UDC 523.24; 521.1/3 OSP

CHANGE IN THE MUTUAL DISTANCE OF CELESTIAL BODIES RESULTING FROM CHANGES IN SOME OF THEIR ORBITAL ELEMENTS

J. Lazović

Institute of Astronomy, Faculty of Sciences, Beograd

Received June 29, 1982

Summary. The formula is developed for the distance change of two celestial bodies, resulting from the changes in the perihelia arguments, longitudes of the ascending nodes and the inclinations of their orbits. For illustration, the formula is applied to a pair of minor planets at their closest approach.

J. Lazović, PROMENA MEĐUSOBNOG RASTOJANJA NEBESKIH TELA ZBOG PROMENA NEKIH NJIHOVIH PUTANJSKIH ELEMENATA — Izveden je izraz za promenu rastojanja između dva nebeska tela zbog promena argumenata perihela, longituda uzlaznih čvorova i nagiba njihovih putanja. Ilustrovan je na primeru para malih planeta oko njihovog proksimiteta.

The driving idea was to set up the formula giving the change in the mutual distance of two celestial bodies, resulting from the changes in some of their elliptical orbital elements, these changes being produced by the gravitational perturbing action. The distance of two celestial bodies is most directly affected by the position changes of their orbital planes, i. e. by $\Delta\Omega$ and Δi. To this, the effect of the changes $\Delta\omega$ in the perihelia arguments should be added. The foregoing statement is substantiated by the results of our earlier works (Lazović, Kuzmanoski, 1979, 1980, 1981). We will, therefore, arrest ourselves on finding the change in the proximity distance of two minor planets in terms of changes in the three orbital elements just specified. The corresponding formulae will now be developed differently than they were in Lazović (1964).

The heliocentric position vector of a celestial body, moving in an elliptic orbit around the Sun, is given by

$$\mathbf{r} = \xi \mathbf{P} + \eta \mathbf{Q}, \tag{1}$$

$$\left.\begin{aligned} \xi &= a(\cos E - e), \quad \eta = b \sin E, \\ b &= a\sqrt{1-e^2} = a\cos\varphi, \quad e = \sin\varphi \end{aligned}\right\} \tag{2}$$

are orbital coordinates in its orbital plane, themselves functions of the eccentric anomaly E. $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$ are unit vectors mutualy perpendicular, the first two being complanar with the orbital plane, while the third is perpendicular to it; **P** is pointed to the perihelion, $\mathbf{R} = [\mathbf{P}\ \mathbf{Q}]$. Expressed in terms of the elliptical orbital elements, in an orthogonal heliocentric ecliptic coordinate system, they assume the form

$$\left.\begin{aligned} \mathbf{P} &= \left\{\begin{array}{l} \cos\omega\ \cos\Omega - \sin\omega\ \sin\Omega\ \cos i \\ \cos\omega\ \sin\Omega + \sin\omega\ \cos\Omega\ \cos i \\ \sin\omega\ \sin i \end{array}\right. \\ \mathbf{Q} &= \left\{\begin{array}{l} -\sin\omega\ \cos\Omega - \cos\omega\ \sin\Omega\ \cos i \\ -\sin\omega\ \sin\Omega + \cos\omega\ \cos\Omega\ \cos i \\ \cos\omega\ \sin i \end{array}\right. \\ \mathbf{R} &= \left\{\begin{array}{l} \sin\Omega\ \sin i \\ -\cos\Omega\ \sin i \\ \cos i \end{array}\right. \end{aligned}\right\} \tag{3}$$

The relative position vector $\vec{\rho}$ of the celestial body j with respect to the body k, both moving in elliptical orbits around the Sun, is

$$\vec{\rho} = \mathbf{r}_j - \mathbf{r}_k. \tag{4}$$

The distance ρ between these two bodies can be calculated from its square

$$\begin{aligned} \rho^2 &= (\mathbf{r}_j - \mathbf{r}_k)^2 = (\mathbf{r}_j\,\mathbf{r}_j) - 2(\mathbf{r}_j\,\mathbf{r}_k) + (\mathbf{r}_k\,\mathbf{r}_k) = \\ &= (x_j - x_k)^2 + (y_j - y_k)^2 + (z_j - z_k)^2. \end{aligned} \tag{5}$$

The scallar vector products are indicated by small brackets. The ecliptic rectangular heliocentric coordinates $x_i,\ y_i,\ z_i,\ i = j,\ k$ of the two bodies are easily determined by the expressions (1), (2) and (3).

