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An investigation of global existence of the solution of fractional reaction-diffusion system

Nabila Barrouk[1], **Iqbal M. Batiha**[2 3 4] **and Adel Ouannas**[5]

Abstract. This paper investigates the existence of the solution for one of the most important fractional partial differential problems called fractional reaction-diffusion system. In particular, with the use of combining the compact semigroup methods and some L^1-estimates, we prove the global existence of the solution for the fractional reaction-diffusion system. Our investigation can be applied to a wide class of fractional partial differential equations even if they contain nonlinear terms in their constructions.

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1. Introduction

In this paper, we aim to study the following nonlinear parabolic system:

$$
(1.1) \qquad \begin{cases} \frac{\partial \psi_1}{\partial t} - d_1 \Delta \psi_1 = f\left(\psi_1, \psi_2\right), & \text{in }]0,+\infty[\times \Omega, \\ \frac{\partial \psi_2}{\partial t} + d_2 \left(-\Delta\right)^{\beta} \psi_2 = g\left(\psi_1, \psi_2\right), & \text{in }]0,+\infty[\times \Omega, \\ \dfrac{\partial \psi_1}{\partial \eta} = \dfrac{\partial \psi_2}{\partial \eta} = 0, \text{ or } \psi_1 = \psi_2 = 0, & \text{in }]0,+\infty[\times \partial\Omega, \\ \psi_1\left(0,.\right) = \psi_{1_0}\left(.\right), \ \psi_2\left(0,.\right) = \psi_{2_0}\left(.\right), & \text{in } \Omega, \end{cases}
$$

where Ω is a regular and bounded domain of $\mathbb{R}^n$ with boundary $\partial\Omega$, $n \geq 1$, $\psi_1 = \psi_1(t,x)$, $\psi_2 = \psi_2(t,x)$ are real-valued functions such that $x \in \Omega$ and $t > 0$, whereas $(-\Delta)^{\delta}$ is a nonlocal operator in which $0 < \delta < 1$, ($\delta = \alpha$ or β), while d_1, d_2 are two nonnegative values, and f, g are two functions so that they are "regular enough" [17, 18]. Herein, it should be noted that the initial data are assumed to be continuous and nonnegative, whereas the local existence of the solution (ψ_1, ψ_2) is classical in times. Such a solution is also nonnegative

[1]Department of Mathematics and Informatics, Mohamed Cherif Messaadia University, Souk Ahras, Algeria, e-mail: n.barrouk@univ-soukahras.dz, ORCID iD: orcid.org/0009-0009-5559-1956
[2]Department of Mathematics, Al Zaytoonah University of Jordan, Amman 11733 Jordan e-mail: ibatiha@inu.edu.jo, ORCID iD: orcid.org/0000-0002-8443-8848
[3]Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman 346, UAE
[4]Corresponding author
[5]Department of Mathematics and Computer Science, University of Larbi Ben M'hidi, Oum El Bouaghi, Algeria, e-mail: dr.ouannas@gmail.com, ORCID iD: orcid.org/0000-0001-9611-2047

whenever ψ_{1_0} and ψ_{2_0} are so. In the same regard, concerning with problem (1.1), we assume the following hypothesis:

- The two functions ψ_{1_0} and ψ_{2_0} are nonnegative such that:

$$\psi_{1_0},\ \psi_{2_0} \in L^1(\Omega). \tag{1.2}$$

- The two functions f and g are quasi-positive, i.e.,

$$f(0,\psi_2) \geq 0,\ g(\psi_1,0) \geq 0,\ \forall \psi_1,\psi_2 \geq 0. \tag{1.3}$$

- There exists a nonnegative constant C independent of (ξ_1,ξ_2) such that:

$$f(\xi_1,\xi_2) + g(\xi_1,\xi_2) \leq C(\xi_1+\xi_2),\ \text{for everything}\ (\xi_1,\xi_2) \in \mathbb{R}_+^2. \tag{1.4}$$

- We have:

$$f(\xi_1,\xi_2) \leq C(\xi_1+\xi_2)\ \text{for all}\ (\xi_1,\xi_2) \in \mathbb{R}_+^2. \tag{1.5}$$

The notion of replacing the anomalous diffusion operator by the standard Laplacian operator $(-\Delta)$ was firstly studied in one-dimensional space. Alikakos [1] showed the existence of global bounded solutions of the considered problem in the case of $f(\psi_1,\psi_2) = -g(\psi_1,\psi_2) = -\psi_1\psi_2^\sigma$, for $1 < \sigma < \dfrac{n+2}{n}$. The extension of this result for $\sigma > 1$ was obtained by Masuda [19]. Then, Haraux and Youkana [15] generalized the result of Masuda via the functional of Lyapunov by putting $f(\psi_1,\psi_2) = g(\psi_1,\psi_2) = -\psi_1\Psi(\psi_2)$, where Ψ is a nonlinear function satisfying the condition:

$$\lim_{\psi_2\to+\infty} \frac{[\log(1+\Psi(\psi_2))]}{\psi_2} = 0.$$

Barabanova [3] generalized the result of Haraux and Youkana by addressing the global existence of nonnegative solutions of a reaction-diffusion equation with exponential nonlinearity. It has been shown later that there is also another very powerful method for dealing with the solvability of the considered problem. This method, which we will use in this work, relies on the compact semigroups. For a better understanding, we refer the reader to the works of Moumeni and Barrouk [20, 21], and for more, see [25, 24, 8, 12]. Later on, a more general model was studied by Haraux and Kirane [14]. They took different diffusion coefficients for the two equations and certain general nonlinear terms. They also proved the existence of global bounded solutions and investigated their asymptotic behavior. Equally, Hnaien et al. [16] proved the existence of a local solution, global existence and asymptotic behavior of solutions for system (1.1) when $f(\psi_1,\psi_2) = -\lambda\psi_1\psi_2$ and $g(\psi_1,\psi_2) = \lambda\psi_1\psi_2 - \mu\psi_2$. However, for further studies about the previous discussion, the reader may consult the references [7, 2, 4, 10, 6, 9, 5]

This paper is organized as follows. In Section 2, we present some definitions and preliminaries. The existence of a local solution, positivity and global existence for particular system are studied in Section 3. In Section 4, we prove the global existence of a solution for the fractional reaction-diffusion system, followed by Section 5 that introduces the conclusions of this work.

