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PSEUDO-HARMONIC SIGNAL VARIATIONS GENERATED BY LABROUSTE TRANSFORMS OF THE CLASS $\Pi(T)$

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Summary: The Labrouste transforms of the class $\Pi(T)$ generate pseudo-harmonic fluctuations of a random process because the series of mutually correlated observations $\{Y_i\}$ are obtained even when the series of white noise $\{X_j\}$ are transformed.

In the case of $\Pi(T) = T_2T_3T_3T_4$ transform amplitudes A of pseudo-harmonic fluctuations satisfy the relation: $A\ /\sqrt{\sigma} = c\ (n)$, where σ is the standard deviation of results in the original series and $c\ (n)$ is a parameter depending on the number of observations n. When n is increased, $c\ (n)$ decreases. For example, if $n = 500$, $c\ (n) = 0.18$ and if $n = 10000$, $c\ (n) = 0.03$.

D. Djurović: REZIME - Labrustove transformacije klase $\Pi(T)$ generišu pseudo-harmonijske fluktuacije slučajnog procesa zato što su rezultati $\{Y_i\}$ u transformisanim serijama uzajamno korelisani čak i kada se transformišu serije belog šuma $\{X_j\}$.

U slučaju transformacije $\Pi(T) = T_2T_3T_3T_4$ amplitude A pseudo-harmonijskih fluktuacija zadovoljavaju relaciju: $A/\sqrt{\sigma} = c(n)$, gde je σ standardna devijacija rezultata u originalnoj seriji, a $c(n)$–parametar koji zavisi samo od broja posmatranja n. Kad n raste, $c(n)$ opada. Na primer, za $n = 500$, $c(n) = 0.18$, a za $n = 10000$, $c(n) = 0.03$.

1. INTRODUCTION

Selective Labrouste transforms (LT) of $\Pi(T) = T_pT_qT_r\dots$ class make possible a separation of a component $s_0 = s_0(t)$ at *a priori* known frequency f_0

from a complex polyharmonic signal $X = X(t)$, where the argument t is usually time. If components with frequencies close to f_0 do not exist in $X(t)$, then by suitable choice of parameters $p, q, r \ldots$, it can be achieved that the small perturbations of $s_0(t)$, if they exist, appear in the filter output. Those perturbations are frequently the main subject of the signal analysis.

The known methods of spectral/periodogram analysis yield a transformation of the time function $X(t)$ into a frequency function $S(f)$ in the course of which informations concerning the so-called local signal perturbations are lost. Even in the case of the known *instantaneous frequency analysis method* (Griffiths 1975) one can identify only great or long-term changes in the spectra because the filter reactions due to the smoothing of data can be too small and unappreciable.

The LT of the $\Pi(s)_L$ class and corresponding parasite effects have been already discussed in Djurović (1989), so the subject of the present paper will be the analysis of LT of the class $\Pi(T)$.

An example of $\Pi(T)$ transform efficiency may be found in an another paper of the same author (Stajić and Djurović, 1986), where the 50–80 days oscillation, whose amplitude is for few orders of magnitude below the noise level, has been extracted from the Wolf number observations.

A brief theoretical survey of class $\Pi(T)$ transforms is given in the Appendix. This is done because it is thought that the relevant literature is not easily available to a majority of readers. During the last few decades LT have been extremely rarely exploited and the original book (Labrouste and Labrouste 1940), in which their detailed description is given, was published long ago.

Since in many experimental works detections of weak signals in the presence of high noise level are frequently attempted, one of the essential questions concerning the efficiency of various mathematical filters is whether they produce pseudo-harmonic fluctuations and, if yes, how large their amplitudes are. The knowledge of this answer makes possible to distinguish whether the obtained result is generated by the filter itself or in the random process really exist a fluctuation observed at the filter output.

2. PSEUDO-HARMONIC FLUCTUATIONS GENERATED BY THE LABROUSTE TRANSFORM $\Pi(T)$

Suppose a time series $\{X_i\}$ $(i = 1, 2, ..., n)$, representing a purely random process or a *white noise*, is given. A series $\{Y_i\}$ from the equation A.1 (Appendix) is called a single LT of class $\Pi(T)$. Evidently, Y_i and Y_{i+j} are not

