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On unit sphere tangent bundles over complex Grassmannians[1]

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Abstract. Let $G_{k,n}(\mathbb{C})$ for $2 \leq k < n$ denote the Grassmann manifold of k-dimensional vector subspaces of $\mathbb{C}^n$. In this paper we show that the total space of the unit sphere tangent bundle $S^{2m-1} \to E \overset{p}{\to} G_{k,n}(\mathbb{C})$ is not formal, where $m = k(n-k)$.

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1. Introduction

We begin by fixing notation and recalling some results on differential graded algebras. All vector spaces and algebras are taken over the field $\mathbb{Q}$ of rational numbers.

Definition 1.1. A graded algebra A is a sum $A = \underset{n\geq 0}{\oplus} A^n$, where A^n is a vector space, together with an associative multiplication $A^i \otimes A^j \to A^{i+j}, x \otimes y \mapsto xy$ and has $1 \in A^0$. It is graded commutative if for any homogeneous elements x and y,

$$xy = (-1)^{|x||y|}yx$$

where $|x| = i$ for $x \in A^i$. If A is a graded algebra equipped with a linear differential map $d : A^n \to A^{n+1}$ such that $d \circ d = 0$ and

$$d(xy) = (dx)y + (-1)^{|x|}x(dy),$$

then (A, d) is called a differential graded algebra and d is called a differential. Moreover, if A is also a graded commutative algebra, then (A, d) is a commutative differential graded algebra (*cdga*). It is said to be connected if $H^0(A) \cong \mathbb{Q}$.

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Definition 1.2. Let $V = \oplus_{i\geq 0}V^i$ with $V^{\text{even}} := \oplus_{i\geq 0}V^{2i}$ and $V^{\text{odd}} := \oplus_{i\geq 1}V^{2i-1}$. A commutative graded algebra A is called free commutative if $A = \wedge V = S(V^{\text{even}})\otimes E(V^{\text{odd}})$, where $S(V^{\text{even}})$ is the symmetric algebra on V^{even} and $E(V^{\text{odd}})$ is the exterior algebra on V^{odd}.

Definition 1.3. A Sullivan algebra is a commutative differential graded algebra $(\wedge V, d)$ where $V = \cup_{k\geq 0}V(k)$ and $V(0) \subset V(1)\cdots$ such that $dV(0) = 0$ and $dV(k) \subset \wedge V(k-1)$. It is called minimal if $dV \subset \wedge^{\geq 2}V$.

If (A, d) is a *cdga* of which the cohomology is connected and finite dimensional in each degree, then there always exists a quasi-isomorphism from a Sullivan algebra $(\wedge V, d)$ to (A, d) [4]. To each simply connected space, Sullivan associates a *cdga* $A_{PL}(X)$ of rational polynomial differential forms on X that uniquely determines the rational homotopy type of X [10]. A minimal Sullivan model of X is a minimal Sullivan model of $A_{PL}(X)$.

Definition 1.4. A morphism of commutative differential graded algebras $f : (\wedge V, d) \to (\wedge V\otimes\wedge W, D)$ is a Koszul-Sullivan extension (KS-extension for short) if $Dv = dv$ for $v \in V$ and $DW \subset \wedge V \otimes \wedge W$.

Let $X \overset{\iota}{\to} E \overset{p}{\to} B$ be a fibration between simply connected spaces with $(\wedge V, d)$ and $(\wedge W, d')$ Sullivan models of B, X respectively, and at least one of $H^*(B;\mathbb{Q})$ and $H^*(X;\mathbb{Q})$ has finite type. Then there is a KS-extension $(\wedge V, d) \overset{\varphi}{\to} (\wedge V \otimes \wedge W, D) \overset{\psi}{\to} (\wedge W, d')$, where φ and ψ are respective models for p and ι, see [4, §15].

Definition 1.5. [5] A simply connected space X is called formal if there is a quasi-isomorphism $(\wedge V, d) \to H^*(\wedge V, d)$, where $(\wedge V, d)$ is the minimal Sullivan model of X.

Examples of formal spaces include spheres, projective complex spaces, homogeneous spaces G/H, where G and H have same rank, and compact Kähler manifolds.

Definition 1.6. [5] Let (A, d) be a *cdga* with cohomology $H^*(A, d)$. Let a, b, and c be cohomology classes in $H^*(A, d)$ whose products $a \cdot b = b \cdot c = 0$. Choose cocycles x, y and z representing a, b and c respectively. Then there are elements v and w such that $dv = xy$ and $dw = yz$. The element

$$vz - (-1)^{|x|}xw$$

is a cocycle whose cohomology class depends on the choice of v and w. Each cohomology class

$$vz - (-1)^{|x|}xw$$

is called a *triple Massey product* of a, b and c. If a triple Massey product is 0 as a cohomology class, then it is said to be trivial.

