Publications of the Department of Astronomy – Beograd, № 17, 1989

UDC 521:51-72;-73 OSP

PSEUDO-HARMONIC FLUCTUATIONS OF A RANDOM PROCESS GENERATED BY LABROUSTE TRANSFORMS OF CLASS $\Pi(s)_L$

D. Djurović, Institute of Astronomy, University of Belgrade

Received May 22,1989

Summary: The Labrouste transforms (Labrouste and Labrouste, 1940) of class $\Pi(s)_L$ generate pseudo-harmonic fluctuations of a random process because the results in transformed series are mutually correlated,even in the cases when original series represent a white noise.

In the case of the analyzed $\Pi(s)_L = (s_6 s_8)_{12}$ transform, appropriate for the filtering of the known 50-day oscillation of the length of the day and the atmospheric angular momentum, the amplitude A of the pseudo-harmonic fluctuation does not exceed 5% of the standard deviation σ of original data.

D. Djurović: Labrustove transformacije (Labrouste and Labrouste, 1940) klase $\Pi(s)_L$ generišu pseudo-harmonijske fluktuacije slučajnog procesa zato što su rezultati u transformisanim serijama uzajamno korelisani, čak i u slučajevima kada originalne serije predstavljaju beli šum.

U slučaju analizirane $\Pi(s)_L = (s_6 s_8)_{12}$ transformacije, koja je pogodna za filtriranje poznate 50-dnevne oscilacije dužine dana i atmosferskog ugaonog momenta, amplituda A pseudo-harmonijske fluktuacije ne prelazi 5% standardne devijacije σ u originalnoj seriji.

1. INTRODUCTION

The well-known methods of the spectral/periodogram analyses, such as the Fourier transforms, the Blackman-Tukey method (Blackman and Tukey, 1958), and the maximum entropy method of spectral analysis (Ulrych and Bishop, 1975), the instantaneous frequency analysis (Griffiths 1975) and the others, based on a hypothesis that the signal composition is strictly sinusoidal are not always convenient for the studies of many phenomena observed in astronomy, meteorology and other sciences. The transformation of time function into the frequency function leads to the lost of information on its "local"features, on so-called perturbations. An example well illustrating the non-conformity of the above methods could be the known quasi-biennial irregularity of the Earth rotation (Iijima and Okazaki, 1966) or the quasi-biennial stratospheric oscillation (QBO) (see, for example Labitzke and Van Loon, 1988), whose deviations from the sinusoidal form are so large that the application of these methods is not recommendable.

The perturbations of sinusoidal signals are frequently the main subject of research. Several classes of Labrouste transforms, especially of the class $\Pi(T) = T_p T_q T_r \ldots$ and the class $\Pi(s) = (s_p s_q s_r \ldots)_L$, with a convenient choice of parameters $p, q, r, \ldots$, yield the possibility to change the filter selectivity and obtain that the useful signal pass the filter, preserving its small perturbations.

The users of Labrouste transforms should be aware of the fact that they generate pseudo-harmonic fluctuations at frequencies of amplification. The estimation of amplitudes of these fluctuations, being the subject of the present paper, is important because we need to know whether the filter output signal is generated by the filter itself, or it really exists in the studied phenomenon.

2. PSEUDO-HARMONIC FLUCTUATIONS GENERATED BY LABROUSTE TRANSFORMS OF $\Pi(s)_L$ CLASS

In the present paper only the Labrouste transform $\Pi(s)_L = (s_6 s_8)_{12}$ will be discussed because it has been used in an earlier work of Djurović and Paquet (1988). An analysis of other transforms would significantly enlarge our work without a possibility to attempt the conclusions of general validity.

During the last few decades Labrouste transforms have rarely been used. On the other hand, the literature on them is not easily available to a wide circle of specialists. For this reason a brief introduction in the theory of $\Pi(s)_L$ transforms is given in Appendix.

In order to simplify the discussion of the origin of the pseudo-harmonic fluctuations, generated by these transforms, only the single $(s_4)_L$ transform of the white noise, given as the series of equidistant data,will be analyzed.