2. Preliminaries

In this part, some preliminaries and overview of the local existence and global existence of a solution for fractional reaction-diffusion system are illustrated. This will pave the way to introduce our findings later on.

Definition 2.1. Let $F(\psi_1, \psi_2) \in X$, where X is a Banach space. The function F is locally Lipschitz if for all $t_1 \geq 0$ and all constant $k > 0$, there exists a constant $L(k, t_1) > 0$ such that the following condition:

$$\left\| F(\psi_1, \psi_2) - F\left(\hat{\psi}_1, \hat{\psi}_2\right) \right\| \leq L \left| (\psi_1, \psi_2) - \left(\hat{\psi}_1, \hat{\psi}_2\right) \right|,$$

is satisfied for all $(\psi_1, \psi_2), \left(\hat{\psi}_1, \hat{\psi}_2\right) \in \mathbb{R} \times \mathbb{R}$ with $|(\psi_1, \psi_2)| \leq k$, $\left|\left(\hat{\psi}_1, \hat{\psi}_2\right)\right| \leq k$ and $t \in [0, t_1]$ for all $t > 0$.

Lemma 2.2. *Let $T(t)$ be a semigroup engendered by the m-dissipative operator A in the Banach space X. Suppose that F is a function locally Lipschitz and $\omega_0 \in X$ is the representation of the initial data. Then the problem:*

$$\left\{ \begin{array}{l} \omega \in C([0, T], D(A)) \cap C^1([0, T], X), \\ \dfrac{d\omega}{dt} - A\omega = F(\omega), \\ \omega(0) = \psi_0. \end{array} \right. \tag{2.1}$$

admits a unique local solution ω verifying

$$\omega(t) = T(t)\omega_0 + \int_0^t T(t - \tau) F(\omega(\tau)) d\tau, \ \forall t \in [0, T].$$

Theorem 2.3. *Consider the following classical boundary-eigenvalue system for the fractional power of Laplacian in Ω with homogeneous Neumann boundary condition:*

$$\left\{ \begin{array}{ll} (-\Delta)^\alpha \varphi_k = \lambda_k^\alpha \varphi_k, & \textit{in } \Omega, \\ \dfrac{\partial \varphi_k}{\partial \eta} = 0, & \textit{on } \Omega, \end{array} \right.$$

where Ω is a open bounded domain in $\mathbb{R}^N$ and

$$D((-\Delta)^\alpha) = \left\{ \psi \in L^2(\Omega), \ \frac{\partial \psi}{\partial \eta} = 0, \ \|(-\Delta)^\alpha \psi\|_{L^2(\Omega)} < +\infty \right\},$$

with

$$\|(-\Delta)^\alpha \psi\|_{L^2(\Omega)}^2 = \sum\nolimits_{k=1}^{+\infty} |\lambda_k^\alpha \langle \psi, \varphi_k \rangle|^2.$$

Then, this system has a countable system of eigenvalues of the Laplacian operator in $L^2(\Omega)$ with homogeneous Neumann boundary condition, where $0 < c \leq \lambda_1 < \lambda_2 < \ldots < \lambda_j < \ldots$ such that $\lambda_j \to \infty$ as $j \to \infty$, and φ_k are the corresponding eigenvectors for $k = 1, 2, \ldots, +\infty$.

In light of the above preliminaries, it should be noted that:

$$(-\Delta)^{\alpha}\psi=\sum_{k=1}^{+\infty}\lambda_k^{\alpha}\left\langle \psi,\varphi_k\right\rangle \varphi_k,$$

for $\psi\in D\left((-\Delta)^{\alpha}\right)$. Accordingly, with the use of the integration by parts, we can obtain the following formula:

$$\int_{\Omega}u\left(x\right)(-\Delta)^{\alpha}v\left(x\right)dx=\int_{\Omega}v\left(x\right)(-\Delta)^{\alpha}u\left(x\right)dx, \tag{2.2}$$

for $u,v\in D\left((-\Delta)^{\alpha}\right)$.

Lemma 2.4. *[11] Let $\theta\in C_0^{\infty}\left(Q\right)$ such that $\theta\geq 0$. Then, there is a nonnegative function $\Phi\in C^{1,2}\left(Q\right)$ such that Φ is a solution of the system:*

$$\begin{cases} -\Phi_t-d\Delta\Phi=\theta & on\ Q,\\ \Phi\left(t,x\right)=0 & on\ \left(0,T\right)\times\partial\Omega,\\ \Phi\left(T,x\right)=0 & on\ \Omega.\end{cases} \tag{2.3}$$

It is worth mentioning here that, for all $q\in\left]1,\infty\right[$ and $q'\in\left]1,\infty\right]$, we have:

$$q\leq q' \text{ and } 2-\left(\frac{1}{q}-\frac{1}{q'}\right)(n+2)>0,$$

which implies that there exists a nonnegative constant c, independent of θ, such that:

$$\left\Vert \Phi\right\Vert_{L^{q'}(Q_T)}\leq c\left\Vert \theta\right\Vert_{L^{q}(Q_T)}. \tag{2.4}$$

