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APPROXIMATE VALUES OF TRUE ANOMALIES OF QUASICOMPLANAR ASTEROIDS IN PROXIMITY

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We have exposed a new graphical-numerical method of determining of approximate values of true anomalies, corresponding to least distance between two elliptical orbits with small mutual inclination. This procedure is especially convenient in the case when the number of pairs of quasicomplanar orbits of asteroids is great. It is applied to the pair of „our" asteroids: 589 *Croatia* and 1564 *Srbija*, which has made possible the comparison of the results here exposed with those previously obtained by another method.

The problem of the determination of approximate values of true anomalies, necessary if solving the proximity of elliptical orbits of two celestial bodies, appears in the general case as well in the case when the two asteroids perform a motion almost in the same plane, **(1)**. *J. Simovljević* has shown the way of obtaining approximate values of eccentric anomalies for quasicomplanar asteroids in proximity, **(2)**, by use of a graphic, which is similar to the idea of Radau's diagram for the solution of Kepler's equation of planetary motion, **(3)**. The problem of determining the proximity being considered in **(1)** by means of true anomalies, we have tried to obtain a corresponding method for direct determination of necessary approximate initial values of these anomalies.

1. In the general case of elliptic motion the equation which determine true anomalies of points of orbits with least reciprocal distance can take the form, **(1)**,

$$\left.\begin{aligned} f(v_1, v_2) &\equiv V_1 \sin v_2 + W_1 \cos v_2 + s_1 \sin v_1 = 0, \\ g(v_1, v_2) &\equiv V_2 \sin v_1 + W_2 \cos v_1 - s_2 \sin v_2 = 0, \end{aligned}\right\} \quad (1)$$

where

$$\left.\begin{aligned} V_i &= B_i - D_i \sin v_i + G_i \cos v_i - \frac{1}{2} K_i \sin 2 v_i - B_i \cos^2 v_i, \\ W_i &= C_i - A_i \sin v_i + F_i \cos v_i + \frac{1}{2} H_i \sin 2 v_i - C_i \cos^2 v_i, \end{aligned} \quad (i = 1, 2) \right\} \quad (2)$$

Quantities on right-hand sides of equations (2): A_i, B_i, C_i, D_i, F_i, G_i, H_i, K_i and s_i depend on orbital elements of two planets. Needed true anomalies v_1 and v_2, which represent the solutions of system (1), are found by successive approximations, starting from approaching values for the two orbits: v_{10} and v_{20}, which are sufficiently close to searched ones. After k numerical corrections we find the values of true anomalies

$$v_{1k} = v_{1\,(k-1)} + \Delta\, v_{1\,(k-1)} \qquad \text{and} \qquad v_{2k} = v_{2\,(k-1)} + \Delta\, v_{2\,(k-1)}, \tag{3}$$

which satisfy system (1) with sufficient accuracy. Numerical corrections of true anomalies are found by means of formulae

$$\left.\begin{aligned} \Delta\, v_{1\,(k-1)} &= \frac{g_{k-1} f'_{v_2\,(k-1)} - f_{k-1} g'_{v_2\,(k-1)}}{f'_{v_1\,(k-1)} g'_{v_2\,(k-1)} - f'_{v_2\,(k-1)} g'_{v_1\,(k-1)}}, \\ \Delta\, v_{2\,(k-1)} &= \frac{f_{k-1} g'_{v_1\,(k-1)} - g_{k-1} f'_{v_1\,(k-1)}}{f'_{v_1\,(k-1)} g'_{v_2\,(k-1)} - f'_{v_2\,(k-1)} g'_{v_1\,(k-1)}} \end{aligned}\right\} \quad (k = 1, 2, 3, ..) \tag{4}$$

in radians. In this way we obtain, with approximate initial values and two numerical corrections, next values for the solutions

$$\left.\begin{aligned} v_{12} &= v_{10} + \Delta\, v_{10} + \Delta\, v_{11} = v_{11} + \Delta\, v_{11}, \\ v_{22} &= v_{20} + \Delta\, v_{20} + \Delta\, v_{21} = v_{21} + \Delta\, v_{21}. \end{aligned}\right\} \tag{5}$$