It is required to find the change $\Delta\rho$ in the mutual distance of the celestial bodies concerned, emerging as a result of the changes in the orbital elements above specified. Suppose the changes in the perihelia arguments, longitudes of the ascending nodes and the inclinations of orbits are small such that squares and higher powers of these changes might be neglected. Then, from (4) it follows

$$\Delta\vec{\rho} = \Delta\mathbf{r}_j - \Delta\mathbf{r}_k. \tag{6}$$

By scalar multiplying (4) and (6) we find

$$\Delta\rho = \frac{1}{\rho}\left\{(\mathbf{r}_j\,\Delta\mathbf{r}_j) + (\mathbf{r}_k\,\Delta\mathbf{r}_k) - (\mathbf{r}_j\,\Delta\mathbf{r}_k) - (\mathbf{r}_k\,\Delta\mathbf{r}_j)\right\}. \tag{7}$$

As a consequence of the cited changes in the orbital elements, the corresponding change in the heliocentric position vector is

$$\Delta\mathbf{r} = \frac{\partial\mathbf{r}}{\partial\omega}\Delta\omega + \frac{\partial\mathbf{r}}{\partial\Omega}\Delta\Omega + \frac{\partial\mathbf{r}}{\partial i}\Delta i,$$

or in the form

$$\Delta \mathbf{r} = \xi \, \Delta \mathbf{P} + \eta \, \Delta \mathbf{Q}, \tag{8}$$

as the elements ω, Ω and i in (1) do appear only in vectors **P** and **Q** (see e.g. Simovljević, 1977). For $\Delta\mathbf{P}$ and $\Delta\mathbf{Q}$ we have

$$\left.\begin{aligned} \Delta \mathbf{P} &= \frac{\partial \mathbf{P}}{\partial \omega} \Delta\omega + \frac{\partial \mathbf{P}}{\partial \Omega} \Delta\Omega + \frac{\partial \mathbf{P}}{\partial i} \Delta i, \\ \Delta \mathbf{Q} &= \frac{\partial \mathbf{Q}}{\partial \omega} \Delta\omega + \frac{\partial \mathbf{Q}}{\partial \Omega} \Delta\Omega + \frac{\partial \mathbf{Q}}{\partial i} \Delta i. \end{aligned}\right\} \tag{9}$$

From (3) we derive

$$\left.\begin{aligned}
\frac{\partial \mathbf{P}}{\partial \omega} &= \left\{\begin{matrix} -\sin\omega\,\cos\Omega - \cos\omega\,\sin\Omega\,\cos i \\ -\sin\omega\,\sin\Omega + \cos\omega\,\cos\Omega\,\cos i \\ \cos\omega\,\sin i \end{matrix}\right\} = \mathbf{Q}, \\
\frac{\partial \mathbf{P}}{\partial \Omega} &= \left\{\begin{matrix} -\cos\omega\,\sin\Omega - \sin\omega\,\cos\Omega\,\cos i \\ \cos\omega\,\cos\Omega - \sin\omega\,\sin\Omega\,\cos i \\ 0 \end{matrix}\right\} = (-P_y,\, P_x,\, 0) = [\mathbf{N}\,\mathbf{P}], \\
\frac{\partial \mathbf{P}}{\partial i} &= \left\{\begin{matrix} \sin\omega\,\sin\Omega\,\sin i \\ -\sin\omega\,\cos\Omega\,\sin i \\ \sin\omega\,\cos i \end{matrix}\right\} = \mathbf{R}\,\sin\omega, \\
\frac{\partial \mathbf{Q}}{\partial \omega} &= \left\{\begin{matrix} -\cos\omega\,\cos\Omega + \sin\omega\,\sin\Omega\,\cos i \\ -\cos\omega\,\sin\Omega - \sin\omega\,\cos\Omega\,\cos i \\ -\sin\omega\,\sin i \end{matrix}\right\} = -\mathbf{P}, \\
\frac{\partial \mathbf{Q}}{\partial \Omega} &= \left\{\begin{matrix} \sin\omega\,\sin\Omega - \cos\omega\,\cos\Omega\,\cos i \\ -\sin\omega\,\cos\Omega - \cos\omega\,\sin\Omega\,\cos i \\ 0 \end{matrix}\right\} = (-Q_y,\, Q_x,\, 0) = [\mathbf{N}\,\mathbf{Q}], \\
\frac{\partial \mathbf{Q}}{\partial i} &= \left\{\begin{matrix} \cos\omega\,\sin\Omega\,\sin i \\ -\cos\omega\,\cos\Omega\,\sin i \\ \cos\omega\,\cos i \end{matrix}\right\} = \mathbf{R}\,\cos\omega,
\end{aligned}\right\} \tag{10}$$

