Publications of the Department of Astronomy — Beograd, № 11, 1981

UDC 524.3—85 OSP

TURBULENT MOTION IN STELLAR ENVELOPES

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Received June 25, 1981

Summary. Stellar envelope is treated in terms of hydrodynamics, as a turbulent, non-stationary, moving medium. The formulation of the problem has been carried out without regard to the way in which the envelope has been formed: the material is ionized hydrogen inside the gravitational field of the star. Heat conduction and the radiation pressure are not considered. The angular momentum is being transferred through turbulence friction while the transfer of energy proceeds through radiation.

Collective mean values of the physical guantities are introduced and acoustic model of turbulence with non-linear pulsations presented. The complete system of equations of turbulent motion adjusted to the mean values, is developed.

T. Angelov, TURBULENTNO KRETANJE U OMOTAČIMA ZVEZDA — Omotač zvezde razmatra se hidrodinamički, kao turbulentna sredina sa nestacionarnim kretanjem. Problem se formuliše nezavisno od načina formiranja omotača: materijal je jonizovan vodonik u gravitacionom polju zvezde, ne razmatraju se toplotno provođenje i pritisak zračenja, uglovni moment prenosi se turbulentnim trenjem, a energija se prenosi zračenjem. Uvode se kolektivne srednje vrednosti fizičkih veličina, daje se model akustičke turbulencije uz nelinearnost pulsacija i izvodi se potpun sistem jednačina turbulentnog kretanja za srednje veličine.

1. INTRODUCTION

Astronomical observations furnish the evidence of existence of turbulence in the stellar atmospheres and in the interstellar medium. The theory of turbulent motion, especially the one dealing with the non-linear pulsations, is not yet complete (Rotta, 1951; Frost, 1960; Lundgren, 1967; Deardorff, 1971; Kolmogorov, 1972; Ievlev, 1973). Concerning the astrophysical conditions, the turbulence mechanism is used with the accretion models of black holes, proton stars and white dwarfs (Shakura, Sunyaev, 1973; Papaloizou and Pringle, 1977; Pringle and Savonije, 1979). Thereby, Keplerian motion of the gas iz considered. The system of equilibrium equations appears to be either uncomplete or with the tension tensor uncomplete. The hardest problem is presented by the unknown viscosity of the accretion envelopes (Pringle and Rees, 1972; Lynden-Bell and Pringle, 1974; Stewart, 1975).

In the present paper the disk-approximated envelope is used in treating the turbulent motion and the problem is formulated without regard to the mode of envelope's fôrming.

2. FORMULATION OF THE PROBLEM

In the present paper cylindrical coordinates (r, φ, z) with the axial symmetry ($\partial/\partial\varphi \equiv 0$) are used in treating the stellar envelope. The coordinate origin coincides with the star's centre, r-axis is in the rotation plane and z-axis is oriented in the sense of the rotation axis. The envelope's mass is much less than that of the star (proper gravitation of the gas in it is neglected). The material is the ionized hydrogen. The envelope's existence is not conditioned by the magnetic field. The heat transfer and the radiation pressure are left out of consideration. The angular momentum transfer mechanism is effected by the turbulence friction and the envelope is considered as a continuum medium with non-stationary processes. The problem's starting equations are:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \boldsymbol{v} = 0 \tag{1}$$

$$\rho \frac{\partial v_r}{\partial t} + \rho \boldsymbol{v} \cdot \nabla v_r - \rho \frac{v_\varphi^2}{r} = -\rho \frac{\partial \psi}{\partial r} - \frac{\partial P}{\partial r} + (\mathrm{Div}\, N)_r \tag{2}$$

$$\rho \frac{\partial v_\varphi}{\partial t} + \rho \boldsymbol{v} \cdot \nabla v_\varphi + \rho \frac{v_r v_\varphi}{r} = (\mathrm{Div}\, N)_\varphi \tag{3}$$

$$\rho \frac{\partial v_z}{\partial t} + \rho \boldsymbol{v} \cdot \nabla v_z = -\rho \frac{\partial \psi}{\partial z} - \frac{\partial P}{\partial z} + (\mathrm{Div}\, N)_z \tag{4}$$