Introducing symbols

$$\left.\begin{aligned} \theta_i &= D_i \cos v_i - G_i \sin v_i + K_i \cos 2\, v_i - B_i \sin 2\, v_i, \\ \nu_i &= A_i \cos v_i - F_i \sin v_i + H_i \cos 2\, v_i - C_i \sin 2\, v_i, \end{aligned}\right\} \quad (i = 1,2) \tag{6}$$

partial derivatives, for (4), are expressed by equations

$$\left.\begin{aligned} f'_{v_1} &= \theta_1 \sin v_2 + \nu_1 \cos v_2 + s_1 \cos v_1, \\ f'_{v_2} &= V_1 \cos v_2 - W_1 \sin v_2, \\ g'_{v_1} &= V_2 \cos v_1 - W_2 \sin v_1, \\ g'_{v_2} &= \theta_2 \sin v_1 + \nu_2 \cos v_1 + s_2 \cos v_2. \end{aligned}\right\} \tag{7}$$

2. We can expect, for approximate values of true anomalies v_{10} and v_{20}, to be relatively close to values of true anomalies of one of two relative nodes, corresponding to the intersection of orbital planes of a chosen pair of celestial bodies. This fact help us for the choice of initial values, for which we take those corresponding to differences smaller in comparison with true anomalies of relative nodes. The determination of initial values for quasicomplanar orbits of asteroids, we are concerned, can be drawn in the precedingly exposed way, **(1)**, but also in the next one, we shall explain here. It is particularly convenient for serial work, if a sequence of pairs or groups of quasicomplanar asteroids is given, **(4)**, **(5)**.

Two asteroids, moving in unperturbed quasicomplanar orbits, have, in positions of least mutual distance, approximately equal heliocentric radii-vectors and longitudes: $r_1 \approx r_2$, $\lambda_1 \approx \lambda_2$. We shall take equalities, instead of approximations, as in the case of the intersection of orbits,

$$\frac{p_1}{1 + e_1 \cos v_1} = \frac{p_2}{1 + e_2 \cos v_2}, \tag{8}$$

$$\pi_1 + v_1 = \pi_2 + v_2, \tag{9}$$

where $\pi_i = \Omega_i + \omega_i$ are the longitudes of the perihelions, and indices $i = 1, 2$ correspond to the first, resp. the second orbit. Solving the two previous equations, with two unknowns, v_1 and v_2, should lead us to the sought solution, supposing that orbital elements of asteroids are known.

With symbols

$$p = p_2 - p_1, \tag{10}$$

$$\pi = \pi_2 - \pi_1, \tag{11}$$

(9) becomes

$$v_1 = \pi + v_2. \tag{12}$$

If we eliminate from (8) first v_2, then v_1, using (12), and having in mind (10) and (11), we get equations

$$\left.\begin{aligned} q_1 \cos v_1 + t_1 \sin v_1 = p, \\ q_2 \cos v_2 - t_2 \sin v_2 = p, \end{aligned}\right\} \tag{13}$$

where

$$\left.\begin{aligned} q_1 &= e_2\, p_1 \cos \pi - e_1\, p_2, & t_1 &= e_2\, p_1 \sin \pi, \\ q_2 &= e_2\, p_1 - e_1\, p_2 \cos \pi, & t_2 &= e_1\, p_2 \sin \pi. \end{aligned}\right\} \tag{14}$$

Putting

$$\left.\begin{aligned} x_1 &= \frac{q_1}{t_1}, & y_1 &= \frac{p}{t_1}, \\ x_2 &= \frac{q_2}{t_2}, & y_2 &= \frac{p}{t_2}, \end{aligned}\right\} \tag{15}$$

equations (13), after dividing the first one by t_1, and the second by t_2, get the form

$$\left.\begin{aligned} y_1 = x_1 \cos v_1 - \sin v_1, \\ y_2 = x_2 \cos v_2 + \sin v_2. \end{aligned}\right\} \tag{16}$$

With symbols

$$\left.\begin{aligned} \xi_1 &= \frac{t_1}{q_1} = \frac{1}{x_1}, & \eta_1 &= \frac{p}{q_1}, \\ \xi_2 &= \frac{t_2}{q_2} = \frac{1}{x_2}, & \eta_2 &= \frac{p}{q_2}, \end{aligned}\right\} \tag{17}$$

equations (13) can be written also in the form

$$\left.\begin{aligned} \eta_1 = \xi_1 \cos (90^\circ - v_1) - \sin (90^\circ - v_1), \\ \eta_2 = \xi_2 \cos (90^\circ - v_2) + \sin (90^\circ - v_2). \end{aligned}\right\} \tag{18}$$

The question if we shall take, for system (13), equations of the form (16) or (18), depends on the values of p and coefficients (14).

