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Fuzzy contractor and nonlinear operator equation in fuzzy normed spaces

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Abstract. The purpose of this article is to study the nonlinear operator theory and the existence problems of solution for some kind of nonlinear operator equations in fuzzy normed space. Also, the concepts of fuzzy contractor and fuzzy contractor couple are investigated. By using these concepts, the existence problems of solutions for nonlinear operator equations with fuzzy contractor or fuzzy contractor couple are discussed.

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1. Introduction

In light of the fact that in many situations the distance between two points is inexact rather than a single real number, Kaleva and Seikkala [15] introduced the idea of fuzzy metric space by describing the distance of points as a fuzzy real number. Since each usual metric space and each Menger probabilistic metric space can be considered as a special case of fuzzy metric space, the study of the fuzzy metric space has attracted many authors and several results for nonlinear mappings have been given in the literature [6], [10], [20]. Schweizer and Sklar [26] provide a variety of examples of probabilistic metric spaces, all of which are from probabilistic origin. Of course, all of which might be thought to be examples of the fuzzy metric spaces. Inspired by the work of Kaleva and Seikkala [15], Felbin [7] introduced and studied the fuzzy normed linear space. It is as important as the concept of Menger probabilistic normed linear space introduced by Serstnev [27] and moreover, each usual normed linear space and each Menger probabilistic normed linear space can still be considered as its special case. Xiao and Zhu [28] studied the linear topological structure of the fuzzy normed linear space and obtained some basic properties. Many authors proved results in fuzzy normed linear spaces including Bag and Samanta [3], Xiao and Zhu [29] and Fang [6].

The contractor theory in Banach spaces established by M. Altman[1] plays a prominent role in the study of existence and uniqueness of solutions for nonlinear operator equations. Inspired by the work of Altman, A.C. Lee and W.J. Padgett [17, 18, 19] investigated random contractor theory as an improved version of Altman's work and studied the existence and uniqueness of solution for

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random operator equations with a random contractor. S.S. Chang [4] and W. J. Zeng [8] introduced the concept of probabilistic contractor. S.S. Chang [4], S.S. Chang, Y.C. Peng [30] and S.S. Chang et al. [5] studied the existence problem of solutions of non linear operator equations with probabilistic contractor in probabilistic metric spaces by using the concept of probabilistic contractor.

More recently, the theory of contractors in fuzzy normed spaces has been advanced in [12, 2, 23, 25]. A lot of research has been developed in regard to single-valued and set valued nonlinear operator equations in Menger probabilistic normed spaces, see e.g. [13, 14, 23, 20, 21, 22]. Most of the considered random normed spaces within the case of set-valued nonlinear operator equations have been non-Archimedean Menger probabilistic normed spaces underneath both the Lukasiewicz t-norm or a t-norm of H-type.

In this paper, we introduce the concept of more general fuzzy contractors in fuzzy normed spaces and show the existence and uniqueness of solutions for set-valued and single-valued nonlinear operator equations in fuzzy normed spaces. Our results enlarge and improve the corresponding consequences of Altman [1], Chang [4], Lee and Pedgett [17, 18, 19] and others.

The plan of this paper is as follows. After introducing (most of) our notation and terminology in Section 2, we consider the existence problem of solutions for the equations with fuzzy contractor in fuzzy normed linear spaces, in Section 3. Section 4 is devoted to introducing the concept of fuzzy contractor couple in fuzzy normed spaces to study the existence and uniqueness of solutions of system of nonlinear operator equations with a fuzzy contractor couple and to discuss the existence problem of common fixed points for a pair of mappings in a fuzzy normed space. In Section 5, we introduce the concept of a more general fuzzy contractor in fuzzy normed spaces and show the existence and uniqueness of solutions for set-valued and single-valued nonlinear operator equations in fuzzy normed spaces.

2. Preliminaries

First we recall some of the basic concepts, which will be used in the sequel.

Definition 2.1. [16, 26] Let $T : [0,1] \times [0,1] \to [0,1]$. Then T is said to be a t-norm if and only if for all $x, y, z \in [0,1]$, we have

(T1) $T(x,y) = T(y,x)$ (commutativity),

(T2) $T(x,y) \le T(x,z)$, if $y \le z$ (monotonicity),

(T3) $T(x,T(y,z)) = T(T(x,y),z)$ (associativity),

(T4) $T(x,1) = x$.

Definition 2.2. [26] A binary operation $T : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if $([0,1],T)$ is a topological monoid with unit 1 such that $T(a,b) \le T(c,d)$ whenever $a \le c, b \le d$ for all $a,b,c,d \in [0,1]$.

Here we present some examples of t-norm:

$$\begin{aligned} T(a,b) &= ab, \quad \text{(product)}\\ T(a,b) &= \min\{a,b\}, \quad \text{(minimum)}\\ T(a,b) &= \max\{a+b-1,0\}, \quad \text{(Lukasiewicz)}\\ T(a,b) &= \frac{ab}{a+b-ab}, \quad \text{(Hamacher)} \end{aligned}$$

Definition 2.3. [24] Let X be a linear space over a field $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) and T be a continuous t-norm. A fuzzy set N in $X \times [0,\infty)$ is called a fuzzy norm on X if and only if for all $x, y \in X$, $t \in \mathbb{R}$ and $c \in \mathbb{K}$.

(FN1) $N(x,t) = 0$ for all $t \leq 0$;

(FN2) $[N(x,t) = 1$, for all $t > 0]$ if and only if $x = 0$;

(FN3) $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$;

(FN4) $N(x+y, t+s) \geq T(N(x,t), N(y,s))$, for all $t, s > 0$;

(FN5) $N(x,.)$ is left continuous and $\lim_{t\to\infty} N(x,t) = 1$.

The triple (X, N, T) will be called fuzzy normed linear space (briefly, FNLS).

Lemma 2.4. *[9] Let (X, N, T) be a FNLS. Then $N(x,.)$ is non-decreasing for all $x \in X$.*

Example 2.5. Let X be a linear space and $\|.\|$ be a norm on X. Let

$$N(x,t) := \begin{cases} 1, & \text{if } |x| < t;\\ 0, & \text{if } |x| \geq t. \end{cases}$$

Then $(X, N, \min)$ is a FNLS. In particular, $(\mathbb{C}, N, T)$ is a FNLS.

Definition 2.6. Let X be a linear space over a field $\mathbb{K}$ with a non-Archimedean valuation $|.|$. A mapping $\|.\| : X \to [0,\infty)$ is said to be a non-Archimedean norm if it satisfies the following conditions:

(N1) $\|x\| = 0$ if and only if $x = 0$;

(N2) $\|cx\| = |c|\,\|x\|$, $c \in \mathbb{K}$ and $x \in X$;

(N3) the strong triangle inequality: $\|x+y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in X$.

Then $(X, \|.\|)$ is called a non-Archimedean normed space. By a complete non-Archimedean normed space, we mean one in which every Cauchy sequence is convergent.

Definition 2.7. Let X be a linear space over a field $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) and T be a continuous t-norm. A fuzzy set N in $X \times [0,\infty)$ is called a fuzzy norm on X if and only if for all $x, y \in X$, $t \in \mathbb{R}$ and $c \in \mathbb{K}$.

