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Oscillation theorems for advanced differential equations

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Abstract. In this paper, we will establish some oscillation criteria for the advanced differential equations

$$u^{'}(t) - \sum_{i=1}^{i=k} q_i(t)\, u^{\alpha}(\tau_i(t)) = 0, \quad \text{for } t \geq t_0$$

where k is an integer and α is a quotient of odd integers, such as $k \geq 1$ and $\alpha \geq 1$. The functions $\{q_i\}_{i\in\{1,\ldots,k\}}$ are continuous positive functions and the arguments $\{\tau_i\}_{i\in\{1,\ldots,k\}}$ are continuous positive functions, such that $\tau_i(t) > t$, for $i \in \{1, ..., k\}$. This study aims to present some new sufficient conditions for the oscillation of solutions to a class of first-order advanced differential equations, using a technique based on a recursive sequence.

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1. Introduction

In this article, we consider the advanced differential equation of the form

$$u^{'}(t) - \sum_{i=1}^{i=k} q_i(t)\, u^{\alpha}(\tau_i(t)) = 0, \quad \text{for } t \geq t_0 \tag{1.1}$$

where k is an integer and α is a quotient of odd integers, such that $k \geq 1$ and $\alpha \geq 1$. The functions $\{q_i\}_{i\in\{1,\ldots,k\}}$, $\{\tau_i\}_{i\in\{1,\ldots,k\}}$ are continuous and positive and they satisfy the conditions stated below:

($\mathcal{H}_1$) $\{\tau_i\}_{i\in\{1,\ldots,k\}} \in \mathcal{C}([t_0,\infty),[t_0,\infty))$ satisfy $\tau_i(t) \geq t$, for $t \geq t_0$ and $\lim_{t\to\infty} \tau_i(t) = \infty$, for $i \in \{1, 2, ..., k\}$,

($\mathcal{H}_2$) $\{q_i\}_{i\in\{1,\ldots,k\}} \in \mathcal{C}([t_0,\infty),[0,\infty))$, such that $Q := \sum_{i=1}^{i=k} q_i \neq 0$ on any interval of the form $[t_0,\infty)$ and $\int_t^{\tau(t)} Q(s)\,ds$ increases on $[t_0,\infty)$, where $\tau(t) := \min\{\tau_i(t) : i \in \{1, ..k\}\}$, for $t \geq t_0$.

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By a solution of (1.1) we mean a nontrivial real-valued function u which is an element of the set $\mathcal{C}^1([T_u,\infty),\mathbb{R})$, $T_u \in [t_0,\infty)$ which satisfies (1.1) on $[T_u,\infty)$. The solutions vanishing in some neighbourhood of infinity will be excluded from our consideration. A solution u of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

Today there has been an increasing interest in obtaining sufficient conditions for oscillation and non oscillation of solutions of advanced type differential equations, we refer the reader to the articles [2, 3, 4, 1, 5, 6, 7, 8, 9, 11, 12, 13] and the references cited therein. So far, there are some results on oscillation of (1.1). In the present work, we study further (1.1) and derive new sufficient oscillation conditions.

2. Oscillation Results

To derive main results in this section, we need the following lemmas.

Definition 2.1. Let us define a sequence of functions by the recurrence relation

$$J_{n+1}(t) := \sum_{i=1}^{i=k} \int_t^{\tau(t)} q_i(s) \exp(J_n(t))\, ds, \quad \text{for } t \geq t_0, \tag{2.1}$$

with

$$J_0(t) := \sum_{i=1}^{i=k} \int_t^{\tau(t)} q_i(s)\, ds, \quad \text{for } t \geq t_0, \tag{2.2}$$

Lemma 2.2. *Assume $(\mathcal{H}_1)-(\mathcal{H}_2)$ hold and $\alpha = 1$. If u is a positive solution of (1.1), then the sequence $\{J_n(t) : n \in \mathbb{N}\}$ converges.*