For $(s_4)_L$ transform the equation (A.1) in Appendix is:

$$Y_i = (X_i + X_{i+2L} + X_{i-2L} + X_{i+4L} + X_{i-4L}) - (X_{i+L} + X_{i-L} + X_{i+3L} + X_{i-3L}). \quad (1)$$

If we compare Y_i with $Y_{i+1}, Y_{i+2}, \ldots, Y_{i+j}$ we can remark that they have no common addends if $j \neq k * L$ $(k = 1, 2, \ldots)$. In other words, they are not correlated. However, for $j = k * L$ and $k \leq 2p$ (in given example $2p = 8$) Y_i and Y_{i+j} are correlated. So, for $j = L$ and $j = 2L$, then it is:

$$Y_{i+L} = -Y_i + X_{i+5L} + X_{i-4L},$$
$$Y_{i+2L} = Y_i + X_{i+6L} + X_{i-3L} - X_{i-4L} - X_{i-3L}.$$

As seen, Y_i and Y_{i+L} are in a negative correlation (anti-correlation), while Y_i and Y_{i+2L} are in a correlation and so on, alternatively. It is also easily seen that Y_i and Y_{i+L} have only 2 uncommon addends, Y_i and Y_{i+2L} have 4 uncommon addends, $\ldots Y_i$ and Y_{i+qL} have $2q$ uncommon addends. Therefore, the covariance function of the random process $Y(t)$ can be written as:

$$C(\tau) = (-1)^k * C_1(\tau) * \delta(\tau - kL), \quad (2)$$

where $C_1(\tau)$ is the absolute value of $C(\tau)$ and $\delta(\tau - kL)$ is the Dirac function.

It is clear from (2) that $C(\tau)$ and the corresponding spectral density $S(\omega)$ are different from these functions for untransformed white noise:

$$C^*(\tau) = \begin{cases} \sigma^2 & , \text{for } \tau = 0, \\ 0 & , \text{for } \tau \neq 0, \end{cases}$$

$$S(\omega) = \sigma^2/\pi.$$

where σ^2 is the white noise variance.

The spectral density estimate of the random process $Y(t)$ for the set of frequencies $f_m = m * f_0$, where $f_0 = 1/4pL$ and $m = 1, 2, \ldots, 2p$, can be computed by using the next formula (e.g.Blackman 1965):

$$S(f_m) = \Delta\tau[C(0) + C(2pL) * \cos m\pi + 2\sum_{q=1}^{2p-1} C(qL) * \cos(qm\pi/2p)] =$$

$$= L[C(0) - (-1)^m C_1(2pL) + \sum_{q=1}^{2p}(-1)^q C_1(qL) * \cos(qm\pi/2p)]. \qquad (3)$$

The upper limit of f_m, the Nyquist frequency, is $f_N = 1/2L$.

Since the number of uncommon X_i in the expressions for Y_i and Y_{i+qL} is $2q$, it can be easily proved that $C_1(qL)$ linearly decreases when q increases:

$$C_1(qL) = C(0) + q * \Delta C$$

where $\Delta C = -C(0)/(2p+1)$.

By the substitution in (3) one obtains:

$$S(f_m) = L\left[C(0) - (-1)^m C_1(2pL) + 2C(0)\sum_{q=1}^{2p}(-1)^q \cos(qm\pi/2p) + \right.$$

$$\left. +2\Delta C\sum_{q=1}^{2p}(-1)^q q\cos(qm\pi/2p)\right] \approx$$

$$\approx L[C(0) + 2C(0) * S_1 + 2\Delta C * S_2].$$

Since it is:

$$C_1(2pL) = C(0) - C(0) * 2p/(2p+1) \approx 0,$$

and sums S_1 and S_2 are defined through the expressions:

$$S_1 = \sum_{q=1}^{2p}(-1)^q \cos qx,$$

$$S_2 = \sum_{q=1}^{2p}(-1)^q q * cosqx,$$

where $x = m\pi/2p$, One can obtain the next expressions for S_1 and S_2 by using the geometric progression and the Moivre formula:

$$S_1 = \left[\cos(4p+1)\frac{x}{2} - \cos\frac{x}{2}\right]/2\cos\frac{x}{2},$$

$$S_2 = \left[-(4p+1)\sin(4p+1)\frac{x}{2} + \sin\frac{x}{2}\right]/4\cos\frac{x}{2} +$$
$$+ \left[\sin\frac{x}{2} * cos(2p+1)\frac{x}{2}\cos px\right]/2\cos^2\frac{x}{2}.$$

If m is even, the following relations are valid:

$$S_1 = 0,$$

$$S_2 = p,$$

$$S(f_m) = L[C(0) + 2p\Delta C] =$$
$$= LC(0)[1 - 2p/(2p+1)] > 0.$$

As seen $S(f_m)$ does not depend on m or it is the same for all even values of m. In other words, $S(f_m)$ is a flat spectrum, even when $Y(t)$ has the components with frequencies f_m.