(FN1) $N(x,t) = 0$ for all $t \leq 0$;

(FN2) $[N(x,t) = 1$, for all $t > 0]$ if and only if $x = 0$;

(FN3) $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$;

(FN4) $N(x+y, \max\{t,s\}) \geq T(N(x,t), N(y,s))$, for all $t, s > 0$;

(FN5) $N(x,.)$ is left continuous and $\lim_{t\to\infty} N(x,t) = 1$.

If N is a non-Archimedean fuzzy norm on X, then (X, N, T) is called a non-Archimedean fuzzy normed space (briefly, N.A FNLS).

Example 2.8. Let $(X, \|.\|)$ be a non-Archimedean normed space, $T(a,b) = ab$ for all $a, b \in [0,1]$. For each $k \in \mathbb{N}$ and for all $x \in X$, consider

$$N(x,t) = \begin{cases} \dfrac{t}{t + k\|x\|}, & \text{if } t > 0; \\ 0, & \text{if } t \leq 0. \end{cases}$$

Then (X, N, T) is a N.A FNLS.

Definition 2.9. [24] Assume that (X, N, T) is a fuzzy normed linear space and $\{x_n\}$ is a sequence in X.

1. The sequence $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that
$$\lim_{t\to\infty} N(x_n - x, t) = 1, \text{ for all } t > 0.$$
In this case x is called the limit of the sequence $\{x_n\}$ and we denote $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

2. The sequence $\{x_n\}$ is called a Cauchy sequence if
$$\lim_{n\to\infty} N(x_{n+p} - x_n, t) = 1$$
for all $t > 0$ and all $p \in \mathbb{N}$.

3. (X, N, T) is said to be complete if every Cauchy sequence in X is convergent to a point in X. A complete fuzzy normed linear space will be called a fuzzy Banach space.

Definition 2.10. [26] Suppose that (X, N, T) is a fuzzy normed linear space.

(a) A sequence $\{x_n\}$ in X is τ-convergent to $x \in X$ if for any $\epsilon > 0$, $0 < \lambda < 1$, there exists a positive integer $k = k(\epsilon, \lambda)$ such that
$$N(x_n - x, \epsilon) > 1 - \lambda$$
whenever $n \geq k$. In this case, we write $x_n \xrightarrow{\tau} x$.

(b) A sequence $\{x_n\}$ in X is a τ-Cauchy sequence if for any $\epsilon > 0$, $0 < \lambda < 1$, there exist a positive integer $k = k(\epsilon, \lambda)$ such that

$$N(x_n - x_m, \epsilon) > 1 - \lambda$$

whenever $n, m \geq k$.

(c) (X, N, T) is said to be τ-complete if every τ-Cauchy sequence in X is τ-convergent to some point in X.

3. Fuzzy contractor and Nonlinear operator Equations in fuzzy normed space

In this section, we consider the existence problem of solutions for the equations with fuzzy contractor in fuzzy normed linear spaces.

Definition 3.1. Let (X, N_1, T) and (Y, N_2, T) be two fuzzy normed spaces, such that T satisfies the following condition:

$$\sup_{t \in (0,1)} T(t, t) = 1. \tag{3.1}$$

Let τ_1, τ_2 be the topologies generated by the family of (ϵ, λ)-neighborhoods on (X, N_1, T) and (Y, N_2, T), respectively. A mapping $S : Dom(S) \subset X \to Y$ is said to be closed if for any sequence $\{x_n\} \subset Dom(S)$, whenever $x_n \xrightarrow{\tau_1} x$, $Sx_n \xrightarrow{\tau_2} y$, we have $x \in Dom(S)$ and $Sx = y$.

Definition 3.2. [11] Let T be a t-norm satisfying the condition (3.1). T is said to be of H-type if the family of functions $\{T^m(t)\}_{m=1}^{\infty}$ is equi-continuous at $t = 1$, where

$$T^1(t) = T(t, t),\ T^m(t) = T(T^{m-1}(t), t),\ t \in [0, 1],\ m = 2, 3, \cdots .$$

Example 3.3. (i) If there exists a strictly increasing sequence $\{b_n\}_{n\in\mathbb{N}}$ in $[0, 1]$ such that $\lim\limits_{n\to\infty} b_n = 1$ and $T(b_n, b_n) = b_n$ for all $n \in \mathbb{N}$, then T is a t-norm of H-type.
(ii) If T is continuous, then there exists a sequence $\{b_n\}_{n\in\mathbb{N}}$ as in (i).
The t-norm $T = \min$ is a trivial example of a t-norm of H-type, but there are t-norms T of H-type with $T \neq \min$ (see, e.g., [11]).

Definition 3.4. A t-norm T is said to have the fixed point property if every contraction mapping on a complete fuzzy normed linear space (X, N, T) has a fixed point.

The following result is easy to prove:

Lemma 3.5. *If for every contraction mapping S on a fuzzy normed linear space (X, N, T) and for each point $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_n = S^n x_0$ is a τ-Cauchy sequence, then S has the fixed point property.*

Theorem 3.6. *Let (X, N, T) be a fuzzy normed linear space such that T is of H-type. Then T has the fixed point property.*

Proof. Let (X, N, T) be a fuzzy normed linear space such that T is of H-type and $S : X \to X$ be a mapping satisfying the following:

$$N(Sx - Sy, kt) \geq N(x - y, t), \; x, y \in X, t \geq 0,$$

where $k \in (0, 1)$ is a constant. (For the sake of convenience, we only consider $k \in (0, 1/2)$. If $k \in (1/2, 1)$, we can prove it similarly). Let $x_0 \in X$ be any given point. Then for any positive integer m and $t \in \mathbb{R}^+$, we have

$$\begin{aligned} N(x_0 - S^{m+1}x_0, 2t) &\geq T(N(x_0 - Sx_0, t), N(Sx_0 - S^{m+1}x_0, t)) \\ &\geq T(N(x_0 - Sx_0, t), N(x_0 - S^m x_0, t)) \\ &\geq \cdots \\ &\geq T^m(N(x_0 - Sx_0, t)). \end{aligned}$$

Hence for any positive integers n, m, we have

$$N(S^n x_0 - S^{n+m}x_0, 2t) \geq T^m(N(x_0 - Sx_0, k^{-n}t).$$

Since T is of H-type, it follows that

$$\lim_{n\to\infty} N(S^n x_0 - S^{n+m}x_0, 2t) = 1, \;\; t > 0,$$

uniformly in m. This implies that $\{S^n x_0\}$ is a Cauchy sequence in X. Therefore, the conclusion follows from Lemma 3.5 immediately. ■

Definition 3.7. [26] A function $\phi : [0, \infty) \to [0, \infty)$ is said to satisfy the condition (Φ) if it is strictly increasing, $\phi(0) = 0$ and $\lim_{n\to\infty} \phi^n(t) = \infty$ for all $t > 0$.

Remark 3.8. It is easy to see that if ϕ satisfies the condition (Φ), then $\phi(t) > t$ for all $t > 0$.