Proof. Let u be an eventually positive solution of (1.1). From (1.1), we have $u'(t) \geq 0$, for $t \geq t_0$. On the other hand, for $i \in \{1, ..., k\}$, we have

$$\begin{aligned} \ln\left(\frac{u(\tau_i(t))}{u(t)}\right) &= \int_t^{\tau_i(t)} \frac{u'(s)}{u(s)} ds = \sum_{m=1}^{m=k} \int_t^{\tau_i(t)} q_m(s) \frac{u(\tau_m(s))}{u(s)} ds \\ &\geq \sum_{m=1}^{m=k} \int_t^{\tau(t)} q_m(s) \frac{u(\tau_m(s))}{u(s)} ds \\ &\geq \sum_{m=1}^{m=k} \int_t^{\tau(t)} q_m(s)\, ds \geq J_0(t), \quad \text{for } t \geq t_0. \end{aligned} \tag{2.3}$$

This means,

$$\frac{u(\tau_i(t))}{u(t)} \geq \exp(J_0(t)), \quad \text{for } t \geq t_0 \text{ and for } i \in \{1, ..., k\}.$$

From (2.3) and the above inequality, we obtain

$$\ln\left(\frac{u(\tau_i(t))}{u(t)}\right) \geq \sum_{m=1}^{m=k} \int_t^{\tau(t)} q_m(s) \exp(J_0(s))\, ds := J_1(t), \quad \text{for } t \geq t_0.$$

By induction, we can see that if

$$\ln\left(\frac{u(\tau_i(t))}{u(x)}\right) \geq J_n(t), \quad \text{for } t \geq t_0 \text{ and for } i \in \{1, ..., k\}.$$

In the same way, we find that the inequality is true for $n+1$. By (2.1) and the above inequality, we conclude that the sequence $\{J_n(t) : n \in \mathbb{N}\}$ is increasing, thus $\{J_n(t) : n \in \mathbb{N}\}$ converges. □

Lemma 2.3. *Assume* $(\mathcal{H}_1)-(\mathcal{H}_2)$ *hold and* $\alpha = 1$*. The sequence* $\{J_n(t) : n \in \mathbb{N}\}$ *defined by* (2.1)*, converges if and only if*

$$\sum_{i=1}^{i=k} \int_t^{\tau(t)} q_i(s)\, ds \leq \frac{1}{e}, \quad \textit{for all } t \geq t_0. \tag{2.4}$$

Proof. Sufficient: Suppose that (2.2) is true. Then

$$J_0(t) \leq \frac{1}{e} = v_0, \quad \text{for all } t \geq t_0,$$

Then, we get

$$\begin{aligned} J_1(t) &\leq \sum_{i=1}^{i=k} \int_t^{\tau(t)} q_i(s) \exp(J_0(t))\, ds \\ &\leq \sum_{i=1}^{i=k} \int_t^{\tau(t)} q_i(s)\, ds \exp(v_0) \leq v_0 \exp(v_0) = v_1. \end{aligned}$$

By induction, we can see that if

$$J_n(t) \leq v_0 \exp(v_n) < 1.$$

In view of Lemma [10, Lemma 1], $\{J_n(t) : n \in \mathbb{N}\}$ converges.
Necessary: Suppose that $\{J_n(t) : n \in \mathbb{N}\}$ converges, then there is a positive real function denoted $J(t)$, such that $J(t) = \lim_{n\to\infty} J_n(t)$, by (2.1), we find that the function J satisfies

$$J(t) = \sum_{i=1}^{i=k} \int_t^{\tau(t)} q_i(s) \exp(J(s))\, ds, \quad \text{for } t \geq t_0. \tag{2.5}$$

By the hypothesis, we have that the function J_0 is increasing on $[t_0, \infty)$, then by induction deduce that functions J_n are increasing on $[t_0, \infty)$, we conclude that the function J increases on $[t_0, \infty)$. By the above equality, we obtain

$$\sum_{i=1}^{i=k} \int_t^{\tau(t)} q_i(s)\, ds \leq J(t) \exp(-J(t)), \quad \text{for } t \geq t_0.$$

