Publications of the Department of Astronomy — Beograd, № 12, 1983.

UDC 523.11 OSP

THE WORLD LINES OF ISOTROPIC EXPANSION IN THE DE SITTER UNIVERSE

I. Lukačević
Institute of Mechanics, Faculty of Sciences, Beograd

Received 20. September 1983.

Summary. The well-known Lemaître coordinate transformation in de Sitter's spacetime, leading to an expanding Euclidean spatial metric, is obtained here as a particular case in a family of transformations. These transformations lead to non-geodesic radial world lines (except in the Lemaître case) of isotropic expansion, with determined consequences for the metric and the acceleration of particles on them.

I. Lukačević, SVETSKE LINIJE IZOTROPNE EKSPANZIJE U DE SITTER-OVOJ VASIONI — Poznata Lemetrova koordinatna transformacija, koja određuje jednu euklidsku metriku u širenju unutar de Siterove Vasione, dobijena je u ovom radu kao poseban slučaj jedne familije transformacija. Te transformacije daju radijalne svetske linije (izuzev u Lemetrovom slučaju) izotropnog širenja, s određenim posledicama po metriku i ubrzanje čestica na njima.

INTRODUCTION

We consider, in this paper, the de Sitter Universe in order to find the radial world lines which expand isotropically; the choice of radial lines being the natural consequence of the spherical symmetry of such a spacetime. Our essential assumption is the linear dependence of the Sitter's coordinate time on the time function of new world lines. The Lemaître solution appears then as a particular case, corresponding to radial geodesics and implying a constant rate of expansion. In the general, non geodesic, case o centrifugal or centripetal acceleration appears along the world lines of isotropic expansion, depending upon the sign of a constant of integration; the expansion of the spacelike part of the metric represents a dilation, although the form of the metric coefficients is complicated by the fact that it depends upon both new variables, representing respectively time and distance functions.

*

* *

We take the de Sitter line element in the form [1]:

$$\varepsilon\, ds^2 = \frac{dr^2}{1-\frac{r^2}{R^2}} + r^2(d^2\vartheta + \sin^2\vartheta\, d\varphi^2) - \left(1-\frac{r^2}{R^2}\right) d\tau^2 \tag{1.1}$$

where r is the radial variable, τ the coordinate time, ϑ, φ the cyclic coordinates. R is the distance from the origin to the »antipodal point« of the Universe. We consider here only the ponderable Universe which corresponds to the part of (1.1) characterized by $r<R$.

We study here the radial motions of virtual observers leaving the origin of coordinates. Their time τ' is assumed to be of a retarded type with respect to τ:

$$\tau' = \tau + \mu(r) \tag{1.2}$$

The fact that we restrict ourselves to radial motions only does not affect the generality of our conclusions because of the spherically forminvariant character of (1.1), where r represents simply the distance from a privileged observer at the origin to any event. Putting

$$x^1 = r, \quad x^2 = \vartheta, \quad x^3 = \varphi, \quad x^4 = \tau \tag{1.3a}$$

$$x'^1 = r', \quad x'^4 = \tau' \quad x'^2 = \vartheta', \quad x'^3 = \varphi', \tag{1.3b}$$

we obtain by (1.2) for the components of the tangent unit vector ξ^α of the world lines τ':

$$\xi\alpha \left\{ \frac{\mu'\sqrt{1-\frac{r^2}{R^2}}}{\sqrt{1-\mu'^2\left(1-\frac{r^2}{R^2}\right)}}, 0, 0, \frac{\mu'\sqrt{1-\frac{r^2}{R^2}}}{\sqrt{1-\mu'^2\left(1-\frac{r^2}{R^2}\right)}} \right\} \tag{1.4}$$

The orthogonal trajectories of (1.2) are expressed through a parameter ψ which is itself a function of r':

$$\psi(r') = \psi(r, \tau) \tag{1.5}$$

The proper expansion tensor $\vartheta_{\alpha\beta}$ for the world lines τ' is the function of the four velocity ξ^α [2]

$$\vartheta_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha + \xi_\alpha \xi^\gamma \nabla_\gamma \xi_\beta + \xi_\beta \xi^\gamma \nabla_\gamma \xi_\alpha \tag{1.6}$$

where ∇ denotes covariant differentiation. We shall now calculate the components of $\vartheta_{\alpha\beta}$ with respect to both coordinate systems (1.3a—b):

$$\vartheta'^{11} = \vartheta^{11}\left(\frac{\partial\psi}{\partial x^1}\right)^2 + 2\vartheta^{14}\frac{\partial\psi}{\partial x^1}\frac{\partial\psi}{\partial x^4} + \vartheta^{44}\left(\frac{\partial\psi}{\partial x^4}\right)^2 \tag{1.7a}$$

$$\vartheta'^{22} = \vartheta^{22}, \qquad \vartheta'^{33} = \vartheta^{33}, \tag{1.7b}$$