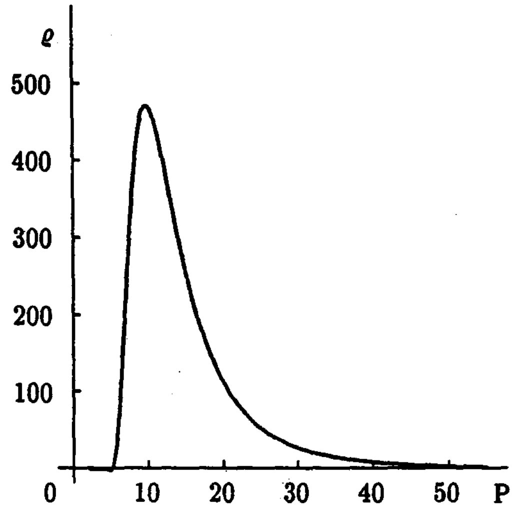

Fig. 1

mutually independent for every $j < m$. This means that by transforming the series $\{X_i\}$ its important property – the mutual independence of X_i vanishes and, consequently, the covariance $C(\tau)$ and the spectral density $S(\omega)$ are not constant, as in the case of the white noise, where it is:

$$C(\tau) = \begin{cases} \sigma_x^2 & ,\tau = 0, \\ 0 & ,\tau \neq 0, \end{cases}$$

$$S(\omega) = \sigma_x^2/\pi.$$

A theoretical estimate of $S(\omega)$ or of the amplitude of the $\Pi(T)$ pseudo-harmonic variation $Y(t)$ is not simple as in the case of LT of the class $\Pi(s)_L$ (Đurović 1989), because $C(\tau)$ is not a linear function of τ. This can be concluded without an exact proof, merely having in mind the fact that the number of uncommon X_i in Y_i and Y_{i+j} is $4j$, which would result in a linear decrease of $C(\tau)$, unless among these $4j$ addends $j-1$ pairs of opposite numbers appeared, resulting in a negative "acceleration" of $C(\tau)$. In the case of multiple $\Pi(T)$, the theoretical estimate of the spectral density/amplitude becomes more complicated. Because of that the scope of the present paper is limited to an experimental estimate of the transform $\Pi(T) = T_2T_3T_3T_4$ only. The selectivity curve of this transform (Fig. 1) has a maximum for the period $P = 9.7$ units. For this reason it is applicable for the filtering of the well-known 50-day oscillation of the length of day, the atmospheric angular momentum, the solar activity parameters and the geomagnetic indices (Djurović and Paquet 1989) from the 5-day data series.

The data for the experimental estimate of the amplitudes of pseudo-harmonic fluctuations generated by $\Pi(T) = T_2T_3T_3T_4$ transform represent the gaussian series of random numbers with the zero mean and different standard deviations $\sigma = 10, 20, \ldots, 100, 200, \ldots, 1000$. For each σ the series of different lenght ($n = 500, 1000, 2000, \ldots, 10000$ data) are analysed.

The periodogram analysis of the transformed series has been performed by the method of Fourier transforms. The results obtained by using this procedure are presented in the Tables 1-3 and in the Fig.2.

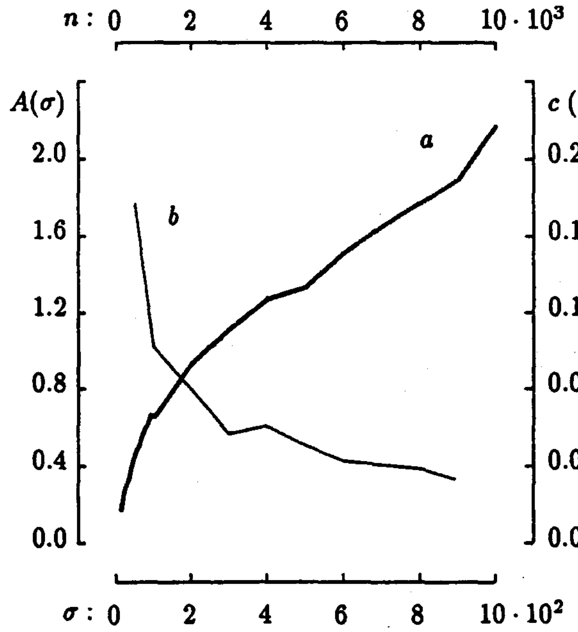

Fig. 2

The dependence of the amplitude A on σ is clearly pronounced (see Fig.2a). With the sufficient accuracy it could be approximated by the expression:

$$A = c\,(n)\sqrt{\sigma}, \tag{1}$$

where the parameter $c\,(n)$ is constant for n given (see Table 1).

It is shown in the Fig. 2b that $c\,(n)$ exponentially decreases when n increases. We have found that the approximation:

$$\log c\,(n) = \alpha + \beta\sqrt{n}$$

TABLE 1

$A/\sqrt{\sigma}$ in function of σ for $n = 5045$.

