.9	0.14	20.2
10	4.05	0.26	0.17	6.3	0.11	20.5
16	5.28	0.17	0.14	7.0	0.06	20.8

Values for P_k^{max}, P_k^{min} and T_k^{min} at the boundary of the convective core (from (35), (36) and (37)), are given in the Table, together with exact values for P_k and T_k directly from the models. M_0 and R_0 are the model parameters (Angelov, 1973), with ($X = 0.650$, $Z = 0.040$), $\mu = 0.643$.

Analysis of the Results and Conclusion

One can see from the Table:

a) $1.6 \leqslant P_k^{max}/P_k\,(\text{mod}) \leqslant 2.4$ (except for $M_0 = 16$, when that ratio is equal to 3.1) and decreases with the decrease of mass. The difference between P_c (mod) and P_1^{max} is little because $2 \lesssim P_c\,(\text{mod})/P_k\,(\text{mod}) \lesssim 4$ (the boundary of the convective core for models considered is $r_k < 0.3\,R$).

b) $3 \lesssim T_k\,(\text{mod})/T_k^{min} \lesssim 4$ and decreases with the increase of mass. Furthermore, $T_k/T^{min} > (1 - q_m^{5/3})$ (relation (38)), i.e. one has surely $T_k > 0.7\ T^{min}$ (for models used $q_m \leqslant 0.4$).

With given M and μ for a models and with a correlation $M - R$ known with a fair accuracy for main-sequence stars, with convective core, it is possible, in the first approximation, to give an analytical estimate of the values of pressure and temperature at the boundary of the core.

REFERENCES

Angelov, T. 1973, Publ. of the Depart. of Astr., Faculty of Sc., University of Beograd, **4**, 10.

Angelov, T. 1975, (Proceedings of the National Conference of Yugoslav Astronomers – 1973, Belgrade), Publ. obs. astr. Beograd, **20**, 110.

Chandrasekhar, S. 1939, Stellar Structure, Dover Publ., University of Chicago Press.

Cox, J. P., Giuli, R. T. 1968, Principles of Stellar Structure, Vol. 1, Gordon and Breach, Sc. Publ., New York.

Kotok, E. V. 1966, Nauč. Inform., Vip. **3**, **1**.