3. Equations (16) and (18) are of the form

$$y = x \cos v + \sin v. \tag{19}$$

We can draw a diagram for this equation, when v varies from 0° to 360° in equidistant intervals. The idea of this diagram is similar to the Radau diagram for solving Kepler's equation of planetary motion. It is symmetric with respect to coordinate axes and coordinate origin, v varying from 0° to 360°. To every value v corresponds one straight line given by equation (19), and the set of these straight lines represents our diagram, by means of which we can, conversely, estimate, with given x and y, approximate values of true anomalies we need. For every point, say in the first quadrant (x, y), we shall have two values of $v : v'$ and v''; for there exist two points of „intersection" of the pair of quasicomplanar orbits of the asteroids. For one of them we can expect corresponding proximity of orbits.

Needed diagram can be drawn at a sufficiently great scale, only for the first quadrant, owing to mentioned symmetry. Use will be made of this unique diagram for every considered pair of quasicomplanar asteroids, which was not the case with graphical determination in **(1)**. So, for a point (x, y) in the different quadrants, approximate values of true anomalies for an orbit are found by the schema:

Quadrant	x	y	v	
I	> 0	> 0	v'	v''
II	< 0	> 0	$180° - v''$	$180° - v'$
III	< 0	< 0	$360° - v'$	$360° - v''$
IV	> 0	< 0	$180° - v''$	$180° + v'$

We obtain, from (19), equations which determine the envelope of the system of v-straight lines

$$\left.\begin{aligned} F(x, y, v) &= x \cos v - y + \sin v = 0, \\ \frac{\partial F}{\partial v} &= - x \sin v + \cos v = 0, \end{aligned}\right\} \tag{20}$$

or, in parametric form

$$x = \operatorname{ctg} v, \qquad y = \operatorname{cosec} v. \tag{21}$$

The condition being always fulfilled

$$x_v'^2 + y_v'^2 \neq 0,$$

there are no singularities, and the envelope is a hyperbola

$$x^2 - y^2 = - 1. \tag{22}$$

If point (x, y) is more distant from hyperbola (22) and coordinate origin, the deviation between v-lines grows, which allows more precision in the evaluation, from the diagram, of the needed initial value of the true anomaly. In fig. 1 is given the idea of the diagram for graphical determination of approximate values of the true anomaly, corresponding to equations (16) or (18); for the latter it is easy to take the value, obtained from the graphic, as the value for $90° - v_i$, and then find the v_i needed, as we shall see from next example. On fig. 1 we have 1 *mm* — 0,025