$$\rho \frac{\partial \varepsilon}{\partial t} + \rho \boldsymbol{v} \cdot \nabla \varepsilon = -P \left(\frac{1}{r} \cdot \frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} \right) + \mathrm{div}\,(N\boldsymbol{v}) - \boldsymbol{v} \cdot \mathrm{Div}\, N - q \tag{5}$$

$$\psi = -\frac{G\mathfrak{M}_*}{R}, \qquad P = \frac{k}{\mu m_H} \rho T, \qquad \varepsilon = \frac{1}{\Gamma - 1} \cdot \frac{P}{\rho}, \quad q = \mathrm{div}\, \boldsymbol{F}_{rad} \tag{6}$$

with $\boldsymbol{v} \cdot \nabla = v_r \frac{\partial}{\partial r} + v_z \frac{\partial}{\partial z}$, $\mu = 0.5$, $\Gamma = 5/3$. The unknown functions are: ρ — the density, $\boldsymbol{v} \equiv (v_r, v_\varphi, v_z)$ — the velocity and T — the temperature. ψ denote the gravitational potential of the star, P — the gas pressure, N — tension tensor, ε — internal energy per gram, $\boldsymbol{F}_{rad}$ — radiation flux, R — the distance of a particular point in the envelope from the star centre, (r, z, t) are Euler's coordinates. The designation Div relates to the divergence of the tensor's. For an arbitrary ortogonal curvilineal coordinate system ξ^i (Diachenko, Imshenik, 1963) we have:

$$(\mathrm{Div}\, N)_k = \frac{1}{h_k} \left[\frac{1}{h_1 h_2 h_3} \cdot \frac{\partial}{\partial \xi^i} \left(\frac{h_1 h_2 h_3 h_k}{h_i} N_{ik} \right) - N_{ii} \frac{\partial \ln h_i}{\partial \xi^k} \right] \tag{7}$$

$$N_{ik} = \varkappa \left\{ \frac{1}{h_k} \cdot \frac{\partial v_i}{\partial \xi^k} + \frac{1}{h_i} \cdot \frac{\partial v_k}{\partial \xi^i} - \frac{1}{h_i h_k} \left(v_i \frac{\partial h_i}{\partial \xi^k} + v_k \frac{\partial h_k}{\partial \xi^i} \right) + \right.$$

$$+2\,\delta_{ik}\frac{v_\lambda}{h_\lambda}\cdot\frac{\partial \ln h_\lambda}{\partial \xi^\lambda}-\frac{2}{3}\,\delta_{ik}\left[\frac{v_\lambda}{h_\lambda}\cdot\frac{\partial \ln (h_1 h_2 h_3)}{\partial \xi^\lambda}+\frac{\partial}{\partial \xi^\lambda}\left(\frac{v_\lambda}{h_\lambda}\right)\right]\Bigg\}. \tag{8}$$

Thus, the viscosity term in the equation of energy:

$$\operatorname{div} N\boldsymbol{v}-\boldsymbol{v}\cdot \operatorname{Div} N = N_{ik}\frac{h_k}{h_i}\cdot\frac{\partial}{\partial \xi^i}\left(\frac{v_k}{h_k}\right)+\frac{v_k}{h_k}\,N_{ii}\,\frac{\partial \ln h_i}{\partial \xi^k} \tag{9}$$

where: $\xi^{1,2,3} = r, \varphi, z$; $h_{1,2,3} = 1, r, 1$; $\varkappa$ — the dynamical coefficient of viscosity.

3. ONE-DIMENSIONAL TREATMENT

The consequences of the envelope forming in the binary systems (e.g. Prendergast and Taam, 1974; Flannery, 1975) suggest the conclusion that its characteristics are most prominent in the vicinity of the orbital (equatorial) plane of the star. The thikness in the Z-direction being low, it is possible to treat the envelope in a disk-approximation, while the assumption $v_z \ll v_r$ ($v_r < v_\varphi$ is real) furnishes, approximately, the isotermic Z-structure of the disk in a hydrostatic equilibrium. Accordingly, the supplementary conditions:

$$v_z \approx 0, \quad \frac{\partial v_r}{\partial z}=0, \quad \frac{\partial v_\varphi}{\partial z}=0, \qquad T(z)=\text{const}, \tag{10}$$

allow to develop the function ψ of gravitational potential in the neighbourhood of the plane of rotation and to solve Eq. (4). For the Z-distribution of density one obtain:

$$\rho(r,z)=\rho(r,o)\exp\left[-\tfrac{1}{2}(z/L)^2\right], \tag{11}$$

$$L=[kT/(A\mu m_H)]^{1/2}, \qquad A(r)=(\partial^2\psi/\partial z^2)_{z=0}. \tag{12}$$