Theorem 1.7. *[5] If X has a non-trivial triple Massey product then X is not formal.*

2. Model of the unit sphere tangent bundle over complex Grassmannians

The complex Grassmannian $G_{k,n}(\mathbb{C})$ is a homogeneous space as $G_{k,n}(\mathbb{C}) \cong U(n)/(U(k) \times U(n-k))$ for $1 \leq k < n$, where $U(n)$ is the unitary group. It is a symplectic manifold of dimension $2m$, where $m = k(n-k)$. The method to compute a Sullivan model of the homogeneous space $G_{k,n}(\mathbb{C})$ is given in detail in [6, 9].

Let $S^{2m-1} \to E \xrightarrow{p} G_{k,n}(\mathbb{C})$ for $2 \leq k < n$ be the unit sphere tangent bundle. A relative minimal model of p is given by

$$(\wedge V, d) \overset{\iota}{\rightarrowtail} (\wedge V \otimes \wedge x_{2m-1}, d') \to (\wedge x_{2m-1}, 0),$$

with $d'v = dv$ for $v \in V$ and $d'x_{2m-1} = z$, as $[z]$ is the Euler class of the tangent bundle [5, Page 82]. Moreover, if $[\omega] \in H^{2m}(\wedge V, d)$ is the fundamental class of $G_{k,n}(\mathbb{C})$, then $[z] = \chi(G_{k,n}(\mathbb{C})) \cdot [\omega]$, where $\chi(G_{k,n}(\mathbb{C}))$ is the Euler characteristic of $G_{k,n}(\mathbb{C})$ (see [2, Proposition 11.24]). As $\chi(G_{k,n}(\mathbb{C})) \neq 0$, there is a quasi-isomorphism

$$(\wedge V \otimes \wedge x_{2m-1}, d') \to (\wedge V \otimes \wedge x_{2m-1}, D),$$

where $Dv = dv$ for $v \in V$ and $Dx_{2m-1} = \omega$.

The unit sphere tangent bundles over complex projective spaces were studied in [1], where it is shown that the total space of the unit sphere tangent bundle over $\mathbb{C}P(n)$ is formal. We extend this study to $G_{k,n}(\mathbb{C})$ for $2 \leq k < n$ and obtain the following result.
We provide here an easier proof for the case $k = 2$ as follows.

Theorem 2.1. *The total space of the unit sphere tangent bundle*

$$S^{2m-1} \to E \to G_{2,n}(\mathbb{C})$$

for $n \geq 4$ is not formal.

Proof. Recall that $G_{2,n}(\mathbb{C})$ is a manifold of dimension $2m$ where $m = 2n-4$. In [7, 3], the cohomology ring $H^*(G_{k,n}(\mathbb{C}), \mathbb{Q})$ has the presentation

$$H^*(G_{k,n}(\mathbb{C}), \mathbb{Q}) = \mathbb{Q}[b_2, \ldots, b_{2k}]/ < h_{n-k+1}, \ldots, h_n >$$

where $< h_{n-k+1}, \ldots, h_n >$ is the ideal generated by the elements h_j for $n-k+1 \leq j \leq n$. Here h_j is the $2j$-degree term in the Taylor series expansion of $(1 + b_2 + \cdots + b_{2k})^{-1}$. As $\{h_{n-k+1}, \ldots, h_n\}$ form a regular sequence in the polynomial algebra $\mathbb{Q}[b_2, \ldots, b_{2k}]$, the minimal Sullivan model of $G_{k,n}(\mathbb{C})$ is

$$(\wedge(b_2, \ldots, b_{2k}, y_{2(n-k)+1}, \ldots, y_{2n-1}), d),$$

where $db_i = 0$ and

$$dy_{2(n-k)+1} = h_{n-k+1}$$
$$\vdots$$
$$dy_{2n-1} = h_n.$$

In particular, the minimal Sullivan model of $G_{2,n}(\mathbb{C})$ is given by

$$(\wedge V, d) = (\wedge(b_2, b_4, y_{2n-3}, y_{2n-1}), d),$$

where $dy_{2n-3} = h_{n-1}$ and $dy_{2n-1} = h_n$ are polynomials in b_2 and b_4. The symplectic class $[b_2] \in H^2(G_{2,n}(\mathbb{C}), \mathbb{Q})$ is such that $[b_2^m]$ is the fundamental class of $G_{2,n}(\mathbb{C})$. As $\chi(G_{2,n}(\mathbb{C})) \neq 0$, a relative minimal model for the unit sphere tangent bundle $S^{2m-1} \to E \to G_{2,n}(\mathbb{C})$ is given by