On the other hand, we have

$$\max\{x \exp(-x) : x \geq 1\} = \frac{1}{e}.$$

By (2.5), deduce that $J(t) \geq 1$, for $t \geq t_0$. From the above, we deduce

$$\sum_{i=1}^{i=k} \int_t^{\tau(t)} q_i(s)\, ds \leq \frac{1}{e}, \quad \text{for } t \geq t_0.$$

This completes the proof. □

Remark 2.4. Assume $(\mathcal{H}_1) - (\mathcal{H}_2)$ hold and $\alpha = 1$. If u is a positive solution of (1.1), then inequality (2.4) is satisfied.

Next, we consider the advanced differential equation (1.1) subject to the initial condition

$$u(t_0) := a > 0. \tag{2.6}$$

Definition 2.5. Let us define a sequence of functions by the recurrence relation (2.7)

$$I_{n+1}^{\alpha}(t) := \left(1 + a^{\alpha-1}(\alpha-1)\sum_{i=1}^{i=k} \int_t^{\tau_i(t)} q_i(s)\, I_n^{\alpha}(s)\, ds\right)^{\frac{\alpha}{\alpha-1}}, \quad \text{for } t \geq t_0,$$

with

$$I_0^{\alpha}(t) := \left(1 + a^{\alpha-1}(\alpha-1)\sum_{i=1}^{i=k} \int_t^{\tau(t)} q_i(s)\, ds\right)^{\frac{\alpha}{\alpha-1}}, \quad \text{for } t \geq t_0, \tag{2.8}$$

where $\alpha > 1$.

Lemma 2.6. *Assume $(\mathcal{H}_1) - (\mathcal{H}_2)$ hold and $\alpha > 1$. If u is a positive solution of* (1.1)*, then the sequence $\{I_n^{\alpha}(t) : n \in \mathbb{N}\}$ converges.*

Proof. Let u be an eventually positive solution of (1.1). From (1.1), we have $u'(t) \geq 0$, for $t \geq t_0$. On the other hand, for $i \in \{1, ..., k\}$, we have

$$\begin{aligned}
\frac{1}{u^{\alpha-1}(t)} - \frac{1}{u^{\alpha-1}(\tau_i(t))} &= (\alpha-1)\int_t^{\tau_i(t)} \frac{u'(s)}{u^{\alpha}(s)} ds \\
&= (\alpha-1)\sum_{m=1}^{m=k} \int_t^{\tau_i(t)} q_m(s)\, \frac{u^{\alpha}(\tau_m(s))}{u^{\alpha}(s)} ds \\
&\geq (\alpha-1)\sum_{m=1}^{m=k} \int_t^{\tau(t)} q_m(s)\, \frac{u^{\alpha}(\tau_m(s))}{u^{\alpha}(s)} ds \qquad (2.9)\\
&> (\alpha-1)\sum_{m=1}^{m=k} \int_t^{\tau(t)} q_m(s)\, ds, \text{ for all } t \geq t_0. \qquad (2.10)
\end{aligned}$$

Since u is increasing on $[t_0, \infty)$, then $u(t) \geq u(t_0) = a$, for all $t \geq t_0$. Hence

$$\frac{u^{\alpha-1}(\tau_i(t))}{u^{\alpha-1}(t)} \geq 1 + a^{\alpha-1}\left(\frac{1}{u^{\alpha-1}(t)} - \frac{1}{u^{\alpha-1}(\tau_i(t))}\right), \quad \text{for all } t \geq t_0. \tag{2.11}$$

From (2.10) and the above inequality, we obtain

$$\begin{aligned}\frac{u^{\alpha}\left(\tau_i\left(t\right)\right)}{u^{\alpha}\left(t\right)} &\geq \left(1+a^{\alpha-1}\left(\alpha-1\right)\sum\nolimits_{m=1}^{m=k}\int_t^{\tau(t)} q_m\left(s\right)ds\right)^{\frac{\alpha}{\alpha-1}} \\ &= I_0^{\alpha}\left(t\right), \quad \text{for all } t\geq t_0.\end{aligned}$$