Definition 3.9. Let (X, N_1, T) and (Y, N_2, T) be two fuzzy normed spaces, $L(Y, X)$ the set of all linear operators from Y to X, $\Theta : X \to L(Y, X)$ and $S : Dom(S) \subset X \to Y$ be two mappings such that $\Theta(x)(y) \subseteq Dom(S)$. Θ is called a fuzzy contractor of S if there exists a function $\phi : [0, \infty) \to [0, \infty)$ satisfying the condition (Φ) such that

$$N_2(S(x + \Theta(x)y) - S(x) - y, t) \geq N_1(y, \phi(t)), \; t \geq 0, x \in Dom(S), y \in Y. \tag{3.2}$$

Theorem 3.10. *Suppose that (X, N_1, T) is a τ_1-complete N.A fuzzy normed space, (Y, N_2, T) is a τ_2-complete fuzzy normed space and T is a t-norm of H-type. Assume that $S : Dom(S) \subset X \to Y$ is a closed operator and $\Theta : X \to L(Y, X)$. If the following conditions are satisfied:*

(i) For all $x \in Dom(S)$ and $y \in Y$,

$$x + \Theta(x)y \in Dom(S), \tag{3.3}$$

(ii) *Θ is the fuzzy contractor of S and ϕ is a function defined by (3.2),*

(iii) *There exists a constant $M > 0$ such that for all $x \in Dom(S)$ and $y \in Y$,*

$$N_1(\Theta(x)y, t) \geq N_2\left(y, \frac{t}{M}\right),\ t \geq 0, \tag{3.4}$$

then, for any given $y_0 \in Y$, the nonlinear operator equation

$$S(x) = y_0 \tag{3.5}$$

has a solution in $Dom(S)$, and, for any given $x_0 \in Dom(S)$, the sequence

$$x_{n+1} = x_n - \Theta(x_n)(Sx_n - y_0) \tag{3.6}$$

τ_1-converges to the solution of (3.5). In addition, if there exists some $x \in X$ such that $\Theta(x) : Y \to X$ is a surjection, then for given $y_0 \in Y$, (3.5) has a unique solution in $Dom(S)$.

Proof. Without loss of generality, we can assume that $y_0 = 0$. If $y_0 \neq 0$, let $G(x) = S(x) - y_0$, $x \in Dom(S)$. Then $Dom(S) = Dom(G)$ and G satisfies all the conditions in Theorem 3.10. Therefore we can turn to discuss $G(x) = 0$. By the condition (i) and (3.6), for each $n = 0, 1, \cdots$, we have $x_n \in Dom(S)$. From condition (ii) and (3.2), it follows that

$$\begin{aligned} N_2(Sx_{n+1}, t) &= N_2(S(x_n - \Theta(x_n)(Sx_n)) - (Sx_n) - (-Sx_n), t) \\ &\geq N_2(Sx_n, \phi(t)) \geq \cdots \\ &\geq N_2(Sx_0, \phi^{n+1}(t)),\ t \geq 0. \end{aligned} \tag{3.7}$$

By the condition (iii) and (3.7), we have

$$\begin{aligned} N_1(x_n - x_{n+1}, t) &= N_1(\Theta(x_n)S(x_n), t) \\ &\geq N_2\left(S(x_n), \frac{t}{M}\right) \\ &\geq N_2\left(Sx_0, \phi^n\left(\frac{t}{M}\right)\right). \end{aligned}$$

Since (X, N_1, T) is a fuzzy normed space, for any m, n with $m > n$, we have

$$\begin{aligned} &N_1(x_n - x_m, t) \\ &\quad\geq T(N_1(x_n - x_{n+1}, t), N_1(x_{n+1} - x_m, t)) \\ &\quad\geq T(N_1(x_n - x_{n+1}, t), T(N_1(x_{n+1} - x_{n+2}, t), \\ &\qquad \underbrace{T(\cdots, T}_{m-n-3}(N_1(x_{m-2} - x_{m-1}, t), N_1(x_{m-1} - x_m, t)) \cdots)) \\ &\quad\geq T\left(N_2\left(Sx_0, \phi^n\left(\frac{t}{M}\right)\right)\right), T\left(N_2\left(Sx_0, \phi^{n+1}\left(\frac{t}{M}\right)\right)\right), \\ &\qquad \underbrace{T(\cdots, T}_{m-n-3}\left(N_2\left(Sx_0, \phi^{m-2}\left(\frac{t}{M}\right)\right), N_2\left(Sx_0, \phi^{m-1}\left(\frac{t}{M}\right)\right) \cdots\right) \end{aligned}$$

for all $t \geq 0$. Since ϕ satisfies the condition (Φ), $\phi(t) > t$ for all $t > 0$. It follows from the above inequality that

$$\begin{aligned} N_1(x_n - x_m, t) \geq T\left(N_2\left(Sx_0, \phi^n\left(\frac{t}{M}\right)\right), N_2\left(Sx_0, \phi^n\left(\frac{t}{M}\right)\right),\right. \\ \left.\underbrace{T(\cdots, T}_{m-n-3}\left(N_2\left(Sx_0, \phi^n\left(\frac{t}{M}\right)\right), N_2\left(Sx_0, \phi^n\left(\frac{t}{M}\right)\right)\cdots\right)\right) \\ = T^{m-n-1}\left(N_2\left(Sx_0, \phi^n\left(\frac{t}{M}\right)\right)\right), \ t \geq 0. \end{aligned}$$

Since T is of H-type and $\phi^n\left(\frac{t}{M}\right) \to \infty$ $(n \to \infty)$, for any given $\lambda \in (0,1)$ and $t > 0$, there exists positive integer $n(t,\lambda)$, as $n \geq n(t,\lambda)$, $m > n$, we have

$$N_1(x_n - x_m, t) \geq T^{m-n-1}\left(N_2\left(Sx_0, \phi^n\left(\frac{t}{M}\right)\right)\right) > 1 - \lambda. \tag{3.8}$$

This implies that $\{x_n\}$ is a Cauchy sequence in X. Let $x_n \xrightarrow{\tau_1} z$. By (3.7), we have

$$\lim_{n\to\infty} N_2(Sx_n, t) = 1, \quad t > 0,$$

i.e., $Sx_n \xrightarrow{\tau_2} \theta$. By the closedness of S, $z \in Dom(S)$ and $Sz = 0$. That is to say, z is a solution of (3.5), and the iterative sequence (3.6) τ_1-converges to z.

Next we prove that if $\Theta(w) : Y \to X$ is surjective, then z is the unique solution of equation (3.5) in $Dom(S)$. In fact, if $z^* \in Dom(S)$ is also a solution of (3.5). By the surjective property, there exists a point $y \in Y$ such that $z^* - z = \Theta(w)y$. Hence we have

$$\begin{aligned} N_2(y,t) &= N_2(S(z^*) - S(z) - y, t) \\ &= N_2(S(z + \Theta(w)y) - S(z) - y, t) \\ &\geq N_2(y, \phi(t)) \geq \cdots \\ &\geq N_2(y, \phi^m(t)), \quad m = 1, 2, \cdots. \end{aligned}$$

Let $m \to \infty$, we have $N_2(y,t) = 1$ for all $t > 0$, i.e., $y = 0$. Hence $z = z^*$. This completes the proof. ■

From Theorem 3.10, we have the following result:

Corollary 3.11. *Let (X, N_1, T), (Y, N_2, T), T, S be the same as in Theorem 3.10. Let $F : Y \to X$ be a linear operator satisfying the following conditions:*

(i) $x + Fx \in Dom(S)$ for all $x \in Dom(S)$ and $y \in Y$,

(ii) $N_1(S(x + Fx) - Sx - y, t) \geq N_2(y, \phi(t))$ for all $x \in Dom(S)$ and $y \in Y$, where ϕ satisfies the condition (Φ),

(iii) there exists a constant $M > 0$ such that for all $x \in Dom(S)$ and $y \in Y$

$$N_1(Fy, t) \geq N_2\left(y, \frac{t}{M}\right), \quad t \geq 0.$$

Then the nonlinear equation (3.5) has a solution in Dom(S), and for any given $x_0 \in$ Dom(S), the iterative sequence

$$x_{n+1} = x_n - F(Sx_n - y_0)$$

τ_1-converges to a solution of (3.5). In addition, if F is surjective, then for each $y_0 \in Y$, the equation (3.5) has a unique solution in Dom(S).