If m is odd, S_1 and S_2 satisfy the next relations:

a) $S_1 = -1.0$,

b) $S_2 < -(p+1)$, for each $m \geq (p+1)$,

c) S_2 depends on m: $S_2 = S_2(m)$.

Spectral density for m odd is:

$$S^*(f_m) = L[-C(0) + 2S_2(m) * \Delta C].$$

Therefore, in the case of m odd the transformation $\Pi(s)_L$ is selective.

After the substitution: $\Delta C = -C(0)/(2p+1)$ in the equations for $S(f_m)$ and $S^*(f_m)$ one obtains:

$$S^*(f_m)/S(f_m) = -[2p+1+2S_2(m)].$$

As the Labrouste transforms of the class $\Pi(s)_L$ enlarge the amplitudes of the sinusoids whose periods are $P_m = 2L/(2m+1)$, $m = 1,3,5,\ldots$ by a factor ϱ (equ. A.6 in Appendix) the following condition must be fulfilled:

$$2p+1+2S_2(m) < -1.$$

From the second property of $S_2(m)$ it follows that the last inequality is satisfied if $m \geq p+1$. Since $m = f_m/f_0 = P_0/P_m$ and $P_0 = 4L$, it must be $P_m \leq 4pL/(p+1)$. The smallest amount of the right-hand side is $4/3 * 2L$. Therefore, it is fulfilled for all odd subharmonics of $2L$.

From the above discussion it follows that Labrouste transforms generate pseudo-harmonic fluctuations because the data in transformed series $\{Y_i\}$, $\{Z_i\},\ldots$ are not mutually independent, even in the case when original series $\{X_i\}$ represent a purely white noise.

A theoretical estimate of amplitudes A of pseudo-harmonic fluctuations generated by multiple Labrouste transforms is not easy. Because of that in the present paper it will be done experimentally.

The data treated are gaussian series of $n \subset [500, 10000]$ random numbers with zero mean and standard deviations $\sigma \subset [10, 1000]$. Such a wide range of σ is taken with the aim to establish more accurate relation between A and σ.

The mentioned series are transformed according to the algorithm $\Pi(s)_L = (s_6 s_8)$ and, after that, their periodograms are computed by the method of Fourier transforms. From the results of periodogram analysis we have remarked the following:

1. For a given amplification period ($P = 2L/(2k+1)$) and a given n, the ratio $R = A/\sigma$ is constant. To illustrate that, in the Table 1 are presented the results for $n = 650$. This n is close to the number of data in series treated in Djurović and Paquet (1988).

According to the above conclusion, for a given n the mean R for $\sigma \subset [10, 1000]$ is considered as a "definitive" one (see Table 2).

The largest R obtained from more than 200 series is under 0.05 (see column R_{max} in Table 2).

2. The ratio R is not the same for all periods of amplification (see Table 2). To the smaller period corresponds the smaller R.
3. The ratio R also depends on n with a tendency of asymptotic change when n increases (Table 2).

Since a white noise can be generated by a superposition of a large number (theoretically, infinite number) of sinusoids with random and mutually independent phases, with a uniform distribution of amplitudes and continuous distribution of frequencies (e.g.Blackman et al.1958), the amplitudes of components are (statistically) larger for σ larger. On the other hand, for a given P the amplification factor of the transform (equ.A.6) is constant. Therefore, the relation $A/\sigma =$ const. is a consequence of the nature of the white noise.

Decreasing of R when n increases can be explained by the fact that the noise enlarges the amplitudes A for an amount ΔA whose mathematical expectation is:

$$E(\Delta A) = \sigma_z \sqrt{(\pi/(2n+1)},$$

where σ_z is standard deviation of the noise in transformed series.

Let A_0 be the true amplitude. The amplitude computed by Fourier transforms is :

$$A = A_0 + \Delta A = A_0 + \text{const}/\sqrt{(2n+1)}.$$

Evidently, when n increases the second addend and, consequently, A decreases.