In addition, for all $\omega_0\in L^1\left(\Omega\right)$ and $h\in L^1\left(Q\right)$, we have the following equalities:

$$\int_{Q}\left(T\left(t\right)\omega_0\left(x\right)\right)\theta dxdt=\int_{\Omega}\omega_0\left(x\right)\Phi\left(0,x\right)dx, \tag{2.5}$$

and

$$\int_{Q}\left(\int_0^t T\left(t-\tau\right)h\left(\tau,x\right)d\tau\right)\theta dxdt=\int_{Q}h\left(\tau,x\right)\Phi\left(\tau,x\right)dxd\tau. \tag{2.6}$$

In this work, the compactness of the operator B should be addressed. From this point of view, we introduce next how we can define this operator, followed by some basic facts related to the compactness.

$$B\left(F\right)\left(t\right)=\int_0^t T\left(t-\tau\right)F\left(\omega\left(\tau\right)\right)d\tau,\ \forall t\in\left[0,T\right].$$

Theorem 2.5 (Dunford–Pettis [13]). *Let $\digamma$ be a bounded set in $L^1\left(\Omega\right)$. Then $\digamma$ has a compact closure in the weak topology $\sigma\left(L^1,L^{\infty}\right)$ if and only if $\digamma$ is equi-integrable, i.e.,*

(a) $\left\{\forall\varepsilon>0,\exists\delta>0:\int_A|f|<\varepsilon,\forall A\subset\Omega,\ \textit{measurable with}\ |A|<\delta,\forall f\in\digamma\right\},$

and

$$(b)\ \left\{\forall\varepsilon>0,\ \exists\omega\subset\Omega,\ \textit{measurable with } |\omega|<\infty:\int_{\Omega\backslash\omega}|f|<\varepsilon,\ \forall f\in F\right\}.$$

Theorem 2.6 ([20, 21, 22, 23])**.** *If for all $t>0$, the operators $T(t)$ are compact, then B is compact of $L^1([0,T],X)$ in $L^1([0,T],X)$.*

Remark 2.7. The semigroup $T(t)$ generated by the operator $d(-\Delta)^{\delta}$ is compact in $L^1(\Omega)$.

3. Study of a particular system

This section is divided into three subsections so that the first one aims to deal with the local existence of a solution for a first-order system derived from system (1.1), then the positivity of such solution will be discussed, followed by exploring the global existence of the solution of the derived system.

3.1. Local existence of a particular system

In this subsection, we intend to discuss the local existence of a solution of a first-order system derived from system (1.1). In this connection, we define the functions $\psi_{1_0}^n$ and $\psi_{2_0}^n$ by:

$$\psi_{1_0}^n=\min(\psi_{1_0},n),\ \text{and } \psi_{2_0}^n=\min(\psi_{2_0},n),$$

for all $n>0$. It is clear that $\psi_{1_0}^n$ and $\psi_{2_0}^n$ verify assumption (1.2), i.e.,

$$\psi_{1_0}^n,\psi_{2_0}^n\in L^1(\Omega)\ \text{and}\psi_{1_0}^n\geq 0,\ \psi_{2_0}^n\geq 0.$$

Now, it is a suitable time to convert system (1.1) to an abstract first-order system in the Banach space $X=L^1(\Omega)\times L^1(\Omega)$ of the form:

$$(3.1)\qquad \begin{cases}\frac{\partial w_n}{\partial t}-Aw_n=F(w_n) & \text{in } [0,T[\times\Omega,\\ \frac{\partial w_n}{\partial\eta}=0,\ \text{or } w_n=0 & \text{in } [0,T[\times\partial\Omega,\\ w_n(0,\cdot)=w_{n_0}(\cdot) & \text{in } \Omega,\end{cases}$$

where $w_n=(\psi_1^n,\psi_2^n)$, $w_{n_0}=\left(\psi_{1_0}^n,\psi_{2_0}^n\right)$, $F=(f,g)$, and the operator A is defined as:

$$A=\begin{pmatrix} d_1\Delta & 0\\ 0 & -d_2(-\Delta)^{\beta}\end{pmatrix},$$

where

$$D(A):=\left\{w_n\in L^1(\Omega)\times L^1(\Omega):\left(\Delta\psi_1^n,(-\Delta)^{\beta}\psi_2^n\right)\in L^1(\Omega)\times L^1(\Omega)\right\}.$$

It is worth mentioning that system (3.1) can be returned back to the shape of system (2.1), which let us confirm that if (ψ_1^n, ψ_2^n) is a solution of system (3.1), then this solution will satisfy the following integral equations:

$$\begin{cases} \psi_1^n(t) = T_1(t)\,\psi_{1_0}^n + \displaystyle\int_0^t T_1(t-\tau)\, f(\psi_1^n(\tau), \psi_2^n(\tau))\, d\tau, \\ \psi_2^n(t) = T_2(t)\,\psi_{2_0}^n + \displaystyle\int_0^t T_2(t-\tau)\, g(\psi_1^n(\tau), \psi_2^n(\tau))\, d\tau, \end{cases} \tag{3.2}$$

where $T_1(t)$ and $T_2(t)$ are the semigroups of contractions in $L^1(\Omega)$ generated by the operator $d_1\Delta$ and $-d_2(-\Delta)^\beta$.