As an application of Theorem 3.10 and Corollary 3.11, we generalize contraction condition of mappings in fuzzy normed linear space and get a fixed point theorem.

Theorem 3.12. *Let (X, N, T) be a τ-complete N.A fuzzy normed space and T be a t-norm of H-type. Let $F : X \to X$ satisfy the following condition:*

$$N(Fx - Fy, t) \geq N(x - y, \phi(t)), \quad t \geq 0.$$

Then F has a unique fixed point, and for any $x_0 \in X$, the iterative sequence $\{x_n\}_{n=1}^{\infty} = \{Fx_{n-1}\}_{n=1}^{\infty}$ τ-converges to this fixed point in X.

Proof. Letting $Sx = x - Fx$, $x \in X$, $\Theta(x) = I$, $x \in X$, I is the identity mapping, it is easy to see that S satisfies all conditions in Theorem 3.10. ■

Theorem 3.13. *Let (X, N_1, T) be a τ_1-complete fuzzy normed space and (Y, N_2, T) be a τ_2-complete fuzzy normed space. Let T be a t-norm of H-type, $S :$ Dom$(S) \subset X \to Y$ be a closed operator and $\Theta : X \to L(Y, X)$ satisfy the following conditions:*

(i) for all $y \in Y$ and $x \in$ Dom(S), $x + \Theta(x)y \in$ Dom(S),

(ii) there exists $q \in (0, 1)$ such that for all $x \in$ Dom(S) and $y \in Y$,

$$N_2(S(x + \Theta(x)y) - Sx - y, t) \geq N_2\left(y, \frac{t}{q}\right), \quad t \geq 0,$$

(iii) there exists a constant $M > 0$ such that

$$N_1(\Theta(x)y, t) \geq N_2\left(y, \frac{t}{M}\right), \quad x \in Dom(S), y \in Y, t \geq 0.$$

Then the conclusions of Theorem 3.10 still hold.

Theorem 3.14. *Let (X, N, T) be a τ-complete fuzzy normed space and T be a t-norm of H-type. Let $F : X \to X$ be a mapping satisfying that there exists $q \in (0, 1)$ such that*

$$N(Fx - Fy, t) \geq N\left(x - y, \frac{t}{q}\right), \quad t \geq 0.$$

Then F has a unique fixed point $z \in X$, and for any $x_0 \in X$, the iterative sequence $\{x_n\}_{n=1}^{\infty} = \{Fx_{n-1}\}_{n=1}^{\infty}$ τ-converges to z.

4. Fuzzy Contractor Couple and Nonlinear Operator Equations in fuzzy normed space

The aim of this section is to introduce the concept of fuzzy contractor couple in a non-Archimedean fuzzy normed space to study the existence and uniqueness of solutions of a system of nonlinear operator equations with a fuzzy contractor couple and to discuss the existence problem of common fixed points for a pair of mappings in a non-Archimedean fuzzy normed space.

Definition 4.1. A mapping $\Theta : Y \to X$ is said to be odd if

$$\Theta(-x) = -\Theta(x), \quad y \in Y.$$

We denote by $O(Y, X)$ the set of all odd mappings from Y to X.

Definition 4.2. Let $S : D \subset X \to 2^Y$ be a set-valued mapping. A single-valued mapping $s : D \subset X \to Y$ is called the selection mapping of S if $s(x) \in S(x)$ for all $x \in D$.

Definition 4.3. Let (X, N_1, T) and (Y, N_2, T) be two N.A fuzzy normed spaces and $\Theta_i : X \to O(Y, X)$, $i = 1, 2$. Let $S, R : D \subset X \to 2^Y$ and $s, r : D \to Y$ be the selection mappings of S and R, respectively. (Θ_1, Θ_2) is called a fuzzy contractor couple of S and R if there exists a function $\phi : [0, \infty) \to [0, \infty)$ satisfying the condition (Φ) and such that for any $t \geq 0$, $x \in D$ and $y \in Y$ the following hold:

$$\begin{aligned}
& N_2(s(x+\Theta_1(x)y) - r(x) - y, t) \\
& \quad \geq \min\{N_2(y, \phi(t)), N_2(s(x+\Theta_1(x)y), \phi(t)) \\
& \qquad N_2(r(x), \phi(t)), N_2(s(x+\Theta_1(x)y) - r(x), \phi(t)), \\
& \qquad N_2(r(x)+y, \phi(t)), N_2(s(x+\Theta_1(x)y) - y, \phi(t))\}, \\
& N_2(r(x+\Theta_2(x)y) - s(x) - y, t) \\
& \quad \geq \min\{N_2(y, \phi(t)), N_2(r(x+\Theta_2(x)y), \phi(t)) \\
& \qquad N_2(s(x), \phi(t)), N_2(r(x+\Theta_2(x)y) - s(x), \phi(t)), \\
& \qquad N_2(s(x)+y, \phi(t)), N_2(r(x+\Theta_2(x)y) - y, \phi(t))\}.
\end{aligned} \tag{4.1}$$

Definition 4.4. Let $S, R : D \subset X \to 2^Y$ be two given set-valued mappings. For given $y_0 \in Y$, if there exists a $z \in D$ such that

$$y_0 \in S(z) \text{ and } y_0 \in R(z), \tag{4.2}$$

then z is called a solution of the system of set-valued mapping equations (4.2).

Theorem 4.5. *Let (X, N_1, T) and (Y, N_2, T) be τ-complete N.A fuzzy normed spaces with $T = \min$. Let $S, R : D \subset X \to 2^Y$ and let s and r be τ-closed selection mappings of S and R, respectively. If $\Theta_i : X \to O(Y, X)$ $(i = 1, 2)$ and the following conditions are satisfied:*

(i) $x + \Theta_i(x)y \in D$ for all $x \in X$ and $y \in Y$,

(ii) (Θ_1, Θ_2) *is the fuzzy contractor couple of* S *and* R,

(iii) *there exists a nonnegative strictly increasing function* $g(t)$ *with* $g(0) = 0$ *such that for any* $x \in D$ *and* $y \in Y$, *the following holds:*

$$N_1(\Theta_i(x)y, t) \geq N_2(y, g(t)), \quad t \geq 0. \tag{4.3}$$

Then for given $y_0 \in Y$, *the following system of nonlinear set-valued mapping equations*

$$y_0 \in S(x) \quad \textit{and} \quad y_0 \in R(x) \tag{4.4}$$

has a solution in D *and for any given* $x_0 \in D$, *the iterative sequence*

$$\begin{aligned} x_{2n+1} &= x_{2n} - \Theta_1(x_{2n})(r(x_{2n}) - y_0) \\ x_{2n+2} &= x_{2n+1} - \Theta_2(x_{2n+1})(s(x_{2n+1}) - y_0) \end{aligned} \tag{4.5}$$