The remaining components of $\vartheta'^{\alpha\beta}$ are equal to zero. The condition of orthogonality of ψ lines with respect to τ' lines is:

$$\xi^\alpha \frac{\partial\psi}{\partial x^\alpha} = 0$$

Therefrom, taking into consideration (1.1) and (1.4) and returning to coordinates r, τ one obtains:

$$\frac{\partial\psi}{\partial\tau} = \mu\left(1-\frac{r^2}{R^2}\right)\frac{\partial\psi}{\partial r} \tag{1.8}$$

We have thus, after some calculus, from (1.1), (1.4) and (1.8), for the expansion tensor:

$$\vartheta'^1_1 = \frac{1-\dfrac{r^2}{R^2}}{1-\mu'^2\left(1-\dfrac{r^2}{R^2}\right)}\left(\vartheta_{11} - 2\mu\vartheta_{14} + \mu^2\vartheta_{44}\right) \tag{1.9}$$

$$\vartheta'^2_2 = \vartheta'^3_3 = -\frac{2}{r}\mu'\left(1-\frac{r^2}{R^2}\right)\sqrt{\frac{1-\dfrac{r^2}{R^2}}{1-\mu'^2\left(1-\dfrac{r^2}{R^2}\right)^2}} \tag{1.10}$$

We shall formulate the condition of isotropic expansion with respect to the world lines τ', that is:

$$\vartheta'^1_1 = \vartheta'^2_2 = \vartheta'^3_3 \tag{1.11}$$

The second of the above equalities being automatically satisfied, (1.11) reduces to one differential equation whose solution determines $\mu(r)$. Explicitly, (1.11) reads:

$$\left(1-\frac{r^2}{R^2}\right)\mu'' + \left(1-\frac{r^2}{R^2}\right)^2 \mu'^3 - \frac{1}{r}\left(1+\frac{r^2}{R^2}\right)\mu = O \qquad (1.12)$$

The general solution of that differential equation is:

$$\mu = \pm R \int \frac{r\,dr}{(R^2-r^2)\sqrt{C_1 R^2 (R-r^2)+1}} + C_2 \qquad (1.13)$$

There are now several possibilities:

1) $C_1 = 0$. Taking the positive sign in (1.12) and the particular value $C_2 = -R \ln R$ for the additive constant we obtain

$$\tau' = \tau + \frac{1}{2} R \ln\left(1-\frac{r^2}{R^2}\right) \qquad (1.14)$$

and from (1.5)

$$r' = \frac{r}{e^{\tau/R}\sqrt{1-\dfrac{r^2}{R^2}}} \qquad (1.15)$$

The world lines which correspond to the parameter τ' are geodesics. Putting $\vartheta' = \vartheta$, $\varphi' = \varphi$ we obtain a line element of the form

$$\varepsilon ds'^2 = e^{2\tau'/R}(dr'^2 + r'^2 d\vartheta'^2 + r'^2 \sin^2\vartheta' d\varphi'^2) - d\tau'^2 \qquad (1.16$$

which represents the well-known Lemaître form [1] of the de Sitter metric. It is easy to verify that world lines (1.14) correspond to the condition $\vartheta'^1_1 = \text{const}$, which is equivalent to $C_1 = 0$.

2) $C_1 > 0$. With this assumption we obtain, putting $a^2 = C_1 R^4 + 1$, from (1.13)

$$\tau' = \tau \pm \frac{1}{2} R \ln\left(D \frac{\sqrt{a^2 - C_1 R^2 r^2} + 1}{\sqrt{a^2 - C_1 R^2 r^2} - 1}\right) \qquad (1.17)$$

and, having solved (1.8), with the use of the new variable $r = \exp.(\psi/aR)$:

$$r' = D' e^{-\tau/a}\left(\frac{\sqrt{a^2 - C_1 R^2 r^2} - 1}{\sqrt{a^2 - C_1 R^2 r^2} + 1}\right)^{\pm\frac{1}{2}} \cdot \left(\frac{a + \sqrt{a^2 - C_1 R^2 r^2}}{a - \sqrt{a^2 - C_1 R^2 r^2}}\right)^{\pm\frac{1}{2a}} \qquad (1.18)$$

D and D' are new constants. Before formulating the line in the terms of variables (1.17) and (1.18) we shall consider the third possibility.

