NUMERICAL DETERMINATION OF APPROXIMATE TRUE ANOMALIES IN THE PROXIMITY OF QUASICOMPLANAR ORBITS OF CELESTIAL BODIES

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A purely numerical method of determination of approximate values of true anomalies for two celestial bodies in the proximity of their osculating quasicomplanar elliptic orbits is given. It consists in the solution of only one corresponding quadratic equation. This method is particularly suited and efficient for serial work when determining the least distances of orbits of quasicomplanar asteroids, when these distances are small. This method is illustrated by an already examined pair of asteroids, which has allowed the comparison with previously obtained results and made possible to point out the advantage and the simplicity of this new method, which represents a further step when computing the proximities.

The determination of the proximity of osculating elliptic orbits of two celestial bodies poses the problem of determining the positions of those bodies, when the mutual distance of their orbits is minimal. When these positions are known, the minimal distance can be determined by a simple calculus.

True anomalies v_1 and v_2 of the positions of proximity for two celestial bodies moving in osculating elliptic orbits can be obtained as the solutions of exact and general transcendent equations of the form

$$f(v_1, v_2) = 0, \quad g(v_1, v_2) = 0. \tag{1}$$

We solve these equations numerically, and after k successive approximations with necessary accuracy, we obtain the solutions in the form

$$v_{1k} = v_{1(k-1)} + \Delta v_{1(k-1)}, \quad v_{2k} = v_{2(k-1)} + \Delta v_{2(k-1)}, \tag{2}$$

$$(k = 1, 2, \ldots).$$

Corresponding expressions for f, g and $\Delta v_{i(k-1)}$, $i = 1, 2$, are given in the previous papers (**1, 2**). Meanwhile, the problem consists in the knowledge of the approximate — initial values of true anomalies v_{10} and v_{20} of the proximity. For quasicomplanar orbits of two bodies moving in almost the same plane, or having a small

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mutual inclination of their orbits, a graphical (**1**) and a numerical-graphical way (**2**) of determining the needed initial true anomalies are given. We shall present a purely numerical method of determining those approximate true anomalies.

In the proximity of quasicomplanar elliptic orbits of two celestial bodies, when their distance is small, their longitudes and radii vectors are approximately equal $\lambda_1 \approx \lambda_2$, $r_1 \approx r_2$, and when we take equalities instead of approximations, as for the intersection of orbits, we have, as in (**2**),

$$\pi_1 + v_1 = \pi_2 + v_2\,, \tag{3}$$

where $\pi_i = \Omega_i + \omega_i$, $(i = 1, 2)$, and

$$\frac{p_1}{1 + e_1 \cos v_1} = \frac{p_2}{1 + e_2 \cos v_2}\,, \tag{4}$$

where index 1 is related to the first and 2 to the second orbit for considered bodies We get from (3) the important relation

$$v_2 = v_1 - \pi, \tag{5}$$

which we shall use further when computing the wanted approximate values of true anomalies. The quantity π represents

$$\pi = \pi_2 - \pi_1\,. \tag{6}$$

When eliminating from (4) v_2 using (5), we get

$$q_1 \cos v_1 + t_1 \sin v_1 = p, \tag{7}$$

where the functions of orbital elements are

$$p = p_2 - p_1\,, \tag{8}$$

$$q_1 = e_2\, p_1 \cos \pi - e_1\, p_2\,, \quad t_1 = e_2\, p_1 \sin \pi\,.$$

Equation (7) is the same as in (**2**), where it was solved graphically. Now we shall solve it numerically. Using

$$\cos v_1 = \sqrt{1 - \sin^2 v_1} \tag{10}$$

we eliminate $\cos v_1$ from (7) and obtain a quadratic equation in $\sin v_1$

$$(q_1^2 + t_1^2) \sin^2 v_1 - 2pt_1 \sin v_1 + (p^2 - q_1^2) = 0, \tag{11}$$

having the solutions:

$$(\sin v_1)_{1,2} = \frac{pt_1 \pm q_1\sqrt{q_1^2 + t_1^2 - p^2}}{q_1^2 + t_1^2}\,, \tag{12}$$

by condition $q_1^2 + t_1^2 - p^2 \geqslant 0$, wherefrom we can obtain, for the first orbit, the following four (or two, for $q_1^2 + t_1^2 - p^2 = 0$) possible values for the required approximate value $v_{10} : (v_{10})_1, (v_{10})_2, (v_{10})_3, (v_{10})_4$, because we do not know yet to which quadrant belongs the possible angle. With these values for v_{10} we can obtain then, from (5), the four (or two) corresponding values for v_{20} for the second orbit. The multivalued character of the possible solutions is eliminated by a simple comparison of values of true anomalies here obtained with exact values of true anomalies for the relative nodes of the orbits (the points in which projections of the orbits on the apparent celestial sphere intersect) of the considered pair of celestial bodies, for it is natural that the proximity of the orbits be near one of their relative nodes (ascending or descending). By means of the least differences of these anomalies we determine which true anomalies, among all possible, ought to be choosen as the required approximate ones, with which we operate further in successive approximations (2).