where $\mathbf{N} = (0,\ 0,\ 1)$ is the unit vector, pointed to the ecliptic north pole. The corresponding vector products are in the square brackets. In view of (10), the formulae (9) become

$$\Delta\mathbf{P} = \mathbf{Q}\,\Delta\omega + [\mathbf{N}\,\mathbf{P}]\,\Delta\Omega + \mathbf{R}\,\sin\omega\;\Delta i,$$

$$\Delta\mathbf{Q} = -\mathbf{P}\,\Delta\omega + [\mathbf{N}\,\mathbf{Q}]\,\Delta\Omega + \mathbf{R}\,\cos\omega\;\Delta i.$$

On substituting these formulae in (8) we obtain

$$\Delta\mathbf{r} = (-\eta\mathbf{P} + \xi\mathbf{Q})\,\Delta\omega + (\xi\,[\mathbf{N}\,\mathbf{P}] + \eta\,[\mathbf{N}\,\mathbf{Q}])\,\Delta\Omega + (\xi\sin\omega + \eta\cos\omega)\,\mathbf{R}\;\Delta i.$$

If now we resort to (1), whence $\xi = (\mathbf{r}\ \mathbf{P})$, $\eta = (\mathbf{r}\ \mathbf{Q})$, and employ the known formula in the theory of vectors $[\mathbf{a}\,[\mathbf{b}\ \mathbf{c}]] = \mathbf{b}\,(\mathbf{a}\ \mathbf{c}) - \mathbf{c}\,(\mathbf{a}\ \mathbf{b})$, taking thereby in account $\mathbf{R} = [\mathbf{P}\ \mathbf{Q}]$, the foregoing expression becomes

$$\Delta\mathbf{r} = [\mathbf{R}\ \mathbf{r}]\,\Delta\omega + [\mathbf{N}\ \mathbf{r}]\,\Delta\Omega + (\mathbf{M}\ \mathbf{r})\,\mathbf{R}\ \Delta i. \tag{11}$$

Therein the unit vector

$$\mathbf{M} = \mathbf{P}\ \sin\omega + \mathbf{Q}\ \cos\omega = (-\sin\Omega\ \cos i,\ \cos\Omega\ \cos i,\ \sin i) \tag{12}$$

is introduced, which is perpendicular to the node line of the orbit of the celestial body concerned. Note that $[\mathbf{N}\ \mathbf{r}] = -y\ \mathbf{n}_1 + x\ \mathbf{n}_2$; $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ are the unit vectors coincident with the axes of the orthogonal heliocentric ecliptic coordinate system. From (1) and (11) we deduce for one and the same body $(\mathbf{r}\ \Delta\mathbf{r}) = 0$. Therefore, (7) is reduced to

$$\Delta\rho = -\frac{1}{\rho}\left\{(\mathbf{r}_k \Delta\mathbf{r}_j) + (\mathbf{r}_j\ \Delta\mathbf{r}_k)\right\}. \tag{13}$$