Theorem 3.1. *There exists $T_{\max} > 0$ such that (ψ_1^n, ψ_2^n) is a local solution of system (3.1), for all $t \in [0, T_{\max}]$.*

Proof. Due to $T_1(t)$ and $T_2(t)$ are semigroups of contractions and as F is locally Lipschitz in which $0 \le \psi_{1_0}^n, \psi_{2_0}^n \le n$, then $\exists T_{\max} > 0$ such that (ψ_1^n, ψ_2^n) is a local solution of system (3.1) on $[0, T_{\max}]$. □

Theorem 3.2. *Let $\psi_{1_0}^n, \psi_{2_0}^n \in L^1(\Omega)$. Then there exists a maximal time $T_{\max} > 0$ and a unique solution $(\psi_1^n, \psi_2^n) \in C\left([0, T_{\max}), L^1(\Omega) \times L^1(\Omega)\right)$ of system (3.1) such that either*

$$T_{\max} = +\infty,$$

or

$$T_{\max} < +\infty \text{ and } \lim_{t \to T_{\max}} \left(\|\psi_1^n(t)\|_\infty + \|\psi_2^n(t)\|_\infty\right) = +\infty.$$

Proof. For arbitrary $T > 0$, we define the following Banach space::

$$E_T := \left\{(\psi_1^n, \psi_2^n) \in C\left([0, T], L^1(\Omega) \times L^1(\Omega)\right) : \|(\psi_1^n, \psi_2^n)\| \le 2\left\|\left(\psi_{1_0}^n, \psi_{2_0}^n\right)\right\|\right\}$$

such that $2\left\|\left(\psi_{1_0}^n, \psi_{2_0}^n\right)\right\| = R$, where $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\Omega)}$ and $\|\cdot\|$ is the norm of E_T defined by:

$$\|(\psi_1^n, \psi_2^n)\| := \|\psi_1^n\|_{L^\infty([0,T], L^\infty(\Omega))} + \|\psi_2^n\|_{L^\infty([0,T], L^\infty(\Omega))}\,.$$

Next, for every $(\psi_1^n, \psi_2^n) \in E_T$, we define:

$$\Psi(\psi_1^n, \psi_2^n) := \left(\Psi_1(\psi_1^n, \psi_2^n), \Psi_2(\psi_1^n, \psi_2^n)\right),$$

where for $t \in [0, T]$. That is,

$$\begin{cases} \Psi_1(\psi_1^n, \psi_2^n) = T_1(t)\,\psi_{1_0}^n + \displaystyle\int_0^t T_1(t-\tau)\, f(\psi_1^n(\tau), \psi_2^n(\tau))\, d\tau, \\ \Psi_2(\psi_1^n, \psi_2^n) = T_2(t)\,\psi_{2_0}^n + \displaystyle\int_0^t T_2(t-\tau)\, g(\psi_1^n(\tau), \psi_2^n(\tau))\, d\tau. \end{cases}$$

Now, we aim to prove the local existence of a solution of system (3.1) by the Banach fixed point theorem. To this purpose, we have to address two claims:
Claim 1: $\Psi\left(\psi_1^n, \psi_2^n\right) \in E_T$.
To address this claim, we let $\left(\psi_1^n, \psi_2^n\right) \in E_T$ to obtain, by maximum principle, the following inequalities:

$$\begin{aligned}\left\|\Psi_1\left(\psi_1^n, \psi_2^n\right)\right\|_\infty &\leq \left\|\psi_{1_0}^n\right\|_\infty + C\left(\left\|\psi_1^n\right\|_\infty + \left\|\psi_2^n\right\|_\infty\right)T \\ &\leq \left\|\psi_{1_0}^n\right\|_\infty + C\left(\left\|\psi_{1_0}^n\right\|_\infty + \left\|\psi_{2_0}^n\right\|_\infty\right)T.\end{aligned}$$

Similarly, we have:

$$\left\|\Psi_2\left(\psi_1^n, \psi_2^n\right)\right\|_\infty \leq \left\|\psi_{2_0}^n\right\|_\infty + C\left(\left\|\psi_{1_0}^n\right\|_\infty + \left\|\psi_{2_0}^n\right\|_\infty\right)T.$$

Consequently, we obtain:

$$\begin{aligned}\left\|\Psi\left(\psi_1^n, \psi_2^n\right)\right\| &\leq \left\|\psi_{1_0}^n\right\|_\infty + \left\|\psi_{2_0}^n\right\|_\infty + 2C\left(\left\|\psi_{1_0}^n\right\|_\infty + \left\|\psi_{2_0}^n\right\|_\infty\right)T \\ &\leq 2\left(\left\|\psi_{1_0}^n\right\|_\infty + \left\|\psi_{2_0}^n\right\|_\infty\right).\end{aligned}$$

Now, we choose T so that $T \leq \frac{\left\|\psi_{1_0}^n\right\|_\infty + \left\|\psi_{2_0}^n\right\|_\infty}{CR}$. Then $\Psi\left(\psi_1^n, \psi_2^n\right) \in E_T$, for $T \leq \frac{\left\|\psi_{1_0}^n\right\|_\infty + \left\|\psi_{2_0}^n\right\|_\infty}{CR}$.
Claim 2: Ψ is a contraction map.
For $\left(\psi_1^n, \psi_2^n\right), \left(\tilde{\psi}_1^n, \tilde{\psi}_2^n\right) \in E_T$, we have:

$$\begin{aligned}\left\|\Psi_1\left(\psi_1^n, \psi_2^n\right) - \Psi_1\left(\tilde{\psi}_1^n, \tilde{\psi}_2^n\right)\right\|_\infty &\leq L\int_0^t \left\|\left(\psi_1^n, \psi_2^n\right) - \left(\tilde{\psi}_1^n, \tilde{\psi}_2^n\right)\right\|_\infty d\tau \\ &\leq LT\left(\left\|\tilde{\psi}_2^n - \psi_2^n\right\|_\infty + \left\|\tilde{\psi}_1^n - \psi_1^n\right\|_\infty\right).\end{aligned}$$