The second form of the equation, which is somewhat shorter than (11), can be obtained by substitutions

$$x_1 = \frac{q_1}{t_1}, \qquad y_1 = \frac{p}{t_1}, \tag{13}$$

which are functions of orbital elements. Then (7) becomes

$$y_1 = x_1 \cos v_1 + \sin v_1 , \tag{14}$$

as in (**2**). Hence, eliminating $\cos v_1$ by means of (10), we obtain second form of the corresponding quadratic equation in $\sin v_1$

$$(1 + x_1^2)\sin^2 v_1 - 2y_1 \sin v_1 + (y_1^2 - x_1^2) = 0, \tag{15}$$

having the solutions

$$(\sin v_1)_{1,2} = \frac{y_1 \pm x_1\sqrt{1 + x_1^2 - y_1^2}}{1 + x_1^2}, \tag{16}$$

by condition $1 + x_1 - y_1^2 \geqslant 0$.

Thus we obtain the required approximate values of the true anomalies v_{10} and v_{20} of the proximity of quasicomplanar orbits of celestial bodies by solving one quadratic equation (11) or (15), i. e. we find solutions by means of (12) or (16) and (5). The calculation is not difficult, and we have avoided the use of a diagram, which could be less precise.

Now we shall illustrate previously exposed numerical procedure by elements already used for the quasicomplanar pair of asteroids $1 \equiv 589$ *Croatia* and $2 \equiv 1564$ *Srbija*, in order to compare the results of this calculation with those previously obtained in (**1**) and (**2**). So for the considered pair we have from (**1**)

$$e_1 = 0.0398179, \quad p_1 = 3.1295420, \quad e_2 = 0.2115994, \quad p_2 = 3.0082028,$$

and by means of (6) and (8)

$$\pi = 12^\circ\!.686, \quad p = -0.1213392,$$

so that expressions (9) and (13) yield the values

$$q_1 = 0.5262632, \quad t_1 = 0.1454263; \quad x_1 = 3.6187622, \quad y_1 = -0.8343690.$$

Then for the solutions (12) of the corresponding quadratic equation (11) we get the values

$$(\sin v_1)_1 = 0.8805767, \quad (\sin v_1)_2 = -0.9989651; \tag{17}$$

and for the solutions (16) of the other form of the corresponding equation (15) which can serve as a control of the calculation, we obtain values among which the first differs only by 1 in the seventh decimal from the value in (17), i. e. our calculation is correct. Further, we obtain from (17) as possible values for the needed approximate true anomaly of the proximity for the first orbit:

$$\begin{gathered}(v_{10})_1 = 61\overset{\circ}{.}712, \quad (v_{10})_2 = 180^\circ - (v_{10})_1 = 118\overset{\circ}{.}288, \\ (v_{10})_3 = 180^\circ + 87\overset{\circ}{.}393 = 267\overset{\circ}{.}393, \quad (v_{10})_4 = 360^\circ - 87\overset{\circ}{.}393 = 272\overset{\circ}{.}607.\end{gathered} \tag{18}$$

Thereafter, using (5), we obtain corresponding possible values for the approximate true anomaly of the proximity for the second orbit:

$$(v_{20})_1 = 49\overset{\circ}{.}026, \quad (v_{20})_2 = 105\overset{\circ}{.}602, \quad (v_{20})_3 = 254\overset{\circ}{.}707, \quad (v_{20})_4 = 259\overset{\circ}{.}921. \tag{19}$$

The values of the true anomalies of the relative nodes of the orbits of choosen asteroids (**3**) are:

$$v_1 = 296\overset{\circ}{.}047, \quad v_2 = 283\overset{\circ}{.}352, \quad v_1' = 116\overset{\circ}{.}047, \quad v_2' = 103\overset{\circ}{.}352, \tag{20}$$

where the first pair of values corresponds to the ascending relative node of the first orbit relatively to the second, and the second pair corresponds to the descending relative node. The difference between values of true anomalies for the two nodes on the same orbit is 180°. Now with values (18) for the first orbit, (19) for the second orbit and those from (20) for relative nodes we form differences, so that for the required approximate values of true anomalies we take those for which these differences are the least. This is the criterion for the determination of the pair of possible values (18) and (19) which ought to be taken for the further calculation of proximity.