τ*-converges to a solution of (4.4).*

Proof. Without loss of generality, we can assume that $y_0 = 0$. In fact, if $y_0 \neq 0$, letting $S_1(x) = \{u - y_0 : u \in S(x)\}$, $R_1(x) = \{u - y_0 : u \in R(x)\}$, then $s_1(x) = s(x) - y_0$, $r_1(x) = r(x) - y_0$, $Dom(S_1) = Dom(R_1) = D$ and S_1, R_1 satisfy all the conditions of Theorem 4.5. Therefore we can turn to discuss the following equations:

$$0 \in S_1(x) \quad \text{and} \quad 0 \in R_1(x).$$

It follows from the condition (i) and (4.5) that for each $n = 0, 1, \cdots$, $x_n \in D$ and the following holds:

$$\begin{aligned}
& N_2(s(x_{2n+1}), t) \\
&= N_2(s(x_{2n} + \Theta_1(x_{2n})(-r(x_{2n}))) - r(x_{2n}) - (-r(x_{2n})), t) \\
&\geq \min\{N_2(r(x_{2n}), \phi(t)), N_2(r(x_{2n+1}), \phi(t)), N_2(r(x_{2n}), \phi(t)), \\
&\quad N_2(s(x_{2n+1}) - r(x_{2n}), \phi(t)), N_2(r(x_{2n}) - r(x_{2n}), \phi(t)), N_2(s(x_{2n+1}) \\
&\quad + r(x_{2n}), \phi(t))\} \\
&= \min\{N_2(r(x_{2n}), \phi(t)), N_2(s(x_{2n+1}), \phi(t)), N_2(s(x_{2n+1} - r(x_{2n}), \phi(t))), \\
&\quad N_2(s(x_{2n+1}) + r(x_{2n}), \phi(t))\} \\
&\geq \min\{N_2(r(x_{2n}), \phi(t)), N_2(s(x_{2n+1}), \phi(t)), N_2(s(x_{2n+1}), \\
&\quad \phi(t)), N_2(r(x_{2n}), \phi(t))\} \\
&= N_2(r(x_{2n}), \phi(t)).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
& N_2(r(x_{2n}), \phi(t)) \\
&= N_2(r(x_{2n-1} + \Theta_2(x_{2n-1})(-s(x_{2n-1}))) - s(x_{2n-1}) - (-r(x_{2n-1})), \phi(t)) \\
&\geq \min\{N_2(s(x_{2n-1}), \phi^2(t)), N_2(r(x_{2n}), \phi^2(t)), N_2(s(x_{2n-1}), \phi^2(t)), \\
&\quad N_2(r(x_{2n}) - s(x_{2n-1}), \phi^2(t)), N_2(s(x_{2n-1}) - s(x_{2n-1}), \phi^2(t)), N_2(r(x_{2n}) \\
&\quad - s(x_{2n-1}), \phi^2(t))\} \\
&\geq \min\{N_2(s(x_{2n-1}), \phi^2(t)), N_2(r(x_{2n}), \phi^2(t))\} \\
&= N_2(s(x_{2n-1}), \phi^2(t)).
\end{aligned}$$

By induction, we can prove that

$$N_2(s(x_{2n+1}),t) \geq N_2(s(x_1),\phi^{2n+1}(t)), \quad t \geq 0, n = 0,1,2,\cdots. \tag{4.6}$$

In the same way, we can prove that

$$N_2(r(x_{2n}),t) \geq N_2(r(x_0),\phi^{2n}(t)), \quad t \geq 0, n = 1,2,\cdots. \tag{4.7}$$

From the condition (ii) and (4.6), (4.7), when n is odd, we have

$$\begin{aligned} N_1(x_{n+1}-x_n,t) &= N_1(\Theta_2(x_n)s(x_n),t) \\ &\geq N_2(s(x_n),g(t)) \geq N_2(s(x_1),\phi^{n-1}(g(t))), \quad t \geq 0. \end{aligned}$$

When n is even, we have

$$\begin{aligned} N_1(x_{n+1}-x_n,t) &= N_1(\Theta_1(x_n)r(x_n),t) \\ &\geq N_2(r(x_n),g(t)) \geq N_2(r(x_0),\phi^{n}(g(t))), \quad t \geq 0. \end{aligned}$$

Therefore, for any positive integers m,n with $m>n$ (without loss of generality), we can assume that n is odd and m is even. In the other cases we can prove it similarly, we have

$$\begin{aligned} & N_1(x_m-x_n,t) \\ &\quad \geq \min\{N_1(x_m-x_{m-1},t), N_1(x_{m-1}-x_n,t)\} \\ &\quad \geq \min\{N_1(x_m-x_{m-1},t), N_1(x_{m-1}-x_{m-2},t),\cdots, N_1(x_{n+1}-x_n,t)\} \\ &\quad \geq \min\{N_2(s(x_1),\phi^{m-2}(t)), N_2(r(x_0),\phi^{m-2}(g(t))),\cdots, \\ &\qquad N_2(r(x_0),\phi^{n+1}(g(t))), N_2(s(x_1),\phi^{n-1}(g(t)))\}, \quad t \geq 0. \end{aligned}$$

Since $\phi(t)$ satisfies the condition (Φ), $\phi(t)>t$ for all $t>0$ and so

$$N_1(x_m-x_n,t) \geq \min\{N_2(r(x_0),\phi^{n+1}(g(t))), N_2(s(x_1),\phi^{n-1}(g(t)))\}, \quad t \geq 0.$$

Again by the condition (Φ), when $n\to\infty$, we have $\phi^n(g(t))\to\infty$ for all $t>0$. Hence we have

$$\lim_{n,m\to\infty} N_1(x_m-x_n,t) = 1, \quad t>0,$$

which implies that $\{x_n\}$ is a τ-Cauchy sequence in X. By the completeness of X, let $x_n \xrightarrow{\tau_1} z$. Letting $n\to\infty$ in (4.6) and (4.7) and using the condition (Φ), we have

$$\lim_{n\to\infty} N_2(s(x_{2n+1}),t) = 1 = \lim_{n\to\infty} N_2(r(x_{2n}),t), \quad t>0.$$

This means that $s(x_{2n+1}) \xrightarrow{\tau_2} 0$ and $r(x_{2n}) \xrightarrow{\tau_2} 0$. By the τ-closedness of s and r, we know that $z\in D$ and $s(z)=0=r(z)$. Thus we have

$$0 \in S(z), \quad 0 \in R(z),$$

which show that z is a solution of (4.4) for $y_0=0$, and the iterative sequence (4.5) (in which $y_0=0$) is τ-convergent to z. This completes the proof. ■

Definition 4.6. Let (X, N_1, T) and (Y, N_2, T) be two N.A fuzzy normed spaces. Let $\Theta_i : X \to O(Y, X)$ and $S, R : D \subset X \to Y$ be two single-valued mappings. (Θ_1, Θ_2) is called a fuzzy contractor couple of S and R if there exists a function $\phi : [0, \infty) \to [0, \infty)$ satisfying the condition (Φ) such that for any $x \in D$ and $y \in Y$, the following hold:

$$
\begin{aligned}
& N_2(S(x + \Theta_1(x)y) - R(x) - y, t) \\
& \quad \geq \min\{N_2(y, \phi(t)), N_2(S(x + \Theta_1(x)y), \phi(t)) \\
& \qquad N_2(R(x), \phi(t)), N_2(S(x + \Theta_1(x)y) - R(x), \phi(t)), \\
& \qquad N_2(R(x) + y, \phi(t)), N_2(S(x + \Theta_1(x)y) - y, \phi(t))\}, \\
& N_2(R(x + \Theta_2(x)y) - S(x) - y, t) \\
& \quad \geq \min\{N_2(y, \phi(t)), N_2(R(x + \Theta_2(x)y), \phi(t)) \\
& \qquad N_2(S(x), \phi(t)), N_2(R(x + \Theta_2(x)y) - S(x), \phi(t)), \\
& \qquad N_2(S(x) + y, \phi(t)), N_2(R(x + \Theta_2(x)y) - y, \phi(t))\}.
\end{aligned} \tag{4.8}
$$