Similarly, we can obtain:

$$\left\|\Psi_2\left(\psi_1^n, \psi_2^n\right) - \Psi_2\left(\tilde{\psi}_1^n, \tilde{\psi}_2^n\right)\right\|_\infty \leq LT\left(\left\|\tilde{\psi}_2^n - \psi_2^n\right\|_\infty + \left\|\tilde{\psi}_1^n - \psi_1^n\right\|_\infty\right).$$

Immediately, the above two estimates imply:

$$\begin{aligned}\left\|\Psi\left(\psi_1^n, \psi_2^n\right) - \Psi\left(\tilde{\psi}_1^n, \tilde{\psi}_2^n\right)\right\|_\infty &\leq 2LT\left(\left\|\tilde{\psi}_2^n - \psi_2^n\right\|_\infty + \left\|\tilde{\psi}_1^n - \psi_1^n\right\|_\infty\right) \\ &\leq \frac{1}{2}\left\|\left(\psi_1^n, \psi_2^n\right) - \left(\tilde{\psi}_1^n, \tilde{\psi}_2^n\right)\right\|,\end{aligned}$$

for $T \leq \max\left(\frac{\left\|\psi_{1_0}^n\right\|_\infty + \left\|\psi_{2_0}^n\right\|_\infty}{CR}, \frac{1}{4L}\right)$. This proves Claim 2, and hence, by the Banach fixed point theorem, we conclude that system (3.1) admits a unique solution $\left(\psi_1^n, \psi_2^n\right) \in E_T$. Furthermore, this solution can be extended on a maximal interval $[0, T_{\max})$ where

$$T_{\max} := \sup\left\{T > 0 : \left(\psi_1^n, \psi_2^n\right) \text{ is a solution to system (3.1) in } E_T\right\}.$$

□

3.2. Positivity of solution of a particular system

In what follows, we intend to prove the positivity of solutions for system (3.1). This would help us to discuss the global existence of such a solution later. In this respect, we introduce the next result.

Lemma 3.3. *Let (ψ_1^n, ψ_2^n) be the solution of system (3.1) such that:*

$$\psi_{1_0}^n(x) \geq 0,\ \psi_{2_0}^n(x) \geq 0,\ x \in \Omega.$$

Then:

$$\psi_1^n(t,x) \geq 0 \text{ and } \psi_2^n(t,x) \geq 0,\ \forall (t,x) \in]0,T[\times \Omega.$$

Proof. Let $\bar{\psi}_1^n(t,x) = 0$ and $\bar{\psi}_2^n(t,x) = 0$ in $]0,T[\times \Omega$. Then $\frac{\partial \bar{\psi}_1^n}{\partial t} = 0$, $\frac{\partial \bar{\psi}_2^n}{\partial t} = 0$ and $\Delta \bar{\psi}_1^n$, $(-\Delta)^\beta \bar{\psi}_2^n = 0$. Now, in accordance with hypothesis (1.3), we can have:

$$\begin{cases} \frac{\partial \psi_1^n}{\partial t} - d_1 \Delta \psi_1^n - f(\psi_1^n, \psi_2^n) = 0 \geq \frac{\partial \bar{\psi}_1^n}{\partial t} - d_1 \Delta \bar{\psi}_1^n - f\left(\bar{\psi}_1^n, \psi_2^n\right), \\ \frac{\partial \psi_2^n}{\partial t} + d_2 (-\Delta)^\beta \psi_2^n - g(\psi_1^n, \psi_2^n) = 0 \geq \frac{\partial \bar{\psi}_2^n}{\partial t} + d_2 (-\Delta)^\beta \bar{\psi}_2^n - g\left(\psi_1^n, \bar{\psi}_2^n\right), \\ \psi_1^n(0,x) = \psi_{1_0}^n(x) \geq 0 = \bar{\psi}_1^n(0,x), \\ \psi_2^n(0,x) = \psi_{2_0}^n(x) \geq 0 = \bar{\psi}_2^n(0,x). \end{cases}$$

Hence, by the comparison theorem, we can obtain:

$$\begin{cases} \psi_1^n(t,x) \geq \bar{\psi}_1^n(t,x), \\ \psi_2^n(t,x) \geq \bar{\psi}_2^n(t,x), \end{cases}$$

where $\psi_1^n(t,x) \geq 0$ and $\psi_2^n(t,x) \geq 0$. □

3.3. Global existence of a particular system

To prove the global existence of the solution of system (3.1), it is enough to find certain estimates related to that solution in $L^1(\Omega)$. In this regard, we introduce the next two results.

Lemma 3.4. *Let (ψ_1^n, ψ_2^n) be a solution of system (3.1). Then, there exists $M(t)$, which depends only on t, such that for all $0 \leq t \leq T_{\max}$, we have:*

$$\|\psi_1^n + \psi_2^n\|_{L^1(\Omega)} \leq M(t).$$

Proof. Actually, we can derive from this result the global existence of the solution (ψ_1^n, ψ_2^n) for system (3.1) given by Theorem 3.1. However, in order to prove this result, we should note that system (3.1) can be rewritten in the form:

$$\begin{cases} \frac{\partial \psi_1^n}{\partial t} - d_1 \Delta \psi_1^n = f(\psi_1^n, \psi_2^n), & \text{in } [0,T[\times \Omega, \\ \frac{\partial \psi_2^n}{\partial t} + d_2 (-\Delta)^\beta \psi_2^n = g(\psi_1^n, \psi_2^n), & \text{in } [0,T[\times \Omega, \\ \frac{\partial \psi_1^n}{\partial \eta} = \frac{\partial \psi_2^n}{\partial \eta} = 0, \text{ or } \psi_1^n = \psi_2^n = 0, & \text{in } [0,T[\times \partial\Omega, \\ \psi_1^n(0,x) = \psi_{1_0}^n(x),\ \psi_2^n(0,x) = \psi_{2_0}^n(x), & \text{in } \Omega. \end{cases} \tag{3.3}$$