σ	$A/\sqrt{\sigma}$	σ	$A/\sqrt{\sigma}$
10	0.061	200	0.066
20	61	300	65
30	64	400	65
40	65	500	60
50	69	600	62
60	69	700	63
70	70	800	63
80	69	900	64
90	70	1000	69
100	67	mean	65

("log" is natural) could be satisfactory. By the least-squares method we have obtained:

$$\alpha = -0.52 \pm 0.15,$$
$$\beta = -0.293 \pm 0.018.$$

The raw and smooth $c(n)$ are presented in the Table 2.

TABLE 2

Raw $c(n)$ and corresponding computed $\hat{c}(n)$ in function of n.

n	$c(n)$		$\hat{c}(n)$	n	$c(n)$		$\hat{c}(n)$
500	0.177	±0.005	0.156	6000	0.043	±0.001	0.044
1000	102	2	114	7000	041	2	040
2000	080	3	081	8000	039	1	037
3000	057	2	066	9000	033	1	035
4000	061	3	056	10000	034	1	032
5000	051	3	049				

From the results in the Table 2 it follows that the square of amplitude/spectral density of the pseudo-harmonic fluctuation generated by $\Pi(T) =$

$T_2T_3T_3T_4$ transform is under 4% of the standard deviation σ. With the increasing n, A decreases. So, for $n > 2000$ it is less than 1%.

The periods P of the pseudo-harmonic fluctuations vary in a relatively wide range. This follows from the results in the Table 3, where the mean (P_m) and extreme ($P_{\min}, P_{\max}$) periods are presented.

TABLE 3

Mean and extreme periods of the pseudo-harmonic fluctuations

n	P_m		$P_{\min}$	$P_{\max}$
500	10.1	+0.3	8.8	12.2
1000	10.4	1	9.5	10.8
2000	9.2	2	7.8	11.0
3000	10.2	4	8.8	13.2
4000	10.5	1	10.0	11.2
5000	10.4	3	8.5	12.0
6000	10.9	3	9.0	12.8
7000	10.7	2	9.8	12.0
8000	10.5	2	9.8	11.8
9000	10.7	3	9.2	12.8
10000	10.5	4	8.8	13.0
mean	10.4			

Bearing in mind that the theoretical period of amplification is 9.7 units, it could be assumed that P_m are biased. However, this assumption should be yet verified.

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APPENDIX

Let the series of equidistant data representing the function $X = X(t)$ be given by:

$$X_{-k}, X_{-k+1}, \ldots, X_0, X_1, \ldots, X_k,$$

with the corresponding arguments t_i, ordered so that $t_i < t_{i+1}$ $(i = -k, -k+1, \ldots, k)$. For the reason of simplicity, let $t_{i+1} - t_i = 1$.

Suppose that each X_i is replaced by:

$$DX_i(m) = X_{i+m} - X_{i-m},$$

where $i \in [-k+m, k-m]$ and m less than k, is a positive integer chosen arbitrarily. In the practical application $m \leq n$.

The number $M = 2m$ will be called *the maximum lag* of the transform.

By the addition of $DX_i(j)$ from $j = 1$ to m we obtain the new series $\{Y_i\}$:

$$\begin{aligned} Y_i &= DX_i(1) + DX_i(2) + \ldots + DX_i(m) = \\ &= \sum_{q=-m}^{q=m} (X_{i+q} - X_{i-q}). \end{aligned} \tag{A.1}$$

In Labrouste's notation (Labrouste et al. 1940) this transform is called *the transform of the class* T_m.

Suppose that $X(t)$ represents a function of the type:

$$X(t) = \sum a_j \sin(\omega_j t + \alpha_j).$$

It was shown in Labrouste and Labrouste (1940) that the amplitudes of sinusoidal components in $X(t)$ are modulated by a factor τ_m, depending on the lag m and the period $P_j = 2\pi/\omega_j$:

$$A_j = \tau_m * a_j,$$

where:

$$\tau_m = 2\frac{\sin((m+1)\pi/P_j)\sin(m\pi/P_j)}{\sin(\pi/P_j)}. \tag{A.2}$$

Evidently,the transform T_m has the selectivity features: several sinusoids are amplified, several - damped. The factor τ_m is called *the selectivity function*.

If the series $\{Y_i\}$ is retransformed in the same way as $\{X_i\}$, with the maximum lag $2p$, the new transform, symbolically written:

$$\Pi(T) = T_m T_p$$

is obtained. By the multiple transformation we obtain the transform:

$$\Pi(T) = T_m T_p T_q \dots$$

The amplitude factor of the above transform is:

$$\varrho = \tau_m \tau_p \tau_q \dots$$