The total density σ, in section r = const, is obtained by integrating (11). The integration of remaining equations of the system (1—6), with respect to z, can be performed now. One obtains:

$$\frac{\partial \sigma}{\partial t}+\boldsymbol{v}\cdot\nabla\sigma+\sigma\left(\frac{\partial v_r}{\partial r}+\frac{v_r}{r}\right)=0 \tag{13}$$

$$\frac{\partial v_r}{\partial t}+\boldsymbol{v}\cdot\nabla v_r-\frac{v_\varphi^2}{r}=-\frac{\partial \Psi}{\partial r}-\frac{\pi}{2\sigma A}\cdot\frac{\partial A}{\partial r}-\frac{1}{\sigma}\cdot\frac{\partial \pi}{\partial r}+\frac{1}{\sigma}(\operatorname{Div} N)_r \tag{14}$$

$$\frac{\partial v_\varphi}{\partial t}+\boldsymbol{v}\cdot\nabla v_\varphi+\frac{v_r v_\varphi}{r}=\frac{1}{\sigma}(\operatorname{Div} N)_\varphi \tag{15}$$

$$\frac{\partial \varepsilon}{\partial t}+\boldsymbol{v}\cdot\nabla\varepsilon=-\frac{\pi}{\sigma r}\cdot\frac{\partial v_r}{\partial r}+\frac{1}{\sigma}[\operatorname{div}(N\boldsymbol{v})-\boldsymbol{v}\cdot\operatorname{Div} N]-Q \tag{16}$$

$$\Psi=-\frac{G\mathfrak{M}_*}{r}, \quad \pi=\frac{kT}{\mu m_H}\sigma, \quad \varepsilon=\frac{1}{\Gamma-1}\cdot\frac{\pi}{\sigma}, \quad Q=\frac{q}{\sigma}, \quad \boldsymbol{v}\cdot\nabla=v_r\frac{\partial}{\partial r}. \tag{17}$$

For a cylindrical configuration in axial symmetry Eqs. (7 — 9), by means of condition (10), become:

$$(\mathrm{Div}\, N)_r = \frac{\partial N_{rr}}{\partial r} + \frac{N_{rr} - N_{\varphi\varphi}}{r},$$

$$(\mathrm{Div}\, N)_\varphi = \frac{\partial N_{r\varphi}}{\partial r} + 2\frac{N_{r\varphi}}{r}, \tag{18}$$

$$\mathrm{div}\, N\mathbf{v} - \mathbf{v}\cdot \mathrm{Div}\, N = N_{rr}\frac{\partial v_r}{\partial r} + N_{\varphi\varphi}\frac{v_r}{r} + N_{r\varphi}\left(\frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r}\right).$$

$$N_{rr} = \eta\left[2\frac{\partial v_r}{\partial r} - \frac{2}{3}\left(\frac{\partial v_r}{\partial r} + \frac{v_r}{r}\right)\right],$$

$$N_{\varphi\varphi} = \eta\left[2\frac{v_r}{r} - \frac{2}{3}\left(\frac{\partial v_r}{\partial r} + \frac{v_r}{r}\right)\right], \tag{19}$$

$$N_{r\varphi} = \eta\left[\frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r}\right],$$

with a surface density $\sigma = \int\limits_{-\infty}^{+\infty} \rho\,(r, z)\, dz$ and $\eta = \sigma\nu$; ν is the kinematical viscosity coefficient.