Concerning the dependence of R on the period of amplification, this dependence is probably due to the fact that the selectivity of transform $\Pi(s)_L$ is

TABLE 1

The ratio $R = A/\sigma$ in function of σ for different amplification period and the series of $n = 650$ random numbers.

σ	P: 3.4	4.8	8.0	24.0
20	0.001	0.007	0.16	0.12
40	2	4	10	12
60	2	7	14	15
80	2	12	9	16
100	2	8	8	22
120	3	4	10	19
140	2	7	12	20
160	2	10	15	15
180	2	12	11	12
200	2	8	16	15
250	3	10	14	18
300	2	9	8	14
350	2	13	9	14
400	2	12	14	11
450	2	7	10	11
500	1	10	12	11
550	2	6	12	11
600	2	9	13	17
650	2	8	16	15
700	2	6	14	14
750	1	7	10	16
800	3	7	12	14
mean	2	8	12	15
σ_1	0.5	2	3	3

σ_1 - standard deviation of R

larger for smaller periods (see Fig.1 in Appendix). Because of that the number of white noise components amplified by the transform is smaller and, definitively, the resulting pseudo-harmonic fluctuation has a smaller amplitude.

TABLE 2

Mean and extreme values of $R = A/\sigma$ in units 0.0001.

$P =$	3.4			4.8			8.0			24.0		
n	R		R_{max}	R		R_{max}	R		R_{max}	R		R_{max}
500	62	±4	230	140	±35	390	92	±26	305	116	±8	264
1000	40	7	97	72	15	281	71	19	172	117	12	410
2000	26	2	64	51	7	178	58	8	174	88	14	290
3000	20	2	78	38	2	142	40	12	193	70	16	270
4000	15	3	47	32	1	125	36	7	124	54	14	213
5000	11	3	29	21	2	73	30	8	76	42	7	160
6000	11	1	54	21	3	47	32	7	117	41	10	132
7000	12	4	35	18	2	64	32	9	107	37	10	108
8000	11	1	57	19	1	65	25	4	64	39	10	114
9000	10	1	30	17	2	52	24	7	89	30	6	113
10000	10	1	38	24	4	138	25	7	70	31	9	106

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This work is part of the research project suported by the Fund for Scientific Research of the S.R. Serbija.

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APPENDIX

Let $X = X(t)$ be a time dependent function digitized as a series:

$$X_0, X_1, \ldots, X_n$$

where $X_k = X(t_k)$, $(k = 0, 1, 2, ..., n)$.

A single Labrouste transform $(s_p)_L$, where the parameters (p, L) are integers, selected according to the practical constraints (this aspect will be explained later on), consists of computing a new series $\{Y_i\}$ according to the expression:

$$Y_i = X_i + X_{i+2L} + X_{i-2L} + X_{i+4L} + X_{i-4L} + \ldots + X_{i+pL} + X_{i-pL} -$$

$$-[X_{i+L} + X_{i-L} + X_{i+3L} + X_{i-3L} + \ldots + X_{i+(p-1)L} + X_{i-(p-1)L}], \qquad (A.1)$$

if p is even, or:

$$Y_i = X_i + X_{i+2L} + X_{i-2L} + \ldots + X_{i+(p-1)L} + X_{i-(p-1)L} -$$

$$-[X_{i+L} + X_{i-L} + \ldots + X_{i+pL} + X_{i-pL}], \qquad (A.2)$$

if p is odd. Moreover, the index $[i]$ must be such that $pL < i < n - pL$.

The same procedure applied to $\{Y_i\}$ series defines a double Labrouste transform $\Pi(s) = (s_p s_q)_L$ and generate the series $\{Z_i\}$:

$$Z_i = Y_i + Y_{i+2L} + Y_{i-2L} + \ldots + Y_{i+qL} + Y_{i-qL} -$$

$$-[Y_{i+L} + Y_{i-L} + Y_{i+3L} + Y_{i-3L} + \ldots + Y_{i+(q-1)L} + Y_{i-(q-1)L}], \qquad (A.3)$$

if q is even, or:

$$Z_i = Y_i + Y_{i+2L} + Y_{i-2L} + \ldots + Y_{i+(q-1)L} + Y_{i-(q-1)L} -$$

$$-[Y_{i+L} + Y_{i-L} + Y_{i+3L} + Y_{i-3L} + \ldots + Y_{i+qL} + Y_{i-qL}], \qquad (A.4)$$

if q is odd.