By considering the first and second equations of system (3.3), we obtain:

$$\frac{\partial}{\partial t}\left(\psi_1^n+\psi_2^n\right)-d_1\Delta\psi_1^n+d_2\left(-\Delta\right)^{\beta}\psi_2^n=f\left(\psi_1^n,\psi_2^n\right)+g\left(\psi_1^n,\psi_2^n\right).$$

By taking into account assumption (1.4), we can have:

$$\frac{\partial}{\partial t}\left(\psi_1^n+\psi_2^n\right)-d_1\Delta\psi_1^n+d_2\left(-\Delta\right)^{\beta}\psi_2^n\leq C\left(\psi_1^n+\psi_2^n\right).$$

Now, by integrating the above inequality over Ω coupled with using the integration by parts to formula (2.2), we get:

$$\int_{\Omega}\left(-\Delta\right)^{\beta}\psi_2^n\left(x\right)dx=0.$$

Consequently, by Green's formula, we can obtain:

$$\int_{\Omega}\Delta\psi_1^n\left(x\right)dx=0,$$

which implies:

$$\int_{\Omega}\frac{\partial}{\partial t}\left(\psi_1^n+\psi_2^n\right)dx\leq C\int_{\Omega}\left(\psi_1^n+\psi_2^n\right)dx\quad\text{or}\quad\frac{\frac{\partial}{\partial t}\displaystyle\int_{\Omega}\left(\psi_1^n+\psi_2^n\right)dx}{\displaystyle\int_{\Omega}\left(\psi_1^n+\psi_2^n\right)dx}\leq C.$$

With the use of integrating the above assertion over $[0,t]$, we find:

$$\ln\int_{\Omega}\left(\psi_1^n+\psi_2^n\right)dx\bigg|_0^t\leq Ct\quad\text{or}\quad\ln\frac{\displaystyle\int_{\Omega}\left(\psi_1^n+\psi_2^n\right)dx}{\displaystyle\int_{\Omega}\left(\psi_{1_0}^n+\psi_{2_0}^n\right)dx}\leq Ct.$$

This immediately leads us to infer the following inequality:

$$\frac{\displaystyle\int_{\Omega}\left(\psi_1^n+\psi_2^n\right)dx}{\displaystyle\int_{\Omega}\left(\psi_{1_0}^n+\psi_{2_0}^n\right)dx}\leq\exp\left(Ct\right),$$

which gives:

$$\Rightarrow\int_{\Omega}\left(\psi_1^n+\psi_2^n\right)dx\leq\exp\left(Ct\right)\int_{\Omega}\left(\psi_{1_0}^n+\psi_{2_0}^n\right)dx$$

$$\Rightarrow\int_{\Omega}\left(\psi_1^n+\psi_2^n\right)dx\leq\exp\left(Ct\right)\int_{\Omega}\left(\psi_{1_0}+\psi_{2_0}\right)dx,\ \text{as if}\ \psi_{1_0}^n\leq\psi_{1_0},\ \psi_{2_0}^n\leq\psi_{2_0}.$$

Now, we assume:

$$M\left(t\right)=\exp\left(Ct\right)\left\|\psi_{1_0}+\psi_{2_0}\right\|_{L^1(\Omega)}.$$

Consequently, we note that as ψ_1^n and ψ_2^n are positives, then we gain:

$$\|\psi_1^n+\psi_2^n\|_{L^1(\Omega)}\leq M(t),\ 0\leq t\leq T_{\max},$$

which completes the proof. □

Lemma 3.5. *For any solution (ψ_1^n,ψ_2^n) of system (3.1), there is a constant $L(t)$, which depends only on t, such that:*

$$\|\psi_1^n+\psi_2^n\|_{L^1(Q)}\leq L(t)\|\psi_{1_0}+\psi_{2_0}\|_{L^1(\Omega)}.$$

Proof. Let us multiply the first equation of system (3.2) by θ, and then integrate the result over Q with using the two equations (2.5) and (2.6). This gives:

$$\begin{aligned}\int_Q\psi_{1_n}\theta dxdt&=\int_Q T_1\psi_{1_0}^n(x)\theta dxdt+\int_Q\left(\int_0^t T_1(t-\tau)f(\psi_1^n,\psi_2^n)d\tau\right)\theta dxdt\\&=\int_\Omega\psi_{1_0}^n(x)\Phi(0,x)dx+\int_Q f(\psi_1^n,\psi_2^n)\Phi(\tau,x)dxd\tau.\end{aligned}$$

In a similar manner, we can find:

$$\int_Q\psi_2^n\theta dxdt=\int_\Omega\psi_{2_0}^n(x)\Phi(0,x)dx+\int_Q g(\psi_1^n,\psi_2^n)\Phi(\tau,x)dxd\tau.$$

Therefore, we obtain:

$$\begin{aligned}&\int_Q(\psi_1^n+\psi_2^n)\theta dxdt=\\&\int_\Omega\left(\psi_{1_0}^n(x)+\psi_{2_0}^n(x)\right)\Phi(0,x)dx+\int_Q\left(f(\psi_1^n,\psi_2^n)+g(\psi_1^n,\psi_2^n)\right)\Phi(\tau,x)dxd\tau\\&\leq\int_\Omega\left(\psi_{1_0}^n(x)+\psi_{2_0}(x)\right)\Phi(0,x)dx+\int_Q C(\psi_1^n+\psi_2^n)\Phi(\tau,x)dxd\tau.\end{aligned}$$