4. TURBULENT MOTION

The turbulence mechanism is used in the accretion model of black holes or proton stars in the binary systems (e.g. Shakura, Sunyaev, 1973). We propose the following model of acoustic turbulence in the stellar envelopes:

$$v_t = \alpha v_s\,(\alpha < 1), \qquad L = r\frac{v_s}{v_\varphi}\,(L < r), \quad \nu < \nu_t, \tag{20}$$

where v_t, v_s — turbulence velocity and the isothermic velocity of the sound, L — maximum length of the turbulence cell (heigth scale in the envelope from (12)), ν, ν_t — molecular and turbulent coefficient of viscosity. It is by the coefficient of α that the non-linearity degree of the pulsation is determined. This coefficient is a function of the distance, yielding for the vicinity of the star's surface and for the envelope's periphery the values:

$$\alpha \ll 1\,(r \gtrsim r_*), \quad \alpha \lesssim 1\,(r \gg r_*). \tag{21}$$

As the semi-empiric formula for $\alpha\,(r)$ cannot be employed, we shall consider $\alpha =$ = const. The condition $\nu < \nu_t$ is satisfied for each point in the envelope, being a consequence of α = const. In reality, as it follows from (21), the turbulence is slowly generated in the star surface vicinity, accordingly $\nu > \nu_t$ is possible (the relation $\nu/\nu_t < 1$ virtually requires $\alpha \lesssim 1$ throughout the whole of the envelope).

Let us apply the basic system of Eqs. (13 — 19), completed by the model (20), to the turbulent motion, introducing the symbols: $v_r \equiv u$, $v_\varphi \equiv v$, $p_\varphi =$

$= rv \equiv p$. Let further each one of the unknown function be represented as a sum of its mean value and the pulsation component ($f = <f> + f'$). Average the system of Eqs. (13 — 19) using, moreover, the development for the mean value and the pulsation component of the product:

$$<gh> = <g><h> + <g'h'>,$$

$$(gh)' = <g>h' + <h>g' + g'h' - <g'h'>.$$

Thereby, the following estimates concerning the Eq. (15) are adopted:

a) $$\beta_1 = \frac{k_1}{k_2} \cdot \frac{<u>}{<v>} \cdot \frac{\sqrt{<\sigma'^2>}/<\sigma>}{\sqrt{<u'^2>}/<v>} \ll 1,$$

b) $$\beta_2 = k_3 \frac{\sqrt{<\eta'^2>}}{<\eta>} \cdot \frac{\sqrt{<v'^2>}}{L\partial<v>/\partial r} \ll 1,$$

c) $$\frac{<\sigma' v'>}{<\sigma><v>} \ll 1,$$

where k_i — correlation coefficients: $k_1(\sigma', v')$, $k_2(u', v')$, $k_3(\eta', \partial v'/\partial r)$, $k_1 \approx \approx k_2$. The estimate *c)* implies the assumption of the turbulent diffusion of the mass (later on — of any amount) to be negligibly low in the φ — direction. In adjusting the Eq. (14) to its mean, the following estimates proved useful:

d) $$\frac{<v'^2>}{<v>^2} \ll 1,$$

e) $$\frac{\dfrac{\partial<\sigma>}{\partial r} \Big/ <\sigma>}{\dfrac{\partial r<\sigma' u'>}{\partial r} \Big/ (r<\sigma' u'>)} \sim 1,$$

f) $$\frac{<\sigma' u'>}{<\sigma><u>} \sim 1,$$ g) $$\frac{<u^2>}{<u'^2>} \gtrsim 1.$$

The estimate g) is nothing else than a consequence of the result f) in which the radial turbulent diffusion of the mass has not been neglected. In averaging the energy equilibrium equation, transformed in such a way as to fit the „motion entalpy":

$$I_0 = \varepsilon + \frac{1}{\Gamma - 1} \cdot \frac{\pi}{\sigma} + \frac{u^2 + v^2}{2} + \Psi,$$

use has been made of the (a — g) results, adopting at the same time the estimate

h) $$\beta_3 \sim \frac{1}{P_{rt}} \cdot \frac{(L/r)^2}{1 + \nu/\nu_t} \ll 1$$

in the case the turbulent Prandtl number $P_{rt} = \nu_t/\lambda_t \gtrsim L/r$ (λ_t is the coefficient of the turbulent conductivity). On introducting the notations