$$(\wedge V, d) \xrightarrow{\iota} (\wedge V \otimes \wedge x_{2m-1}, D) \to (\wedge x_{2m-1}, 0),$$

with $Dv = dv$ for $v \in V$ and $Dx_{2m-1} = b_2^m$. We show that $H^*(E, \mathbb{Q})$ contains a non zero triple Massey product by hypothesis on n and $m \geq 4$. As $Dx_{2m-1} = b_2^m$, we have $H^*(\iota)([b_2]) \cdot H^*(\iota)([b_2^{m-1}]) = 0$ in $H^*(E, \mathbb{Q})$. Either $2n \equiv 2 \pmod 4$ or $2n \equiv 0 \pmod 4$. $dy_{2n-1} = b_2 s$, where $s \notin < b_2 >$ and $[s] \in H^{2n-2}(G_{2,n}(\mathbb{C}), \mathbb{Q})$ as $2n \equiv 2 \pmod 4$ implies h_n does not contain a power of b_4. $[s] \in H^{2n-2}(G_{2,n}(\mathbb{C}), \mathbb{Q})$ is the non-zero class of smallest degree such that $H^*(\iota)([b_2]) \cdot H^*(\iota)([s]) = 0$. Moreover, if $2n \equiv 0 \pmod 4$, then $dy_{2n-3} = b_2 r$, where $r \notin < b_2 >$ and $[r] \in H^{2n-4}(G_{2,n}(\mathbb{C}), \mathbb{Q})$ is the non-zero class of smallest degree such that $H^*(\iota)([b_2]) \cdot H^*(\iota)([r]) = 0$. On the one hand, assume that $2n \equiv 2 \pmod 4$, then $dy_{2n-1} = b_2 s$ and the element

$$s x_{2m-1} - b_2^{m-1} y_{2n-1}$$

is a cocycle of degree $2(m+n) - 3$ which cannot be a coboundary for degree reasons. Hence, the triple Massey product set $\langle H^*(\iota)([b_2^{m-1}]), H^*(\iota)([b_2]), H^*(\iota)([s])\rangle$ is non-trivial. Similarly, on the other hand, if $2n \equiv 0 \pmod 4$, then $dy_{2n-3} = b_2 r$ and the element

$$r x_{2m-1} - b_2^{m-1} y_{2n-3}$$

is a cocycle of degree $2(m+n) - 5$ which cannot be a coboundary for degree reasons. Thus, the triple Massey product set $\langle H^*(\iota)([b_2^{m-1}]), H^*(\iota)([b_2]), H^*(\iota)([r])\rangle$ is non-trivial. Thus E is not formal. □

Example 2.2. The minimal Sullivan model of $G_{2,4}(\mathbb{C})$ is given by

$$(\wedge(b_2, b_4, y_5, y_7), d),$$

where

$$db_2 = db_4 = 0, \;\; dy_5 = -b_2^3 + 2b_2 b_4, \;\; dy_7 = b_2^4 - 3b_2^2 b_4 + b_4^2$$

as h_j is the $2j$-th degree term in the Taylor expansion of $(1 + b_2 + b_4)^{-1}$ [6, 8]. With $\chi(G_{2,4}(\mathbb{C})) = 5$, the total space of the unit sphere bundle $S^7 \to E \to G_{2,4}(\mathbb{C})$ will have a relative minimal model of the form

$$(\wedge(b_2, b_4, y_5, y_7, a_7), D)$$

with $Db_i = 0$, $Dy_5 = b_2(b_2^2 - 2b_4)$, $Dy_7 = b_4 b_2^2 - b_4^2$ and $Da_7 = b_2^4$. Take $a = H^*(\iota)([b_2^3])$, $b = H^*(\iota)([b_2])$ and $c = H^*(\iota)([b_2^2 - 2b_4])$ cohomology classes in $H^*(E, \mathbb{Q})$ whose products $a \cdot b = b \cdot c = 0$. The triple Massey product set $\langle a, b, c\rangle$ is represented by the cocycle

$$(b_2^2 - 2b_4)a_7 - b_2^3 y_5$$

of degree 11 which cannot be a coboundary for degree reasons. Thus, the triple Massey product set $\langle a, b, c\rangle$ is non-trivial.