Using Holder inequality yields:

$$\begin{aligned}&\int_Q(\psi_1^n+\psi_2^n)\theta dxdt\\&\leq\|\psi_{1_0}+\psi_{2_0}\|_{L^1(\Omega)}.\|\Phi(0,x)\|_{L^\infty(Q)}+C\|\psi_1^n+\psi_2^n\|_{L^1(Q)}.\|\Phi\|_{L^\infty(Q)}\\&\leq\left(\|\psi_{1_0}+\psi_{2_0}\|_{L^1(\Omega)}+C\|\psi_1^n+\psi_2^n\|_{L^1(Q)}\right).\|\Phi\|_{L^\infty(Q)}\\&\leq\max(1,C)\left(\|\psi_{1_0}+\psi_{2_0}\|_{L^1(\Omega)}+\|\psi_1^n+\psi_2^n\|_{L^1(Q)}\right).\|\Phi\|_{L^\infty(Q)}\\&\leq L_1(t)\left(\|\psi_{1_0}+\psi_{2_0}\|_{L^1(\Omega)}+\|\psi_1^n+\psi_2^n\|_{L^1(Q)}\right).\|\theta\|_{L^\infty(Q)},\end{aligned}$$

where $L_1(t)\geq\max(c,cC)$. Since θ is arbitrary in $C_0^\infty(Q)$, then we have:

$$\|\psi_1^n+\psi_2^n\|_{L^1(Q)}\leq L_1(t)\left(\|\psi_{1_0}+\psi_{2_0}\|_{L^1(\Omega)}+\|\psi_1^n+\psi_2^n\|_{L^1(Q)}\right).$$

Hence, by taking $L(t)=\frac{L_1(t)}{1-L_1(t)}$, we get:

$$\|\psi_1^n+\psi_2^n\|_{L^1(Q)}\leq L(t)\|\psi_{1_0}+\psi_{2_0}\|_{L^1(\Omega)},$$

which finishes the proof of this result. □

4. Global existence of a solution of the main system

In this section, we will provide one of the main results of this work. In particular, with the help of using the four assumptions (1.2)-(1.5), we will explore the global existence of a solution for system (1.1).

Theorem 4.1. *Suppose that the hypotheses (1.2)-(1.5) are satisfied. Then, there exists a solution* (ψ_1, ψ_2) *of the following two integral equations:*

$$(4.1)\quad \begin{cases} \psi_1(t) = T_1(t)\psi_{1_0} + \int_0^t T_1(t-\tau) f(\psi_1(\tau), \psi_2(\tau))\, d\tau, \ \forall t \in [0, T[\,, \\ \psi_2(t) = T_2(t)\psi_{2_0} + \int_0^t T_2(t-\tau) g(\psi_1(\tau), \psi_2(\tau))\, d\tau, \ \forall t \in [0, T[\,, \end{cases}$$

where $\psi_1, \psi_2 \in C\left([0, +\infty[\,, L^1(\Omega)\right)$, $f(\psi_1, \psi_2), g(\psi_1, \psi_2) \in L^1(Q)$ *such that* $Q = (0, T) \times \Omega$, *for all* $T > 0$, *and where* $T_1(t)$ *and* $T_2(t)$ *are the semigroups of contractions in* $L^1(\Omega)$ *generated by* $d_1\Delta$ *and* $-d_2(-\Delta)^\beta$.

Proof. Let $T_d(t)$ be a compact semigroup of contractions engendered by the operator $d(-\Delta)^\delta$. We define the operator B as:

$$B : (w_0, h) \to T_d(t) w_0 + \int_0^t T_d(t-\tau) h(\tau)\, d\tau.$$

It should be noticed that this operator is compact $L^1(Q_T) \times L^1(Q_T)$ in $L^1(Q_T)$. This is because it is a result of adding two compact operators in $L^1(Q_T)$, see Theorem 2.6. Consequently, there is a subsequence $\left(\psi_{1_j}^n, \psi_{2_j}^n\right)$ of (ψ_1^n, ψ_2^n) such that $\left(\psi_{1_j}^n, \psi_{2_j}^n\right)$ converges towards (ψ_1, ψ_2) in $L^1(Q) \times L^1(Q)$. Let us now show that $\left(\psi_{1_j}^n, \psi_{2_j}^n\right)$ is a solution of system (3.2), i.e.,

$$(4.2)\quad \begin{cases} \psi_{1_j}^n(t, x) = T_1(t)\psi_{1_0}^n + \int_0^t T_1(t-\tau) f\left(\psi_{1_j}^n(\tau), \psi_{2_j}^n(\tau)\right) d\tau, \\ \psi_{2_j}^n(t, x) = T_2(t)\psi_{2_0}^n + \int_0^t T_2(t-\tau) g\left(\psi_{1_j}^n(\tau), \psi_{2_j}^n(\tau)\right) d\tau. \end{cases}$$

Thus, it is enough to show that (ψ_1, ψ_2) satisfies system (4.1). To this aim, it is clear to note that if $j \to +\infty$, we gain:

$$\psi_{1_0}^n \to \psi_{1_0}, \ \psi_{2_0}^n \to \psi_{2_0},$$

and

$$(4.3)\quad \begin{cases} f\left(\psi_{1_j}^n, \psi_{2_j}^n\right) \to f(\psi_1, \psi_2), \\ g\left(\psi_{1_j}^n, \psi_{2_j}^n\right) \to g(\psi_1, \psi_2) \end{cases} \quad \text{a.e.}$$