$$-<u'v'> = \nu_t\left(\frac{\partial <v>}{\partial r} - \frac{<v>}{r}\right),$$

$$-<u'u'> = \nu_t\left(\frac{4}{3}\cdot\frac{\partial <\sigma u>/<\sigma>}{\partial r} - \frac{2}{3}\cdot\frac{<\sigma u>/<\sigma>}{r}\right), \tag{22}$$

$$<\eta> = <\sigma>\nu$$

and on modifying the model (20) by $\nu \ll \nu_t$, the system (13 — 16), after several transformations, can be expressed solely by the mean values of the physical quantities. Denote further $<\sigma u>/<\sigma> \equiv <u>$ and write simply f instead of $<f>$. Then the Eqs. (13 — 16) for the mean values can be written in the form:

$$\frac{\partial\sigma}{\partial t} + \frac{1}{r}\cdot\frac{\partial r\sigma u}{\partial r} = 0 \tag{23}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} - \frac{v^2}{r} = a - \frac{1}{\sigma}\cdot\frac{\partial\pi}{\partial r} + \frac{1}{r\sigma}\cdot\frac{\partial r W_{rr}}{\partial r} \tag{24}$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial r} = \frac{1}{r\sigma}\cdot\frac{\partial r^2 W_{r\varphi}}{\partial r} \tag{25}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + (\Gamma - 1)\frac{T}{r}\cdot\frac{\partial ru}{\partial r} = (\Gamma - 1)\frac{\mu m_H}{k\sigma}\left\{W_{rr}\frac{\partial u}{\partial r} + rW_{r\varphi}\frac{\partial(v/r)}{\partial r} - \sigma Q\right\} \tag{26}$$

where

$$a = -\frac{\partial\Psi}{\partial r} - \frac{\pi}{2\sigma A}\frac{\partial A}{\partial r}, \tag{27}$$

while the mean values of the components of the turbulence tension are

$$W_{rr} = \sigma\nu_t\left(\frac{4}{3}\cdot\frac{\partial u}{\partial r} - \frac{2}{3}\cdot\frac{u}{r}\right), \tag{28}$$

$$W_{r\varphi} = \sigma\nu_t\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right). \tag{29}$$

The Eqs. (23 — 29) can be used for describing the turbulent motion in the stellar envelopes.

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This work is a part of the research project of the Basic Organization of Associated Labour for Mathematics, Mechanics and Astronomy of the Belgrade Faculty of Sciences, funded by the Republic Community of Sciences of Serbia.

REFERENCES

Deardorff, J.W.: 1971, *Journ. Comp. Phys.*, **7**, 120.
Diachenko, V.F., Imshenik, V.S.: 1963, *Žur. vič. mat. i mat. fiz.*, **3**, 915.
Flannery, P.B.: 1975, *Astrophys. J.*, **201**, 661.
Frost, V.A.: 1960, *DAN SSSR*, Tom **133**, № 4.
Ievlev, V.M.: 1973, *DAN SSSR*, Tom **208**, № 5.
Kolmogorov, A.N.: 1972, *Izv. AN SSSR*, Tom **6**, № 1—2.
Lundgren, T.S.: 1967, *Phys. Fluids*, **10**, 969.
Lynden-Bell, D. and Pringle, J.E.: 1974, *Monthly Notices Roy. Astron. Soc.*, **168**, 603.
Papaloizou, J. and Pringle, J.E.: 1977, *Monthly Notices Roy. Astron. Soc.*, **181**, 441.
Prendergast, K. and Taam, R.: 1974, *Astrophys. J.*, **189**, 125.
Pringle, J.E. and Rees, M.J.: 1972, *Astron. Astrophys.*, **21**, 1.
Pringle, J.E and Savonije, *G. J.*: 1979, *Monthly Notices Roy. Astron. Soc.*, **187**, 777.
Rotta, J.: 1951, *Z. Phys.*, **126**, 6.
Rotta, J.: 1951, *Z. Phys.*, **131**, 1.
Shakura, N.I., Sunyaev, R.A.: 1973, *Astron. Astrophys.*, **24**, 337
Stewart, J.M.: 1975, *Astron. Astrophys.*, **42**, 95.