From (2.9), (2.11) and the above inequality, we obtain

$$\frac{u^{\alpha-1}\left(\tau_i\left(t\right)\right)}{u^{\alpha-1}\left(t\right)}\geq 1+a^{\alpha-1}\left(\alpha-1\right)\sum\nolimits_{m=1}^{m=k}\int_t^{\tau(t)} q_m\left(s\right)I_0^{\alpha}\left(s\right)ds, \quad \text{for all } t\geq t_0,$$

or

$$\begin{aligned}\frac{u^{\alpha}\left(\tau_i\left(t\right)\right)}{u^{\alpha}\left(t\right)} &\geq \left(1+a^{\alpha-1}\left(\alpha-1\right)\sum\nolimits_{m=1}^{m=k}\int_t^{\tau(t)} q_m\left(s\right)I_0^{\alpha}\left(t\right)ds\right)^{\frac{\alpha}{\alpha-1}} \\ &= I_1^{\alpha}\left(t\right), \quad \text{for } t\geq t_0.\end{aligned}$$

By induction, we can see that

$$\frac{u^{\alpha}\left(\tau_i\left(t\right)\right)}{u^{\alpha}\left(t\right)}\geq I_n^{\alpha}\left(t\right), \quad \text{for } t\geq t_0 \text{ and for } i\in\left\{1,...,k\right\}.$$

In the same way, we find that the inequality is true for $n+1$. We conclude that the sequence $\left\{I_n^{a}\left(t\right):n\in\mathbb{N}\right\}$ is increasing and bounded, then $\left\{I_n^{\alpha}\left(t\right):n\in\mathbb{N}\right\}$ converges. □

Lemma 2.7. *The sequence* $\left\{I_n^{\alpha}\left(t\right):n\in\mathbb{N}\right\}$ *defined by* (2.7) *converges if and only if*

$$\sum\nolimits_{i=1}^{i=k}\int_t^{\tau(t)} q_i\left(s\right)ds\leq \frac{a^{1-\alpha}}{\alpha^{\frac{\alpha}{\alpha-1}}}, \tag{2.12}$$

where $\alpha>1$.

Proof. Suppose that $\left\{I_n^{\alpha}\left(t\right):n\in\mathbb{N}\right\}$ converges. Then there is a positive real function denoted $I^{\alpha}\left(t\right)$, such that $I^{\alpha}\left(t\right)=\lim\limits_{n\to\infty}I_n^{\alpha}\left(t\right)$, by (2.7), we find that the function I^{α} satisfies

(2.13)

$$I^{\alpha}\left(t\right)=\left(1+a^{\alpha-1}\left(\alpha-1\right)\sum\nolimits_{i=1}^{i=k}\int_t^{\tau(t)} q_i\left(s\right)I^{\alpha}\left(s\right)ds\right)^{\frac{\alpha}{\alpha-1}}, \quad \text{for } t\geq t_0.$$

By the hypothesis, we have that the function I_0^{α} is increasing on $[t_0,\infty)$, then by induction deduce that functions I_n^{α} are increasing on $[t_0,\infty)$, we conclude that the function I^{α} increases on $[t_0,\infty)$. By the above equality, we obtain

$$a^{\alpha-1}\left(\alpha-1\right)\sum\nolimits_{i=1}^{i=k}\int_t^{\tau(t)} q_i\left(s\right)ds\leq \frac{\left(I^{\alpha}\left(t\right)\right)^{1-\frac{1}{\alpha}}-1}{I^{\alpha}\left(t\right)}, \quad \text{for } t\geq t_0.$$

On the other hand, we have

$$\sup\left\{\frac{x^{1-\frac{1}{\alpha}}-1}{x}:x\geq 1\right\}=\frac{\alpha-1}{\alpha^{\frac{\alpha}{\alpha-1}}},$$

By (2.13), deduce that $I^{\alpha}(t)\geq 1$, for $t\geq t_0$, which means that

$$\sum_{i=1}^{i=k}\int_t^{\tau(t)}q_i(s)\,ds\leq\frac{a^{1-\alpha}}{\alpha^{\frac{\alpha}{\alpha-1}}},\quad\text{for } t\geq t_0.$$

This completes the proof. □

Now, we establish some sufficient conditions which guarantee that every solution u of (1.1) oscillates on $[t_0,\infty)$.