Thus to show that (ψ_1, ψ_2) verifies system (4.1), it remains to show that:

$$\begin{cases} f\left(\psi_{1_j}^n, \psi_{2_j}^n\right) \to f(\psi_1, \psi_2), \\ g\left(\psi_{1_j}^n, \psi_{2_j}^n\right) \to g(\psi_1, \psi_2) \end{cases}$$

in $L^1(Q)$ when $j \to +\infty$. For this purpose, we integrate the equations of system (3.1) over Q while taking (2.2) into account. In other words, we have:

$$\int_Q (-\Delta)^\beta \psi_{2_j}^n dxdt = 0,$$

Consequently, by Green's formula, we can obtain:

$$\int_\Omega \Delta\psi_1^n(x)\, dx = 0,$$

This immediately yields:

$$\begin{aligned} \int_\Omega \psi_{1_j}^n dx - \int_\Omega \psi_{1_0}^n dx &= \int_Q f\left(\psi_{1_j}^n, \psi_{2_j}^n\right) dxdt, \\ \int_\Omega \psi_{2_j}^n dx - \int_\Omega \psi_{2_0}^n dx &= \int_Q g\left(\psi_{1_j}^n, \psi_{2_j}^n\right) dxdt, \end{aligned}$$

such that:

$$-\int_Q f\left(\psi_{1_j}^n, \psi_{2_j}^n\right) dxdt \leq \int_\Omega \psi_{1_0} dx, \tag{4.4}$$

and

$$-\int_Q g\left(\psi_{1_j}^n, \psi_{2_j}^n\right) dxdt \leq \int_\Omega \psi_{2_0} dx. \tag{4.5}$$

Now, let us assume:

$$\begin{aligned} N_n &= C\left(\psi_{1_j}^n + \psi_{2_j}^n\right) - f\left(\psi_{1_j}^n, \psi_{2_j}^n\right), \\ M_n &= C\left(\psi_{1_j}^n + \psi_{2_j}^n\right) - f\left(\psi_{1_j}^n, \psi_{2_j}^n\right) - g\left(\psi_{1_j}^n, \psi_{2_j}^n\right). \end{aligned}$$

Then, according to (1.4) and (1.5), it is clear that N_n and M_n are positives. Thus, with using of (4.4) and (4.5), we obtain:

$$\begin{aligned} \int_Q N_n dxdt &\leq C\int_Q \left(\psi_{1_j}^n + \psi_{2_j}^n\right) dxdt + \int_\Omega \psi_{1_0} dx, \\ \int_Q M_n dxdt &\leq C\int_Q \left(\psi_{1_j}^n + \psi_{2_j}^n\right) dxdt + \int_\Omega (\psi_{1_0} + \psi_{2_0})\, dx. \end{aligned}$$

This with the use of Lemma 3.5 give:

$$\int_Q N_n dxdt < +\infty, \ \int_Q M_n dxdt < +\infty,$$

which implies:

$$\int_Q \left| f\left(\psi_{1_j}^n, \psi_{2_j}^n\right)\right| dxdt \leq C \int_Q \left(\psi_{1_j}^n + \psi_{2_j}^n\right) dxdt + \int_Q N_n dxdt < +\infty,$$

and

$$\int_Q \left| g\left(\psi_{1_j}^n, \psi_{2_j}^n\right)\right| dxdt \leq C \int_Q \left(\psi_{1_j}^n + \psi_{2_j}^n\right) dxdt + \int_Q M_n dxdt < +\infty.$$

Now, we assume:

$$h_n = N_n + C\left(\psi_{1_j}^n + \psi_{2_j}^n\right), \quad \Psi_n = M_n + C\left(\psi_{1_j}^n + \psi_{2_j}^n\right),$$

It should be noted here that h_n and Ψ_n are positives in $L^1(Q)$ such that:

$$\left| f\left(\psi_{1_j}^n, \psi_{2_j}^n\right)\right| \leq h_n \text{ a.e and } \left| g\left(\psi_{1_j}^n, \psi_{2_j}^n\right)\right| \leq \Psi_n \text{ a.e.}$$

Combining this result with (4.3) yields, based on the dominated convergence, the following assertions:

$$\begin{matrix} f\left(\psi_{1_j}^n, \psi_{2_j}^n\right) \to f(\psi_1, \psi_2) \\ g\left(\psi_{1_j}^n, \psi_{2_j}^n\right) \to g(\psi_1, \psi_2) \end{matrix} \quad \text{in } L^1(Q).$$

Accordingly, by passing in the limit $j \to +\infty$ of (4.2) in $L^1(Q)$, we obtain:

$$\begin{cases} \psi_1(t) = T_1(t)\psi_{1_0} + \int_0^t T_1(t-\tau) f(\psi_1(\tau), \psi_2(\tau)) d\tau, \\ \psi_2(t) = T_2(t)\psi_{2_0} + \int_0^t T_2(t-\tau) g(\psi_1(\tau), \psi_2(\tau)) d\tau. \end{cases}$$

Hence, (ψ_1, ψ_2) satisfies system (4.1), and consequently (ψ_1, ψ_2) is a solution of the system (1.1). □

5. Conclusions

In this paper, we have investigated the existence of a solution of the fractional reaction-diffusion system. For instance, the compact semigroup methods coupled with some L^1-estimates have been used to prove the global existence of a solution of the fractional reaction-diffusion system. Throughout attaining our purpose, we have introduced and derived several theoretical results related to the existence theory.

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