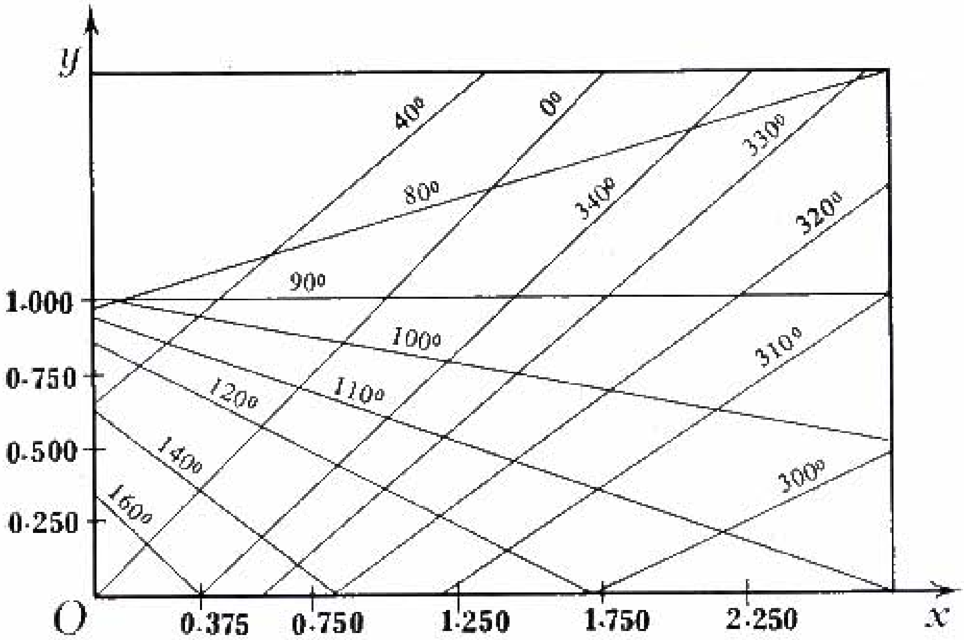

Fig. 1

for orthogonal coordinates, and v-lines are drawn for some choosen values. There are some technical difficulties to give here the whole diagram at the scale 1 *mm* — — 0,001 of the value of orthogonal coordinates in the first quadrant, of which we have made use for the effective application to a given pair of asteroids, but we shall give for it corresponding parts of the diagram. For those parts of the diagram which contain mutually closer lines, as for the lower left part, one can draw, on millimetre paper, the domain at a greater scale.

So we first form numerically, from given orbital elements of the quasi-complanar pair of asteroids, the system of equations (16) or (18), and from those two we choose the one which corresponds to less values of orthogonal coordinates. We can even form only respectively one of the equations of the two systems, for we can make use of equation (12), from which we obtain the second needed anomaly, if the first is given by the diagram, which gets a further reduction of the computation. Second, from the equation determined, of the form (19), with coordinates (x, y) from the diagram, using given schema for the quadrant, we estimate true anomalies.

4. We shall apply now the method of determination of true anomalies to asteroids previously considered, **(1)**, with orbital elements already used, which will allow us to compare the results obtained by the different procedures. It will be the quasicomplanar pair 589 *Croatia* and 1564 *Srbija*; for the first asteroid we shall use index 1 and for the second one index 2. We obtain for them from **(1)**:

$$e_1 = 0.0398, \qquad p_1 = 3.1295, \qquad e_2 = 0.2116, \qquad p_2 = 3.0082;$$

and using (10) and (11) we have

$$p = -0.1213, \qquad \pi = 12^\circ.686,$$

so that in our example equations (13) become

$$0.5263 \cos v_1 + 0.1454 \sin v_1 = -0.1213,$$
$$0.5454 \cos v_2 + 0.0263 \sin v_2 = -0.1213.$$

Corresponding equations (16) are

$$-0.834 = 3.620 \cos v_1 + \sin v_1,$$
$$-4.612 = 20.738 \cos v_2 + \sin v_2,$$

and equations (18) are

$$\left.\begin{aligned} -0.230 &= 0.276 \cos (90^\circ - v_1) + \sin (90^\circ - v_1), \\ -0.222 &= 0.048 \cos (90^\circ - v_2) + \sin (90^\circ - v_2). \end{aligned}\right\} \qquad (23)$$

Comparing the last two systems of equations, we conclude that system (23) is more convenient in finding approximate values of the proximity of true anomalies, because of less corresponding numerical values. From (23) we obtain the values of the coordinates

$$\xi_1 = 0.276, \qquad \eta_1 = -0.230,$$
$$\xi_2 = 0.048, \qquad \eta_2 = -0.222.$$

By means of the positions of these two points in the diagram of v-lines (at the scale 1 *mm* = 0.001, with intervals of 1° for true anomalies) we find, due to the form of the arguments in (23) and the reduction of the positions of points to corresponding positions in the first quadrant represented in the scheme given, these possible approximate values are (Fig. 2 and 3)

$$\left.\begin{aligned} v'_{10} &= 118^\circ.2, \qquad & v'_{20} &= 105^\circ.6; \\ v''_{10} &= 272^\circ.6, \qquad & v''_{20} &= 259^\circ.9. \end{aligned}\right\} \qquad (24)$$

So, e. g., we have computed v'_{10} from $90^\circ - v'_{10} = 180^\circ - 151^\circ.8$, where the value $151^\circ.8$ is obtained from the graphic. We could have shorten our computation by forming the first equation of the system (23) only, from which we should obtain only coordinates ξ_1 and η_1, by means of which we could determine from the diagram only v'_{10} and v''_{10} for the first orbit, and needed values for the second orbit can be obtained by means of equation (12): $v'_{20} = 105^\circ.5$ and $v''_{20} = 259^\circ.9$. We see that the first value only differs by a unity at the first decimal of the value previously obtained, which is at the limit of accuracy and is not of importance, especially because we are working with approximate values of v.