By the same way a triple Labrouste transform $\Pi(s)_L = (s_p s_q s_r)_L$ can be performed.

Let suppose that the function $X(t)$ is composed of N sinusoids:

$$X(t) = \sum_{j=1}^{N} A_j \sin(2\pi f_j t + \alpha_j). \qquad (A.5)$$

Each Labrouste transform $(s_u)_L$, for $u = p, q, r, \ldots$ modifies the amplitude of sinusoids by a factor :

$$\varrho(u, f) \equiv \varrho(u) = \cos[(2u+1)L\pi/P] : \cos(\pi L/P). \qquad (A.6)$$

Therefore, the total modification of the amplitude by a multiple Labrouste transform $(s_p s_q s_r \ldots)_L$ is:

$$\varrho = \varrho(p) * \varrho(q) * \varrho(r) \ldots$$

Having in mind that ϱ is also dependent of the frequency, a Labrouste transform is selective; for this reason it is sometimes named "filter".

For $P_k = 2L/(2k+1)$, $k = 0, 1, 2, \ldots, m$, the function ϱ reaches its maximum values:

$$\varrho_{\max} = (2p+1)(2q+1)(2r+1)\ldots$$

Two examples of the function ϱ are presented in Fig.A.1.

The choice of the parameter L is determined by the period P of the sinusoid selected to be extracted from $X(t)$. It is such that $2L$ is the integer closest to P.

To obtain the best damping of the other sinusoids the parameters $p, q, r \ldots$ are to be chosen according to the next relations:

$$2q/3 \le p \le q - 1, \ \text{for } (s_p s_q)_L,$$

$$\left.\begin{array}{l} 3r/5 \le p \le q-1 \\ 3r/4 \le p \le r-1 \end{array}\right\}, \ \text{for } (s_p s_q s_r)_L,$$

$$\left.\begin{array}{l} 4s/7 \le p \le q-1 \\ 2s/3 \le q \le r-1 \\ 4s/5 \le r \le s-1 \end{array}\right\}, \ \text{for } (s_p s_q s_r s_s)_L.$$

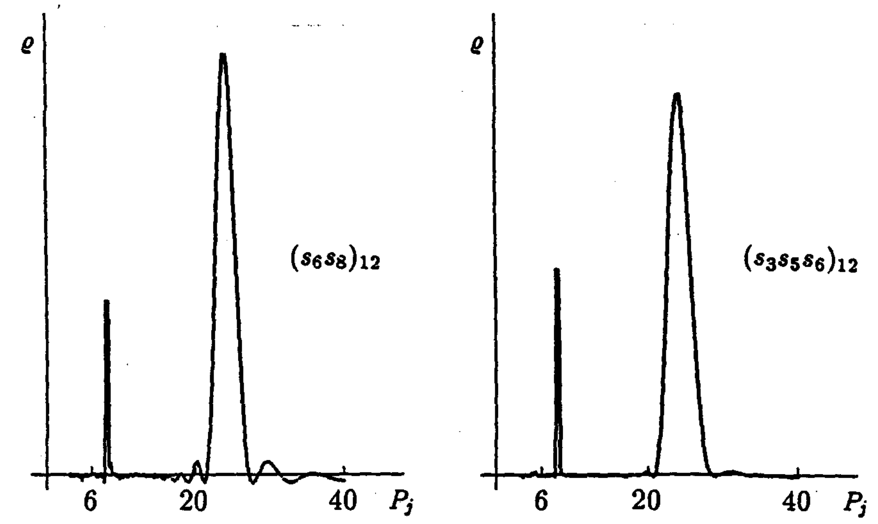

Fig A.1 Amplitude modulation by the labrouste filtering (s_ps_q)

For a single Labrouste transform $(s_u)_L$, ϱ is equal zero for P simultaneously satisfying two conditions:

$$P = (2u+1)2L/(2k+1) \text{ and } P \neq 2L/(2k'+1),$$

where k and k' are arbitrary chosen integers.

Therefore, the choice of u depends on the degree of amplification ($\varrho_{\max} = 2u+1$) and on the number of data accepted to be lost, at two ends of the record.

For a multiple transform $(s_ps_qs_r\ldots)_L$ the total number of data point lost is $2L*(p+q+r+\ldots)$.