3) $0 > C_1 > \frac{1}{R^4}$. Putting $K_1 = -C_1$ and $b^2 = 1 - K_1 R^4$ one obtains expres-

$$D = D'^{2b} = \frac{1-b}{1+b} \tag{1.19}$$

sions similar to (1.17) and (1.18). Then, choosing the particular values and negative expressions at the right hand sides of (1.17) and (1.18) one finally obtains:

$$\tau' = \tau + \frac{1}{2} R \ln \left[\frac{(1-b)(1+\sqrt{b^2+K_1 R^2 r^2})}{(1+b)(1-\sqrt{b^2+K_1 R^2 r^2})} \right] \tag{1.20}$$

$$r' = e^{-\tau/bR} \left[\frac{(1-b)(1+\sqrt{b^2+K_1 R^2 r^2})}{(1+b)(1-\sqrt{b^2+K_1 R^2 r^2})} \right]^{\frac{1}{2b}} \cdot \left(\frac{\sqrt{b^2+K_1 R^2 r^2}-b}{\sqrt{b^2+K_1 R^2 r^2}+b} \right)^{\frac{1}{2}} \tag{1.21}$$

which leads to a metric element of the form:

$$\varepsilon\, ds'^2 = \frac{1}{(r'^2 e^{2\tau'/bR}+1)} \left\{ \frac{4\, b^2}{K_1 R^2} r'^2 e^{2\tau'/bR} [dr'^2 + r'^2 (d\vartheta'^2 + \sin^2\vartheta'\, d\varphi'^2)] - \right.$$
$$\left. - (r'^2 e^{2\tau'/bR} - 1)^2 d\tau'^2 \right\} \tag{1.22}$$

The signs and the values of the constants in (1.19) are chosen as to satisfy conditions

$$r = \Leftrightarrow \tau' = \tau, \qquad r > 0 \Leftrightarrow \tau' > \tau$$
$$r' \to R \Leftrightarrow (\tau' \to \infty,\ r' \to \infty), \ \frac{\partial r'}{\partial r} > 0 \tag{1.23}$$

which are natural; the nev time τ' appears to be hastened with respect to τ, which is acceptable.

*

* *

Let us consider the behaviour of the metric coefficients in (1.22). One obtains, from (1.20), (1.21), that:

$$r'^2 e^{2\tau'/bR} = \frac{\sqrt{b^2+K_1 R^2 r^2}-b}{\sqrt{b^2+K_1 R^2 r^2}+b} < \frac{1-b}{1+b} < 1 \tag{2.1}$$

One has from (1.22):

$$g'_{11} = \frac{4\, b^2 r'^2 e^{2\tau'/bR}}{K_1 R^2 (r'^2 e^{2\tau'/bR}+1)^2} = \frac{4\, b^2}{K_1 R^2 \left(2 + r'^2 e^{2\tau'/bR} \frac{1}{r'^2 e^{2\tau'/bR}} \right)} \tag{2.2}$$

As the variables change inside the hypersphere $r'^2 e^{2\tau'/bR} < 1$ we obtain, for a fixed r', that g'_{11} dilates with the increase of τ'. But as the expression for g'_{11} is in fact the multiplier of the whole spacelike part of the metric (1.22), we conclude to the dilatation of space with respect to the world lines τ'. The rate of expansion does no more depend only upon time as in the case of the Lemaître line element. A somewhat curious feature of the general solution of the differential equation (1.12) with respect to chosen parameters r', τ' is that the maximal values they can take are reciprocally limited by the condition (2.1), from which the limit appears to be inside the »antipodal« hypersurface $r = R$. The last metric coefficient is subjected to contraction, which means, in particular, that τ' is, at the difference from the Lemaître metric, not running at the same rate everywhere in spacetime.

We can remark that there is a particular solution for $K_I = 1/R^4$, but then ξ^α becomes an indetermined vector, which means that the world lines ξ^α are light rays.

The world lines of the coordinate time of the Lemaître line element (1.16) are geodesics, where as the corresponding world lines of (1.22) are not. As centrifugal acceleration already appears along de Sitter's radial geodesics ([3] p. 58), let us examine the acceleration vector of the four velocity ξ^α:

$$W_\alpha = \xi^\beta \nabla_\beta \xi_\alpha \tag{2.3}$$

Taking into consideration (1.1), (1.4) and (1.20) one obtains, for the radial component W_1, the only spacelike component to be different from zero:

$$W_1 = \xi^1 \left(\frac{\partial \xi_1}{\partial r} - \frac{1}{2} \xi^1 \frac{\partial g_{11}}{\partial r} \xi_1 \right) - \frac{1}{2} \frac{\partial g_{44}}{\partial r} \left(\xi^4 \right)^2 = \tag{2.4}$$

$$= \frac{K_I R^2 r}{b^2 \left(1 - \frac{r^2}{R^2} \right)} > 0$$

Thus, the radial acceleration is positive along the world lines τ'; the particles are subjected to a centrifugal acceleration on those lines. In the case characterized by $C_I > 0$, i.e. with a τ' given by (1.17), the acceleration is negative or centripetal. So that case, being unrealistic, has to be discarded.

*

* *

This work is part of a research project supported by the Fund for Scientific Research of the SR Serbia.

REFERENCES

[1] Ch. Moeller: The theory of relativity, Chapter XIII, *Clarendon Press* (1972).
[2] I. Lukačević: *Europhysics Journal (Fizika),* Vol 12, Suppl 3, pp 1—8 (1980).
[3] C.W. Kilmister: General theory of relativity, pp 58—60, *Pergamon Press* (1973).