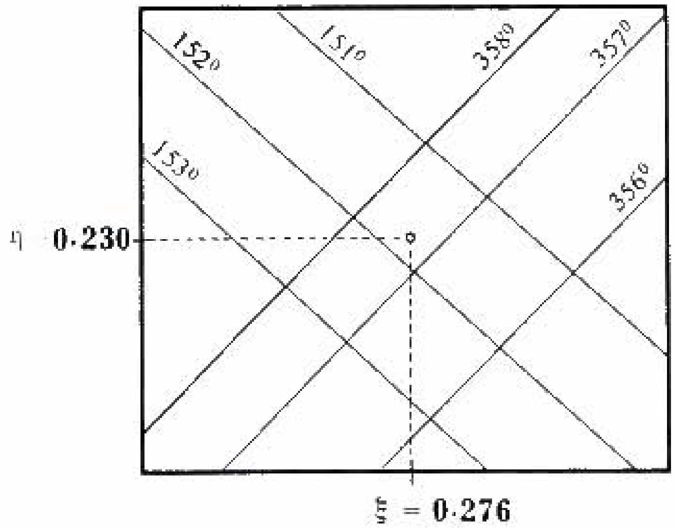

Fig. 2

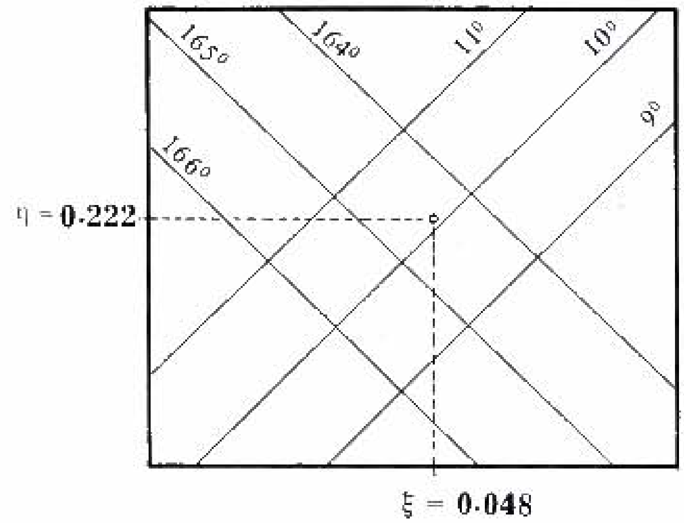

Fig. 3

The values of the true anomalies of the relative nodes of the orbits of chosen asteroids are

$$v_1 = 296^\circ.0, \quad v_2 = 283^\circ.4; \qquad v_1' = 116^\circ.0, \quad v_2' = 103^\circ.4, \tag{25}$$

where the first pair of values corresponds to the ascending relative node of the first orbit relatively to the second, and the second pair is for the descending relative node. The difference between these values for the two nodes is 180°, which allows us to obtain easily one value from the other.

Comparing the values of (24) and (25) for the differences of true anomalies we obtain, in the sense

$$\left.\begin{array}{l}\text{the first point — relative descending node: } \Delta v_1' = 2^\circ.2,\ \Delta v_2' = 2^\circ.2;\\ \text{relative ascending node — the second point: } \Delta v_1 = 23^\circ.4,\ \Delta v_2 = 23^\circ.5.\end{array}\right\} \quad (26)$$

From (26) we conclude that the values for the first point correspond to the proximity and that it is close to the relative descending node of the first orbit relatively to the second, here given differences being notably less.