It is now easy to determine, by (11) and (13), the change in the distance between two bodies j and k, due to the change in only one of the considered orbital elements

$$\left.\begin{aligned}
\Delta\rho_{\omega_j} &= -\frac{1}{\rho}\left(\mathbf{r}_k\left[\mathbf{R}_j\ \mathbf{r}_j\right]\right)\Delta\omega_j, & \Delta\rho_{\omega_k} &= -\frac{1}{\rho}\left(\mathbf{r}_j\left[\mathbf{R}_k\ \mathbf{r}_k\right]\right)\Delta\omega_k,\\
\Delta\rho_{\Omega_j} &= -\frac{1}{\rho}\left(\mathbf{r}_k\left[\mathbf{N}\ \mathbf{r}_j\right]\right)\Delta\Omega_j, & \Delta\rho_{\Omega_k} &= -\frac{1}{\rho}\left(\mathbf{r}_j\left[\mathbf{N}\ \mathbf{r}_k\right]\right)\Delta\Omega_k,\\
\Delta\rho_{i_j} &= -\frac{1}{\rho}\left(\mathbf{M}_j\,\mathbf{r}_j\right)\left(\mathbf{R}_j\,\mathbf{r}_k\right)\Delta_{i_j}, & \Delta\rho_{i_k} &= -\frac{1}{\rho}\left(\mathbf{M}_k\ \mathbf{r}_k\right)\left(\mathbf{R}_k\ \mathbf{r}_j\right)\Delta i_k.
\end{aligned}\right\} \tag{14}$$

The sum of the individual changes (14) supplies the total change $\Delta\rho$.

Accordingly, we are enabled to determine, by way of (13), making use of (1), (2), (3), (5), (11), (12), introducing the indices $i = j$, k for two celestial bodies involved, or else by (14), the looked for change in their mutual distance, resulting from the small changes in the perihelia arguments, longitudes of the ascending nodes and the inclinations of their orbits.

As an example of the numerical application of the foregoing formulae we take the quasicomplanar minor planet pair j = (589) Croatia and k = (1564) Srbija — of specific interest to us because they bear the names of two largest federal republics of Yugoslavia. (1564) Srbija is the very first minor planet discovered at the Belgrade Observatory. To make the calculation more simple we take the same values of the respective orbital elements as were taken in our earlier papers (Lazović, 1964, 1967). With these elements we obtained the least (proximity) distance between respective orbits $\rho_{\min} = 0.000498$ AU. For their proximity positions (ecliptic, 1950.0) we have:

j = (589) Croatia

$$\mathbf{r} = (-2.8847737,\ 1.3382700,\ -0.2480838)$$
$$[\mathbf{R}\ \mathbf{r}] = (-1.3610505,\ -2.8332760,\ 0.5426977)$$
$$[\mathbf{N}\ \mathbf{r}] = (-1.3382700,\ -2.8847737,\ 0)$$
$$(\mathbf{M}\ \mathbf{r})\,\mathbf{R} = (-0.0030481,\ -0.2480651,\ -1.3027243)$$

k = (1564) Srbija

$$\mathbf{r} = (-2.8847622,\ 1.3383597,\ -0.2475942)$$
$$[\mathbf{R}\ \mathbf{r}] = (-1.3610055,\ -2.8308750,\ 0.5551353)$$
$$[\mathbf{N}\ \mathbf{r}] = (-1.3383597,\ -2.8847622,\ 0)$$
$$(\mathbf{M}\ \mathbf{r})\,\mathbf{R} = (-0.0053714,\ -0.2475359,\ -1.2754614)$$

Using these values we obtain, by way of (11) and (13) or (14), the following expression for the resulting change in the mutual distance of the quoted minor planets in the zone of their proximity

$$\begin{aligned}\Delta\rho = {} & 0.0084047\,\Delta\omega_j + 0.0045965\,\Delta\omega_k + \\ & + 0.5505107\,\Delta\Omega_j - 0.5505106\,\Delta\Omega_k + \\ & + 1.3259202\,\Delta i_j - 1.2986254\,\Delta i_k.\end{aligned} \tag{15}$$