Theorem 4.7. *Let (X, N_1, T) and (Y, N_2, T) be τ-complete N.A fuzzy normed spaces and $T = \min$. Let $S, R : D \subset X \to Y$ be two τ-closed single-valued operators and $\Theta_i : X \to O(Y, X)$ $(i = 1, 2)$. If the following conditions are satisfied:*

(i) $x + \Theta_i(x)y \in D$ *for all* $x \in D$ *and* $y \in Y$,

(ii) (Θ_1, Θ_2) *is the fuzzy contractor couple of* S *and* R,

(iii) there exists a nonnegative strictly increasing function $g(t)$ *with* $g(0) = 0$ *such that for any* $x \in D$ *and* $y \in Y$, *the following holds:*

$$N_1(\Theta_i(x)y, t) \geq N_2(y, g(t)), \quad t \geq 0. \tag{4.9}$$

Then for given $y_0 \in Y$, the following system of nonlinear operator equations

$$Sx = y_0 \quad \text{and} \quad Rx = y_0 \tag{4.10}$$

has a solution in D, and for any given $x_0 \in D$, the iterative sequence

$$
\begin{aligned}
x_{2n+1} &= x_{2n} - \Theta_1(x_{2n})(R(x_{2n}) - y_0) \\
x_{2n+2} &= x_{2n+1} - \Theta_2(x_{2n+1})(S(x_{2n+1}) - y_0)
\end{aligned} \tag{4.11}
$$

τ_1-converges to a solution of (4.10).

Especially, if there exists some $w \in X$ such that either $\Theta_1(w)$ or $\Theta_2(w)$ is a surjection, then for given $y_0 \in X$, the system (4.10) of equations has a unique solution in D.

Proof. As a special case of Theorem 4.5, it is easy to see that the the preceding conclusion of Theorem 4.7 is true.

Now we prove the second conclusion. Suppose that there exists some $w \in D$ such that either Θ_1 or Θ_2 is surjection (without loss of generality we assume that $\Theta_1(w)$ is surjective) and there exist two solution z, z^* of (4.10). By the surjective property of $\Theta_1(w)$, there exists an $y \in Y$ such that

$$z^* - z = \Theta_1(w)y.$$

Hence we have

$$\begin{aligned} N_2(y,t) &= N_2(Rz^* - Sz - y, t) = N_2(R(z + \Theta(w)y) - Sz - y, t) \\ &\geq \min\{N_2(Rz^*, \phi(t)), N_2(Sz, \phi(t)), N_2(y, \phi(t)), \\ &\quad N_2(Rz^* - Sz, \phi(t)), N_2(Sz + y, \phi(t)), N_2(Rz^* - y, \phi(t))\} \\ &= N_2(y, \phi(t)) \geq \cdots \\ &\geq N_2(y, \phi^n(t)), \quad n = 1, 2, \cdots, t \geq 0. \end{aligned}$$

Letting $n \to \infty$, we have $N_2(y,t) = 1$ for all $t > 0$, which implies that $y = 0$, i.e., $z = z^*$. This completes the proof. ■

Now, we show the existence of a common fixed points for a pair of mappings in fuzzy normed spaces.

Theorem 4.8. *Let (X, N, T) be a τ-complete N.A fuzzy normed space with $T = \min$. Suppose that $F, G : X \to X$ satisfy the following condition:*

$$\begin{aligned} &N(Gx - Fy, t) \\ &\quad \geq \quad \min\{N(x - y, \phi(t)), N(x - Gx, \phi(t)), N(y - Fy, \phi(t)), \\ &\qquad\quad N(x - Fy, \phi(t)), N(y - Gx, \phi(t)), N((x - Gx) - (y - Fy), \phi(t)) \end{aligned} \tag{4.12}$$

for all $t \geq 0$, where $\phi : [0, \infty) \to [0, \infty)$ satisfies the condition (Φ). Then F and G have a unique common fixed point z in X and for any $x_0 \in X$ the iterative sequence

$$x_{2n+1} = F(x_{2n}) \quad \text{and} \quad x_{2n+2} = G(x_{2n+1}) \tag{4.13}$$

τ-converges to the point z.

Proof. Letting $S(x) = x - F(x), R(x) = x - G(x)$ for all $x \in X$, and $\Theta_i = I_X$, $x \in X$, $i = 1, 2$, we prove that S, R, Θ_1, Θ_2 satisfy all the conditions of Theorem 4.7.

In fact, it is obvious that the conditions (i) and (ii) in Theorem 4.7 are satisfied. Besides, for any $x, y \in X$ and any $t \geq 0$, we have

$$\begin{aligned}
&N(S(x+\Theta_1(x)y)-R(x)-y,t)\\
&\quad= N(S(x+y)-R(x)-y,t)\\
&\quad= N(x+y-F(x+y)-x+Gx-y,t)\\
&\quad= N(Gx-F(x+y),t)\\
&\quad\geq \min\{N(x-(x+y),\phi(t)),N(x-Gx,\phi(t)),N((x+y)\\
&\qquad -F(x+y),\phi(t)),\\
&\qquad N(x-F(x+y),\phi(t)),N(x+y-Gx,\phi(t)),\\
&\qquad N(x-Gx-((x+y)-F(x+y)),\phi(t))\}\\
&\quad= \min\{N(y,\phi(t)),N(R(x),\phi(t)),N(S(x+y),\phi(t)),\\
&\qquad N(S(x+y)-y,\phi(t)),\\
&\qquad N(R(x)-y,\phi(t)),N(S(x+y)-R(x),\phi(t))\}
\end{aligned}$$

$$\begin{aligned}
&N(R(x+\Theta_2(x)y)-S(x)-y,t)\\
&\quad= N((x+y)-G(x+y)-x+Fx-y,t)\\
&\quad= N(G(x+y)-Fx,t)\\
&\quad\geq \min\{N(x+y-x,\phi(t)),N((x+y)-G(x+y),\phi(t)),\\
&\qquad N(x-Fx,\phi(t)),\\
&\qquad N(x+y-Fx,\phi(t)),N(x-G(x+y),\phi(t)),\\
&\qquad N(x+y-G(x+y)-(x-Fx),\phi(t))\}\\
&\quad= \min\{N(y,\phi(t)),N(R(x+y),\phi(t)),N(S(x),\phi(t)),\\
&\qquad N(S(x)+y,\phi(t)),\\
&\qquad N(R(x+y)-y,\phi(t)),N(R(x+y)-S(x),\phi(t))\}.
\end{aligned}$$

This implies that the condition (ii) in Theorem 4.7 is satisfied.