So we take for approximate values of the true anomalies of the chosen pair of quasicomplanar asteroids 589 and 1564

$$v_{10} = 118^\circ.2, \qquad v_{20} = 105^\circ.6. \quad (27)$$

With these values and orbital elements and corresponding values $A_i, B_i, \ldots K_i, s_i$ as in **(1)**, and by means of equations (2), (1), (6), (7), (4) and (3) we find

$$\begin{array}{ll} f_0 = -0.0048156, & g_0 = 0.0046552;\\ \Delta v_{10} = 0^\circ.0980, & \Delta v_{20} = 0^\circ.0028;\\ v_{11} = 118^\circ.2980, & v_{21} = 105^\circ.6028;\\ f_1 = 0.0000014, & g_1 = -0.0000009;\\ \Delta v_{11} = -0^\circ.0003, & \Delta v_{21} = -0^\circ.0003;\\ v_{12} = 118^\circ.2977, & v_{22} = 105^\circ.6025. \end{array} \quad (28)$$

The corrections of true anomalies being 0°.0000 with values given above, so exact values for the positions in the proximity are $v_1 = v_{12}$, $v_2 = v_{22}$, which is the same as the result obtained in **(1)**.

We shall give the results for the same pair of asteroids, with orbital elements used here, obtained by the previous graphic way with corrections from **(1)**, in order to illustrate with an example the advantages of the method exposed here. We have had in **(1)**:

$$\begin{array}{ll} v_{10} = 118^\circ.6, & v_{20} = 105^\circ.4;\\ f_0 = 0.0256062, & g_0 = -0.0251122;\\ \Delta v_{10} = -0^\circ.3142, & \Delta v_{20} = 0^\circ.1909;\\ v_{11} = 118^\circ.2858, & v_{21} = 105^\circ.5909;\\ f_1 = -0.0000098, & g_1 = -0.0000089;\\ \Delta v_{11} = 0^\circ.0119, & \Delta v_{21} = 0^\circ.0116;\\ v_{12} = 118^\circ.2977, & v_{22} = 105^\circ.6025. \end{array}$$

There we had also for the next corrections of true anomalies 0°.0000, so that exact needed values were: $v_1 = v_{12}$, $v_2 = v_{22}$.

The differences between exact and approximate values of true anomalies of the proximity are:

by the new method $v_1 - v_{10} = 0^\circ.098,\ v_2 - v_{20} = 0^\circ.003,$

by the method **(1)** $v_1 - v_{10} = -0^\circ.302,\ v_2 - v_{20} = 0^\circ.203.$

The sums of the squares of these differences are:

here 0.010, from **(1)** 0.132.

5. The method of determination of the approximate values of true anomalies of quasicomplanar asteroids proposed here is very useful in the case when we take, as arguments for solving the problem of proximity, the true anomalies. This method is particularly efficient when computing the shortest distances for a great number of pairs or groups of quasicomplanar asteroids, because necessary initial — approximate data of true anomalies is obtained from one diagram, so there is no need to draw the orbits for every pair as in **(1)**. We can find v_{10} only from the diagram, and then determine v_{20} by means of (12), which shortens the computation. Initial values, obtained in that way, are closer to exact ones; equations (1) are satisfied with more accuracy, and the process of numerical corrections converges more rapidly. In the example given the sum of squares of the differences between the exact and the approximate values of true anomalies is about 13 times less when applying this method instead of the previous one. If we would compute the true anomalies with three decimals, as in *E. M. P.* for the angular values of orbital elements, our example shows that with initial values, obtained with more accuracy, as in this paper, it is enough to compute only the first numerical corrections, which is not the case with the results in **(1)**.

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REFERENCES

1. Лазович, Й. П. 1967, Определение кратчайшего расстояния между орбитами астероидов с малым взаимным наклоном, *Бюллетень Института теоретической астрономии*, XI, **1**, 57 — 62, Академия наук СССР, Ленинград.

2. Simovljević, J. L. Određivanje približnih vrednosti ekscentričnih anomalija kvazikomplanarnih planetoida u proksimitetu. *Privatno saopštenje.*

3. Mišković, V. V. *Predavanja iz opšte astronomije.*

4. Lazović, J. 1970, Pairs of quasicomplanar asteroids, *Publications of the Department of Astronomy*, Faculty of Sciences. University of Beograd, No. **2**, 11 — 23, Beograd.

5. Lazović, J. 1970, Groups of qasicomplanar asteroids, *Publications of the Department of Astronomy*, Faculty of Sciences, University of Beograd, No. **2**, 25 — 27, Beograd.