If we next assume the changes in the considered orbital elements of these two minor planets as:

$$\Delta\omega_i = \pm 1^\circ = \pm 0.0174533 \text{ rad},\ \Delta\Omega_i = \pm 0.^\circ1,\ \Delta i_i = \pm 0.^\circ1 = \pm 0.0017453 \text{ rad},\ i = j,\ k,$$

then, making use of (15), i. e. (14), the individual changes in earlier found the proximity distance (in AU), due to the change in only one of the considered orbital elements, turns out to:

$$\Delta\rho_{\omega_j} = \pm 0.0001467, \qquad \Delta\rho_{\Omega_j} = \pm 0.0009608, \qquad \Delta\rho_{i_j} = \pm 0\ 0023141,$$
$$\Delta\rho_{\omega_k} = \pm 0.000\ 802, \qquad \Delta\rho_{\Omega_k} = \mp 0.0009608, \qquad \Delta\rho_{i_k} = \mp 0\ 0022665.$$

It is evident that the mutual distance of the two minor planets is mostly affected by the inclinations changes, less by the changes in the longitudes of the ascending nodes and the least by the changes in the arguments of perihelia of asteroid orbits. Convenient combination of these values can result in a negative $\Delta\rho$, i. e. in a proximity distance shorter than 0.000498 AU, found originally. In Table I are presented some of such values, and the corresponding values of $\rho = 0.000498 + \Delta\rho$ (in units 10^{-6} AU), associated with the minor planet pair j = (589) Croatia and k = (1564) Srbija.

TABLE I

$\Delta\omega$		$\Delta\Omega$		Δi		$-10^6\Delta\rho$	$10^6\rho$
j	k	j	k	j	k		
-1°	$+1^\circ$	$-0^\circ.1$	$-0^\circ.1$	$+0^\circ.1$	$+0^\circ.1$	19	479
-1	$+1$	-0.1	-0.1	-0.1	-0.1	114	384
-1	-1	$+0.1$	$+0.1$	$+0.1$	$+0.1$	179	319
-1	-1	$+0.1$	$+0.1$	-0.1	-0.1	275	223

On the other hand, assuming

$$\Delta\omega_i = \pm 30'' \ \Delta\Omega_i = \pm 30' = \pm 0.0001454 \text{ rad}, \Delta i_i = \pm 15'' = \pm 0.0000727 \text{ rad}, \quad i = j, k.$$

we obtain by way of terms in (15) the following partial changes of earlier found the mutual proximity distance of this asteroid pair:

$$\Delta\rho_{\omega_j} = \pm 0.0000012, \quad \Delta\rho_{\Omega_j} = \pm 0.0000800, \quad \Delta\rho_{i_j} = \pm 0.0000964,$$
$$\Delta\rho_{\omega_k} = \pm 0.0000007, \quad \Delta\rho_{\Omega_k} = \mp 0.0000800, \quad \Delta\rho_{i_k} = \mp 0.0000944.$$

By performing suitable combination one would get corresponding values listed in Table II.

TABLE II

$\Delta\omega$		$\Delta\Omega$		Δi		$-10^6\Delta\rho$	$10^6\rho$
j	k	j	k	j	k		
$-30''$	$+30''$	$-30''$	$+30''$	$-15''$	$-15''$	163	335
-30	-30	$+30$	$+30$	-15	$+15$	193	305
$+30$	-30	-30	$+30$	-15	$+15$	350	148
-30	-30	-30	$+30$	-15	$+15$	353	145

These results make it clear that even slight changes in the orbital elements of an asteroid pair, if only in three of them: ω, Ω and i, are capable of producing considerable change in their mutual proximity distance. That is why it becomes necessary, if the determination of proximity of minor planets, of the corresponding positions and distance, is to be more accurate, to repeat the proximity calculus. Account should, thereby, be taken of the effects produced by the major planets, but simultaneously, of the mutual perturbing action of the minor planets themselves over the time of their closest approaching, in particular if their closeness is prolonged. These are, obviously, the circumstances by which stronger effects of the gravitational interaction are to be reckoned with.

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This work iz part of the research project supported by the Fund for Scientific Research of the S. R. Serbia.

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