For the general case, a Sullivan model of $G_{k,n}(\mathbb{C})$ for $1 \leq k < n$ is given by (see [9])

$$(\wedge(b_2, b_4, \ldots, b_{2k}, x_2, x_4, \ldots, x_{2(n-k)}, y_1, y_3, \ldots, y_{2n-1}), d)$$

with

$$db_i = 0 = dx_j,\ dy_{2p-1} = \sum_{p_1+p_2=p} b_{2p_1} \cdot x_{2p_2},\ \ 1 \leq p \leq n.$$

Lemma 2.3. *For $2 \leq k < n$ and $n \geq 2k$, the minimal Sullivan model of $G_{k,n}(\mathbb{C})$ is given by*

$$(\wedge(b_2, \ldots, b_{2k}, y_{2(n-k)+1}, \ldots, y_{2n-1}), d),\ dy_{2n-1} = b_{2k} r$$

where $r \notin < b_{2k} >$. It is enough to choose $n \geq 2k$ as $G_{k,n}(\mathbb{C})$ is homeomorphic to $G_{n-k,n}(\mathbb{C})$.

Proof. Consider the Sullivan model

$$(\wedge(b_2, b_4, \ldots, b_{2k}, x_2, x_4, \ldots, x_{2(n-k)}, y_1, y_3, \ldots, y_{2n-1}), d)$$

of $G_{k,n}(\mathbb{C})$ for $2 \leq k < n$,

$$\begin{aligned} dy_1 &= b_2 + x_2 \\ dy_3 &= b_4 + x_4 + b_2 x_2 \\ &\ \vdots \\ dy_{2n-1} &= b_{2k} x_{2(n-k)}. \end{aligned}$$

The model is not minimal as the linear part is not zero. To find its minimal Sullivan model, we make a change of variable $t_2 = b_2 + x_2$ and replace x_2 by $t_2 - b_2$ wherever it appears in the differential. This gives an isomorphic Sullivan algebra

$$(\wedge(b_2, t_2, b_4, \ldots, b_{2k}, x_4, \ldots, x_{2(n-k)}, y_1, y_3, \ldots, y_{2n-1}), d)$$

where

$$\begin{aligned} dy_1 &= t_2 \\ dy_3 &= b_4 + x_4 + b_2(t_2 - b_2) \\ &\ \vdots \\ dy_{2n-1} &= b_{2k} x_{2(n-k)}. \end{aligned}$$

As the ideal generated by y_1 and t_2 is acyclic, the above Sullivan algebra is quasi-isomorphic to

$$(\wedge(b_2, b_4, \dots, b_{2k}, x_4, \dots, x_{2(n-k)}, y_3, \dots, y_{2n-1}), d)$$

where

$$\begin{aligned} dy_3 &= b_4 + x_4 - b_2^2 \\ &\vdots \\ dy_{2n-1} &= b_{2k}x_{2(n-k)}. \end{aligned}$$

One continues in this fashion and make another change of variable, $t_4 = b_4 + x_4 - b_2^2$ and replace x_4 by $t_4 - b_4 + b_2^2$ wherever it appears in the differential and do so until they reach a change of variable of the form

$$\begin{aligned} t_{2(n-k)} &= b_{2(n-k)} + x_{2(n-k)} + \alpha \text{ for } n = 2k, \text{ or} \\ t_{2(n-k)} &= x_{2(n-k)} + \beta \text{ for } n > 2k, \end{aligned}$$

where $\alpha \in \wedge(b_2, \dots, b_{2(k-1)})$, $\beta \in \wedge(b_2, \dots, b_{2k})$ and replace

$$x_{2(n-k)} = \begin{cases} t_{2(n-k)} - b_{2k} + \alpha & \text{for } n = 2k, \\ t_{2(n-k)} + \beta & \text{for } n > 2k, \end{cases}$$

wherever it appears in the differential. This gives an isomorphic Sullivan algebra

$$(\wedge(b_2, \dots, b_{2k}, y_{2(n-k)-1}, y_{2(n-k)+1}, \dots, y_{2n-1}), d)$$

where

$$\begin{aligned} dy_{2(n-k)-1} &= t_{2(n-k)} \\ &\vdots \\ dy_{2n-1} &= b_{2k}x_{2(n-k)}. \end{aligned}$$

As the ideal generated by $t_{2(n-k)}$ and $y_{2(n-k)-1}$ is acyclic, we get the minimal Sullivan model

$$(\wedge(b_2, \dots, b_{2k}, y_{2(n-k)+1}, \dots, y_{2n-1}), d)$$

with

$$dy_{2n-1} = b_{2k}r$$

where $r \in \wedge(b_2, \dots, b_{2k})$ and $[r] \neq 0$ in $H^*(G_{k,n}(\mathbb{C}), \mathbb{Q})$ as $|r| = 2(n-k)$ and there is no coboundary of degree less than $2(n-k)$. In particular, $[r] \neq [b_{2k}]$. □

Theorem 2.4. *More generally, if $2 \leq k < n$, then the total space of the unit sphere tangent bundle*

$$S^{2m-1} \to E \to G_{k,n}(\mathbb{C})$$

is not formal, where $m = k(n-k)$.