Next, since $\Theta_1 = \Theta_2 = I_X$ is surjective, by Theorem 4.7, the iterative sequence

$$x_{2n+1} = x_{2n} - \Theta_1(x_{2n})(Rx_{2n}) = Fx_{2n},$$
$$x_{2n+2} = x_{2n+1} - \Theta_2(x_{2n+1})(Sx_{2n+1}) = Gx_{2n+1}$$

τ-converges to the unique solution $z \in X$ of the system of equations

$$S(x) = 0 \quad \text{and} \quad R(x) = 0.$$

Hence we have $z = Fz, z = Gz$, which imply that z is the unique fixed point of F and G in X. This completes the proof. ■

5. Existence and uniqueness problems of solutions for set-valued and single-valued nonlinear operator equations in fuzzy normed space

In this section, we introduce the concept of more general fuzzy contractors in fuzzy normed spaces and show the existence and uniqueness of solutions for set-valued and single-valued nonlinear operator equations in fuzzy normed spaces. The results in this section extend and improve the corresponding results of Altman [1], Chang et al. [5], [30], Lee and Pedgett [17].

Let (X, N, T) be a fuzzy normed space with a t-norm satisfying the condition (3.1) and Ω_X be a family of all non-empty τ-closed fuzzy bounded subsets of X. For any given $A, B \in \Omega_X$, define the fuzzy functions $N(A, B, .)$ and $N(A, .)$ by

$$N(A,B,t) = \sup_{s<t} T\{\inf_{a\in A}\sup_{b\in B} N(a-b,s), \inf_{b\in B}\sup_{a\in A} N(a-b,s)\}$$

and

$$N(A,t) = \sup_{s<t}\sup_{a\in A} N(a,s), \quad s,t\in\mathbb{R},$$

respectively.

Then, from the definition of $N(A, B, .)$ and $N(A, .)$, we have the following:

Lemma 5.1. *Let (X, N, T) be a fuzzy normed space (resp., a N.A fuzzy normed space) with a t-norm satisfying the condition (3.1) and $A \in \Omega_X$. Then we have the following:*

1. $N(A, 0) = 0$,
2. $N(A, t) = 1$ *for all* $t > 0$ *if and only if* $\theta \in A$,
3. $N(cA, t) = N\left(A, \frac{t}{|c|}\right)$ *for all* $c \in \mathbb{R}$ *and* $c \neq 0$,
4. *for any* $A, B \in \Omega_X$ *and* $\theta \in B$, $N(A, t) \geq N(A, B, t)$ *for all* $t \in \mathbb{R}$,
5. *If t-norm T is continuous, then we have*

$$N(A + x, t_1 + t_2) \geq T(N(x, t_1), N(A, t_2))$$

for all $t_1, t_2 \in \mathbb{R}^+$ *and* $x \in X$.

Definition 5.2. Suppose that (X, N_1, T) and (Y, N_2, T) are two fuzzy normed spaces with the t-norm T satisfying the condition (3.1). Let τ_1 and τ_2 be the topologies induced by the family of (ϵ, λ)-neighborhoods on (X, N_1, T) and (Y, N_2, T), respectively. A set-valued mapping $S : Dom(S) \subset X \to \Omega_Y$ (resp. a single-valued mapping $S : Dom(S) \subset X \to Y$) is said to be τ-closed if for any $x_n \in Dom(S)$ and $y_n \in S(x_n)$ (resp., $y_n = S(x_n)$), whenever $x_n \xrightarrow{\tau_1} x$ and $y_n \xrightarrow{\tau_2} y$, we have $x \in Dom(S)$ and $y \in S(x)$ (resp., $y = S(x)$).

Assume that (X, N_1, T) is a τ_1-complete N.A fuzzy normed space, (Y, N_2, T) is a τ_2-complete fuzzy metric space, T is a t-norm of H-type, and Ω_Y is a nonempty family of τ_2-closed fuzzy bounded subsets of Y. Let $S : Dom(S) \subset X \to \Omega_Y$ (resp., $S : Dom(S) \subset X \to Y$) be a nonlinear set-valued (resp., single-valued) mapping and $\Theta : X \to L(Y, X)$, where $L(Y, X)$ denotes the space of linear operators from Y into X. Let $\phi : [0, \infty) \to [0, \infty)$ satisfy the condition

(Φ) and $u \in Y$ be a given point. Then Θ is called a fuzzy contractor of S with respect to u if for all $x \in Dom(S)$ and $y \in \{y \in Y : x + \Theta(x)y \in Dom(S)\}$,

$$\begin{aligned} N_2(S(x+\Theta(x)y), S(x)+y, t) \geq \min\{&N_2(y, \phi(t)), N_2(S(x)-y, \phi(t)), \\ &N_2(S(x+\Theta(x))-u, \phi(t))\}, \qquad t \geq 0. \end{aligned} \tag{5.1}$$

$$\begin{aligned} \text{(resp., } N_2(S(x+\Theta(x)y) - S(x) - y, t) \geq \min\{&N_2(y, \phi(t)), N_2(S(x)-u, \phi(t)), \\ &N_2(S(x+\Theta(x)y)-u, \phi(t))\}). \end{aligned} \tag{5.2}$$

Remark 5.3. It follows from (5) of Lemma 5.1 that if T is a continuous t-norm with $T(t,t) \geq t$ for all $t \in [0,1]$, then (5.1) is equal to the following:

$$\begin{aligned} &N_2(S(x+\Theta(x)y), S(x)+y, t) \\ &\quad \geq \min\{N_2(y, \phi(t)), N_2(S(x)-u, \phi(t)), \\ &\qquad N_2(S(x+\Theta(x)y)-u, \phi(t)), N_2(S(x)+y-u, 2\phi(t)), \\ &\qquad N_2(S(x+\Theta(x)y)-y-u, 2\phi(t))\}, \ t \geq 0. \end{aligned}$$

If (Y, N_2, T) is also a N.A fuzzy normed space, then (5.1) is equal to the following:

$$\begin{aligned} &N_2(S(x+\Theta(x)y), S(x)+y, t) \\ &\quad \geq \min\{N_2(y, \phi(t)), N_2(S(x)-u, \phi(t)), \\ &\qquad N_2(S(x+\Theta(x)y)-u, \phi(t)), N_2(S(x)+y-u, \phi(t)), \\ &\qquad N_2(S(x+\Theta(x)y)-y-u, \phi(t))\}, \ t \geq 0. \end{aligned}$$

For the single-valued mapping S, we have similar inequalities which are equal to (5.2).

Now we are ready to show the existence and uniqueness of solutions for the set-valued nonlinear equation

$$u \in S(x). \tag{5.3}$$

Theorem 5.4. *Let (X, N_1, T) be a τ_1-complete N.A. fuzzy normed space, (Y, N_2, T) be a τ_2-complete fuzzy metric space, and T be a t-norm of H-type. Let $S : Dom(S) \subset X \to \Omega_Y$ be a τ-closed set-valued mapping. Suppose that $\Theta : X \to L(Y, X)$ satisfies the following conditions:*

1. *$x + \Theta(x)y \in Dom(S)$ for all $x \in Dom(S)$ and $y \in Y$,*
2. *Θ is a fuzzy contractor of S with respect to u, i.e., Θ satisfies the condition (5.1),*
3. *there exists a constant $M > 0$ such that, for any $x \in Dom(S)$ and $y \in Y$,*

$$N_1(\Theta(x)y, t) \geq N_2(y, t/M), \qquad t \geq 0,$$

4. *for any $A, B \in \Omega_Y$ and $a \in A$, there exists a point $b \in B$ such that*

$$N_1(a-b,t) \geq N_1(A-B,t), \qquad t \geq 0.$$

Then the nonlinear set-valued operator equation (5.3) has a solution w in $Dom(S)$. Further, the sequence $\{x_n\}$ defined by

$$x_{n+1} = x_n - \Theta(x_n)y_n$$

τ_1-converges to the solution w in the topology τ_1.

