A NOTE ON SOME GENERAL RELATIONS BETWEEN THE ANOMALIES IN THE TWO-BODY PROBLEM

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In the paper (**1**) *R. Broucke* and *P. Cefola* propose four simple equations between the true, eccentric and mean anomaly in the Keplerian motion:

$$\operatorname{tg}\frac{1}{2}(v-E)=\frac{\sin v}{\beta+\cos v}=\frac{\sin E}{\beta-\cos E}, \tag{1}$$

$$v-M=2\operatorname{arc\,tg}\left(\frac{\sin E}{\beta-\cos E}\right)+e\sin E=$$

$$=2\operatorname{arc\,tg}\left(\frac{\sin v}{\beta+\cos v}\right)+\frac{e\sqrt{1-e^2}\sin v}{1+e\cos v}, \tag{2}$$

$$\beta=\frac{1}{e}\left(1+\sqrt{1-e^2}\right).$$

The equations (1) may be used in numerical calculations, with no trouble when v or E are near to $\pm 90°$, because we have always that $\frac{1}{2}(v-E)<90°$. The equations (2) are formally very simple in comparison with the Fourier series usually used. Meanwhile, all the relations (1) and (2) are special cases of the general expressions for so called first class anomalies in the two-body problem. The first idea for such a generalisation is due to *M. F. Subbotin*, in the paper (**2**), and later it has been enlarged and completed by the author of this note in (**3**).

The equation for the definition of the first class anomaly W,

$$\operatorname{tg}\frac{1}{2}W=q\operatorname{tg}\frac{1}{2}E, \tag{3}$$

q being the characteristic parameter of W, directly gives that

$$W_2 - W_1 = 2 \operatorname{arc\,tg} \left(\frac{\sin W_2}{C + \cos W_2} \right) = 2 \operatorname{arc\,tg} \left(\frac{\sin W_1}{C - \cos W_1} \right), \qquad (4)$$

$$C = \frac{q_2 + q_1}{q_2 - q_1},$$

or, in the case of three different anomalies W_1, W_2 and W_3 (with parameters q_1, q_2 and q_3 respectively),

$$W_2 - W_1 = 2 \operatorname{arc\,tg} \left(\frac{F \sin W_3}{1 + G \cos W_3} \right), \qquad (5)$$

$$F = \frac{q_3 (q_2 - q_1)}{q_3^2 + q_1 q_2}, \qquad G = \frac{q_3^2 - q_1 q_2}{q_3^2 + q_1 q_2}.$$

The equations (1) follow immediately from (4) for the parameters characterizing the true ($q_2 = \sqrt{(1 + e^2)/(1 - e^2)}$) and the eccentric ($q_1 = 1$) anomaly.

In some cases is better to introduce, instead of q, the parameter γ as the characterizing parameter of the first class anomalies, so that

$$q = \sqrt{\frac{\pm A + \gamma}{\pm A - \gamma}}, \qquad A = + \sqrt{e'^2 + \gamma^2}, \qquad e'^2 = 1 - e^2. \qquad (6)$$

Then the upper sign corresponds to the „fundamental" anomaly W in (3), with characteristic parameter γ:

$$\operatorname{tg} \frac{1}{2} W = \sqrt{\frac{A + \gamma}{A - \gamma}} \operatorname{tg} \frac{1}{2} E = q \operatorname{tg} \frac{1}{2} E,$$

and the lower sign gives the „reciprocal" anomaly W', for the same parameter γ:

$$\operatorname{tg} \frac{1}{2} W' = \sqrt{\frac{A - \gamma}{A + \gamma}} \operatorname{tg} \frac{1}{2} E = \frac{1}{q} \operatorname{tg} \frac{1}{2} E.$$

In such a way we obtain from (4) and (6) that

$$W - W' = 2 \operatorname{arc\,tg} \left(\frac{\gamma \sin W}{A + \gamma \cos W} \right),$$

or, by the aid of (5),

$$W - W' = 2 \operatorname{arc\,tg} \left(\frac{F_1 \sin W_1}{1 + G_1 \cos W_1} \right),$$

$$F_1 = \frac{2\gamma}{e'}\frac{q_1}{q_1^2+1} = \frac{\gamma}{\gamma_1}, \qquad G_1 = \frac{q_1^2-1}{q_1^2+1} = \frac{\gamma_1}{\pm A_1},$$

when W_1 is another anomaly, different from W and W'. The last relation is the simplest one when W_1 is the eccentric anomaly ($q_1 = 1$):

$$W - W' = 2 \operatorname{arc\,tg}\left(\frac{\gamma}{e'}\sin E\right).$$

The finite form of the relation between the first class anomalies W and the time (i. e. the mean anomaly M) is the general Kepler's equation

$$M = 2 \operatorname{arc\,tg}\left(\sqrt{\frac{\pm A - \gamma}{\pm A + \gamma}}\,\operatorname{tg}\frac{1}{2}W\right) - \frac{e\,e'\sin W}{\pm A + \gamma\cos W}, \tag{7}$$

as it has been deduced in (**3**). Combining this equation with (3) and making use of (6), it is simple to put it in the form

$$W - M = 2 \operatorname{arc\,tg}\left(\frac{\sin W}{B + \cos W}\right) + \frac{e\,e'\sin W}{\pm A + \gamma\cos W},$$

$$B = \frac{1}{\gamma}(\pm A + e').$$

From this equation we obtain the second one in (2) ($W = v$ for $\gamma = e$). If W_1 and W_2 are two different anomalies, we have

$$W_1 - M = 2 \operatorname{arc\,tg}\left(\frac{P\sin W_2}{Q + \cos W_2}\right) + \frac{e\,e'\sin W_2}{\pm A_2 + \gamma_2\cos W_2},$$

$$P = \frac{q_2(q_1 - 1)}{q_2^2 - q_1}, \qquad Q = \frac{q_2^2 + q_1}{q_2^2 - q_1}.$$

This equation has the simplest form when $W_2 = E$, i. e. $\gamma_2 = 0$:

$$W_1 - M = 2 \operatorname{arc\,tg}\left(\frac{\sin E}{Q - \cos E}\right) + e\sin E,$$

$$Q = \frac{q_1 + 1}{q_1 - 1} = \frac{1}{\gamma_1}(\pm A_1 - e')$$

The special case for this is the first equation (2), when $W_1 = v$, i. e. $\gamma_1 = e$.

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