Proof. The minimal Sullivan model of $G_{k,n}(\mathbb{C})$ is given by $(\wedge V, d) = (\wedge(b_2, \ldots, b_{2k}, y_{2(n-k)+1}, \ldots, y_{2n-1}), d)$ and $(\wedge x_{2m-1}, 0)$ is the model of S^{2m-1}. Let $[b_{2k}^*]$ be the Poincaré dual of $[b_{2k}]$ in $H^*(G_{k,n}(\mathbb{C}), \mathbb{Q})$ and $\omega = b_{2k}b_{2k}^*$. Since $\chi(G_{k,n}(\mathbb{C})) \neq 0$, a relative minimal model for the unit sphere tangent bundle $S^{2m-1} \to E \to G_{k,n}(\mathbb{C})$ is given by

$$(\wedge V, d) \overset{\iota}{\rightarrowtail} (\wedge V \otimes \wedge x_{2m-1}, D) \to (\wedge x_{2m-1}, 0),$$

with $Dv = dv$ for $v \in V$ and $Dx_{2m-1} = \omega$. By Lemma 2.3, there is $[r] \in H^{2n-2k}(G_{k,n}(\mathbb{C}), \mathbb{Q})$ the class of smallest degree such that $H^*(\iota)([b_{2k}]) \cdot H^*(\iota)([r]) = 0$ in $H^*(E;\mathbb{Q})$, where $r \notin < b_{2k} >$. We show that the triple Massey product $\langle H^*(\iota)([b_{2k}^*]), H^*(\iota)([b_{2k}]), H^*(\iota)([r])\rangle$ in $H^*(E;\mathbb{Q})$ is not trivial. It is represented by the cocycle

$$rx_{2m-1} - b_{2k}^* y_{2n-1}.$$

To show that it is not a coboundary, we use an argument in the Leray-Serre spectral sequence for the unit sphere tangent bundle $S^{2m-1} \to E \to G_{k,n}(\mathbb{C})$. In [4, Chapter 18], the Leray-Serre spectral sequence is obtained by filtering $(\wedge V \otimes \wedge x_{2m-1}, D)$ by the degree of $\wedge V$; that is,

$$F^p(\wedge V \otimes \wedge x_{2m-1}) = (\wedge V)^{\geq p} \otimes \wedge x_{2m-1},\ p = 0, 1, 2, \ldots$$

and the associated bigraded module is given by

$$\begin{aligned} E_0^{p,q} &= (\wedge V)^{\geq p} \otimes \wedge x_{2m-1}/(\wedge V)^{\geq (p+1)} \otimes \wedge x_{2m-1} \\ &\cong (\wedge V)^p \otimes \wedge x_{2m-1}. \end{aligned}$$

Moreover, $d_0 = 0$, $d_1 = d$ and $E_2^{p,*} = H^p(\wedge V, d) \otimes \wedge x_{2m-1}$. Thus, $[rx_{2m-1} - b_{2k}^* y_{2n-1}] \cong [rx_{2m-1}]$ at $E_2^{2(n-k),q}$ and we have $E_2 = E_3 = \cdots = E_{2m}$. In particular, $E_{2m}^{2(n-k),2m-1} \cong H^{2(n-k)}(\wedge V, d) \otimes \mathbb{Q} < x_{2m-1} >$. Moreover, $d_{2m} : E_{2m}^{2(n-k),2m-1} \to E_{2m}^{2(n-k)+2m,0}$ is zero, for degree reasons. Hence, the element

$$rx_{2m-1} \in E_{2m}^{2(n-k),2m-1}$$

is a d_{2m}-cocycle. Moreover, it cannot be a d_{2m}-coboundary because $E_{2m}^{2(n-k)-2m,4m-2} = 0$. Hence the class

$$[rx_{2m-1}]$$

is not zero at $E_{2m+1} = E_\infty$. This is a non zero triple Massey product. Therefore, E is not formal. □

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