Theorem 2.8. *Assume* $(\mathcal{H}_1)-(\mathcal{H}_2)$ *hold and* $\alpha=1$. *For all sufficiently large* $t_1\geq t_0$, *assume that*

$$\sum_{i=1}^{i=k}\int_t^{\tau(t)}q_i(s)\,ds>\frac{1}{e},\quad\textit{for } t\geq t_1. \tag{2.14}$$

Then any solution of (1.1) *is oscillatory.*

Proof. Suppose that (1.1) has a nonoscillatory solution u on $[t_0,\infty)$. Since $-u$ is also a solution of (1.1), we can confine our discussion only to the case where the solution u is an eventually positive solution of (1.1). We may assume without loss of generality that there exists $t_1\geq t_0$, such that

$$u(t)>0\quad\text{and}\quad u(\tau_i(t))>0,\ \text{for all } t\geq t_1 \text{ and } i\in\{1,2,...,k\}.$$

This means that equation (1.1) has a positive solution u on $[t_1,\infty)$.

$$u^{'}(t)-\sum_{i=1}^{i=k}q_i(t)\,u(\tau_i(t))=0,\quad\text{for } t\geq t_1$$

By Lemma 2.2 and Lemma 2.3, we obtain

$$\sum_{i=1}^{i=k}\int_t^{\tau(t)}q_i(s)\,ds\leq\frac{1}{e},\quad\text{for } t\geq t_1.$$

which contradicts (2.14). This completes the proof. □

Applying the previous result, we deduce the following corollaries.

Corollary 2.9. *Assume* $(\mathcal{H}_1)-(\mathcal{H}_2)$ *hold and* $\alpha=1$, *and assume that*

$$\liminf_{t\to\infty}\sum_{i=1}^{i=k}\int_t^{\tau(t)}q_i(s)\,ds>\frac{1}{e}.$$

Then any solution of (1.1) *is oscillatory.*

Corollary 2.10. *Assume* $(\mathcal{H}_1)-(\mathcal{H}_2)$ *hold, that* $\alpha=1$, *and assume that*

$$\limsup_{t\to\infty}\sum_{i=1}^{i=k}\int_t^{\tau(t)} q_i(s)\,ds>1.$$

Then any solution of (1.1) *is oscillatory.*

Theorem 2.11. *Assume* $(\mathcal{H}_1)-(\mathcal{H}_2)$ *hold and* $\alpha>1$. *For all sufficiently large* $t_1\ge t_0$, *assume that*

$$\sum_{i=1}^{i=k}\int_t^{\tau(t)} q_i(s)\,ds>\frac{a^{1-\alpha}}{\alpha^{\frac{\alpha}{\alpha-1}}},\quad \text{for } t\ge t_1. \tag{2.15}$$

Then any solution of (1.1)-(2.6) *is oscillatory.*

Proof. Suppose that (1.1) has a nonoscillatory solution u on $[t_0,\infty)$. Since $-u$ is also a solution of (1.1), we can confine our discussion only to the case where the solution u is eventually positive solution of (1.1). We may assume without loss of generality that there exists $t_1\ge t_0$, such that

$$u(t)>0\quad\text{and}\quad u(\tau_i(t))>0,\text{ for all } t\ge t_1 \text{ and } i\in\{1,2,...,k\}.$$

By Lemma 2.6 and Lemma 2.7, we obtain

$$\sum_{i=1}^{i=k}\int_t^{\tau(t)} q_i(s)\,ds\le\frac{a^{1-\alpha}}{\alpha^{\frac{\alpha}{\alpha-1}}},\quad \text{for } t\ge t_0.$$

which contradicts (2.15). This completes the proof. □

As a Theorem of the previous result, we deduce the following corollarie.