As we obtain, in the case considered, for the least differences

$$\Delta v_1' = (v_{10})_2 - v_1' = 2\overset{\circ}{.}241, \quad \Delta v_2' = (v_{20})_2 - v_2' = 2\overset{\circ}{.}250, \tag{21}$$

we conclude that values $(v_{10})_1$, $(v_{20})_1$, $(v_{10})_3$, $(v_{20})_3$, $(v_{10})_4$, $(v_{20})_4$ are left out, because the differences between them and (20) for relative nodes are greater than (21), so that here the proximity is near the descending relative node (at a distance of approximately 2° from it), as we have found already in our previous considerations for the considered pair of asteroids.

We have thus for the required approximate values of true anomalies of proximity

$$v_{10} = (v_{10})_2 = 118\overset{\circ}{.}288, \quad v_{20} = (v_{20})_2 = 105\overset{\circ}{.}602. \tag{22}$$

Let us remark that the same result is obtained when eliminating $\sin v_1$ from equation (7) or (14), and solving corresponding quadratic equations in $\cos v_1$.

We already know (**1**), (**2**) and (**3**), that exact values of true anomalies for the proximity of the considered pair of asteroidal orbits are

$$v_1 = 118\overset{\circ}{.}2977, \quad v_2 = 105\overset{\circ}{.}6025, \tag{23}$$

so that the differences between these values and the approximate values are:

by the new method $\quad v_1 - v_{10} = 0\overset{\circ}{.}010, \quad v_2 - v_{10} = 0\overset{\circ}{.}001,$

i. e. $\quad 0\overset{\circ}{.}0097, \quad 0\overset{\circ}{.}0005,$

by the method (**2**) $\quad v_1 - v_{10} = 0\overset{\circ}{.}098, \quad v_2 - v_{20} = 0\overset{\circ}{.}003,$

by the method (**1**) $\quad v_1 - v_{10} = - 0\overset{\circ}{.}302, \quad v_2 - v_{20} = 0\overset{\circ}{.}203.$

The sums of the squares of these differences are:

here 0.000 (i. e. 0.0001), from (**2**) 0.010, from (**1**) 0.132.

So here exposed purely numerical procedure of finding the approximate true anomalies in proximity of quasicomplanar orbits gives considerably lesser deviations. In the considered example we have obtained by this new method, the precision of one hundredth of a degree, while the previous methods gave the precision of $0\overset{\circ}{.}1$ and $0\overset{\circ}{.}3$. This new method has also the advantage in that drawing and graphical determining is not necessary, so that it is particularly suitable for serial work. It gives more approximate solutions than previous ones (**1**, **2**), and for the considered pair we see that the sum of squares of deviations is 100 times less than for method (**2**).

Let us examin now how approximate values (22) of true anomalies for the proximity of the considered pair of asteroids satisfy exact equations (1), and which values are obtained in further numerical corrections (2), from formulae (**1**) and (**2**). We find so

$$\begin{aligned} f_0(v_{10}, v_{20}) &= -0.0004638, & g_0(v_{10}, v_{20}) &= 0.0004478; \\ \Delta v_{10} &= 0\overset{\circ}{.}0097, & \Delta v_{20} &= 0\overset{\circ}{.}0006; \\ v_{11} &= 118\overset{\circ}{.}2977, & v_{21} &= 105\overset{\circ}{.}6026. \end{aligned}$$

When comparing these values v_{11} and v_{21} with exact ones (23), already known, we see that only the second, v_{21}, differs form v_2 by 1 in the fourth decimal.

Thus we conclude that here exposed method of determining of approximate true anomalies gives values which satisfy better exact equations (1), for which obtained values are about 10 times less than those obtained by the method from (**2**), and the first numerical corrections Δv_{i0} are about 10 and 5 times less than those from (**2**). We see that it was sufficient now, for the pair considered, to determine the first numerical corrections only. So we have come, in our case, to the definitive — exact solutions by one numerical approximation only, while the previous method (**2**) needed the determination of two numerical approximation, by means of two pairs of numerical corrections Δv_{i0} and Δv_{i1}, $i = 1, 2$. It is why this new method is not only simpler, but also more efficient, being shorter and leading faster to solutions required.

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