Corollary 2.12. *Assume* $(\mathcal{H}_1)-(\mathcal{H}_2)$ *hold and that* $\alpha>1$ *is such that*

$$\liminf_{t\to\infty}\sum_{i=1}^{i=k}\int_t^{\tau(t)} q_i(s)\,ds>\frac{a^{1-\alpha}}{\alpha^{\frac{\alpha}{\alpha-1}}}.$$

Then any solution of (1.1)-(2.6) *is oscillatory.*

Next, we give an example to illustrate our main result.

Example 2.13. Consider the delay differential equation

$$x'(t)-\sum_{i=1}^{i=k}x(t+i)=0,\quad \text{for all } t\ge 0. \tag{2.16}$$

Here, $k\in\mathbb{N}$, $\alpha=1$, $q_i(t)=1$, $\tau_i(t)=t+i>t$, for all $i\in\{1,2,...,n\}$, and $\tau(t)=t+1$.
Then $(\mathcal{H}_1)-(\mathcal{H}_2)$ holds. On the other hand, we have

$$\sum_{i=1}^{i=k}\int_t^{\tau(t)} q_i(s)\,ds=\frac{k}{2}(k+1)>\frac{1}{e},\quad \text{for all } t\ge 0.$$

Thus, (2.14) holds. By Theorem 2.8, equation (2.16) is oscillatory.

Example 2.14. Consider the delay differential equation

$$x^{'}(t) - tx^3(t+1) = 0, \quad \text{for all } t \geq 0. \tag{2.17}$$

subject to the initial condition

$$u(0) = a \geq 0. \tag{2.18}$$

Here, $k = 1$, $\alpha = 3 > 1$, $q_1(t) = t$, and $\tau(t) = \tau_1(t) = t + 1 > t$. Then $(\mathcal{H}_1) - (\mathcal{H}_2)$ holds. On the other hand, we have

$$\int_t^{\tau(t)} q(s)\, ds = \frac{1}{2}(2t+1) \geq \frac{1}{2}, \quad \text{for all } t \geq 0.$$

If $u(0) = a > 0.620$, then (2.15) holds. By Theorem 2.11, equation (2.17)-(2.18) is oscillatory.

3. Conclusion

In this paper, we use the recursive sequence we have constructed to establish some new oscillation results of first-order linear dynamic equations with damping. Our results not only unify the oscillation of differential equations but also improve the differential equations established in [10]. However, this problem remains largely open, for future research.

Remark 3.1. For $\alpha > 1$, we pose $\psi_a(\alpha) = a^{1-\alpha}\alpha^{\frac{\alpha}{1-\alpha}}$, we have $\lim_{\alpha\to 1}\psi_a(\alpha) = \frac{1}{e} = \psi_a(1)$, then, we can summarize the two conditions (2.14) and (2.15) which guarantee the oscillation of the equation (1.1) in the cases $\alpha = 1$ and $\alpha > 1$, respectively. Meaning, we get,

$$\sum_{i=1}^{i=k} \int_t^{\tau(t)} q_i(s)\, ds > \psi_a(\alpha), \quad \text{for } t \geq t_1.$$

Remark 3.2. If we consider an advanced differential equation on time scale of the form

$$u^{\Delta}(t) - \sum_{i=1}^{i=k} q_i(t)\, u^{\alpha}(\tau_i(t)) = 0, \quad \text{for } t \geq t_0 \tag{3.1}$$

on an arbitrary time scale $\mathbb{T}$ with $\sup \mathbb{T} = \infty$. Thus, equation (1.1) becomes a special case of equation (3.1) in a case $\mathbb{T} = \mathbb{R}$. From the method given in this paper, one can obtain some oscillation criteria for (3.1). It means obtaining generalizations of Theorems 2.8 and 2.11. The details are left to the reader.

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