Proof. (i) The case of $u = \theta$: In this case, (5.1) can be written as follows:

$$\begin{aligned} N_1(S(x+\Theta(x)y), S(x)+y,t) \geq \min\{&N_1(y,\phi(t)), N_1(S(x),\phi(t)), \\ &N_1(S(x+\Theta(x)y,\phi(t)), \quad t \geq 0. \end{aligned} \tag{5.4}$$

For any given $x_0 \in Dom(S)$, take $y_0 \in S(x_0)$ and let $x_1 = x_0 - \Theta(x_0)y_0$. By the assumption (1), we have $x_1 \in Dom(S)$. Replacing x and y by x_0 and $-y_0$ in (5.4), respectively, from (4) of Lemma 5.1 and $\theta \in S(x_0) - y_0$, we have

$$\begin{aligned} N_1(S(x_1),t) &\geq N_1(S(x_1) - S(x_0) + y_0, t) \\ &= N_1(S(x_0 - \Theta(x_0)y_0 - S(x_0) + y_0, t) \\ &\geq \min\{N_1(y_0,\phi(t)), N_1(S(x_0),\phi(t)), N_1(S(x_1),\phi(t))\} \\ &= \min\{N_1(y_0,\phi(t)), N_1(S(x_1),\phi(t))\}, \qquad t \geq 0. \end{aligned}$$

Hence

$$N_1(y_0,\phi(t)) \leq N_1(S(x_1),\phi(t)), \qquad t \geq 0. \tag{5.5}$$

By the assumption (4), for $\theta \in S(x_0) - y_0$, there exists a point $y_1 \in S(x_1)$ such that

$$N_1(y_1,t) \geq N_1(S(x_1) - S(x_0) + y_0, t), \qquad t \geq 0.$$

Hence, by (5.4) and (5.5), we have $N_1(y_1,t) \geq N_1(y_0,\phi(t))$ for all $t \geq 0$.

Let $x_2 = x_1 - \Theta(x_1)y_1$. By the same method as stated above, there exists a point $y_2 \in S(x_2)$ such that

$$N_1(y_2,t) \geq N_1(y_1,\phi(t)) \geq N_1(y_0,\phi^2(t)), \quad t \geq 0.$$

Inductively, we obtain two sequences $\{x_n\}$ in $Dom(\Theta)$ and $\{y_n\}$ in Y such that

$$x_{n+1} = x_n - \Theta(x_n)y_n \tag{5.6}$$

$$y_n \in S(x_n) \tag{5.7}$$

$$N_1(y_n,t) \geq N_1(y_0,\phi^n(t)), \quad t \geq 0. \tag{5.8}$$

By the assumption (3), (5.6) and (5.8), we have

$$N_1(x_n - x_{n+1}, t) = N_1(\Theta(x_n)y_n, t) \geq N_2(y_n, t/M) \geq \cdots \geq N_2(y_0, \phi^n(t/M))$$

for all $t \geq 0$. Hence, for any integers $m, n\,(m > n)$,

$$\begin{aligned} N_1(x_n - x_m, t) &\geq T(N_1(x_n - x_{n+1}, t), N_1(x_{n+1} - x_m, t) \\ &\geq T(N_1(x_n - x_{n+1}, t), T(N_1(x_{n+1} - x_{n+2}, t), \\ &\underbrace{T(\cdots}_{m-n-3}, T(N_1(x_{m-2} - x_{m-1}, t), N_1(x_{m-1} - x_m, t)) \cdots)) \\ &\geq T(N_2(y_0, \phi^n(t/M)), T(N_2(y_0, \phi^{n+1}(t/M)), \\ &\underbrace{T(\cdots}_{m-n-3}, T(N_2(y_0, \phi^{m-2}(t/M)), N_2(y_0, \phi^{m-1}(t/M)) \cdots)). \end{aligned}$$

for all $t \geq 0$. Since ϕ satisfies the condition (Φ), $\phi(t) > t$ and so, we have

$$\begin{aligned} N_1(x_n - x_m, t) &\geq T(N_2(y_0, \phi^n(t/M)), T(N_2(y_0, \phi^n(t/M)), \\ &\underbrace{T(\cdots}_{m-n-3}, T(N_2(y_0, \phi^n(t/M)), N_2(y_0, \phi^n(t/M)) \cdots)) \\ &= T^{m-n-1}(N_2(y_0, \phi^n(t/M))), \qquad t \geq 0. \end{aligned}$$

Since T is of H-type, $\phi^n(t/M) \to \infty$ for all $t > 0$ as $n \to \infty$ and so, for all $\lambda \in (0, 1)$ and $t > 0$, there exists an integer $n(t, \lambda)$, $n \geq n(t, \lambda)$, $m > n$, such that

$$N_1(x_n - x_m, t) \geq T^{m-n-1}(N_2(y_0, \phi^n(t/M)) > 1 - \lambda.$$

This means that the sequence $\{x_n\}$ is a τ-Cauchy sequence in X. Since (X, N_1, T) is a τ_1-complete fuzzy normed space, let $x_n \xrightarrow{\tau_1} w$. Since ϕ satisfies the condition (Φ), from (5.8), we have

$$\lim_{n\to\infty} N_2(y_n, t) = 1, \qquad t > 0,$$

i.e., $y_n \xrightarrow{\tau_2} \theta$. Therefore, from the τ-closedness of S and (5.7), we have $w \in Dom(S)$ and $\theta \in S(w)$, i.e., w is a solution of (5.2).

(ii) the case of $u \neq \theta$: Let $R(x) = S(x) - u$ for $x \in Dom(S)$. Then $Dom(S) = Dom(R)$ and S satisfying (5.1) is equal to R satisfying (5.4). Therefore, by using the case of $u = \theta$, we can show the existence of solution for the nonlinear set-valued operator equation $\theta \in R(x)$. This completes the proof. ■

For the nonlinear single-valued operator equation

$$u = S(x), \tag{5.9}$$

we also have the following:

Theorem 5.5. *Let (X, N_1, T) be a τ_1-complete N.A. fuzzy normed space, (Y, N_2, T) be a τ_2-complete fuzzy metric space, and T be a t-norm of H-type. Let $S : Dom(S) \subset X \to Y$ be a τ-closed single-valued operator and $\Theta : X \to L(Y, X)$ be such that*

1. *$x+\Theta(x)y \in Dom(S)$ for all $x \in Dom(S)$ and $y \in Y$,*

2. *Θ is a fuzzy contractor of S with respect to u, i.e., Θ satisfies the condition (5.2),*

3. *there exists a constant $M > 0$ such that, for any $x \in Dom(S)$ and $y \in Y$,*

$$N_1(\Theta(x)y, t) \geq N_2(y, t/M), \qquad t \geq 0,$$

Then operator equation (5.9) has a solution w in $Dom(S)$ and for any given $x_0 \in Dom(S)$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = x_n - \Theta(x_n)(S(x_n) - u) \tag{5.10}$$

converges to the solution w of the equation (5.9) in the topology τ_1. If $\Theta(w) : Y \to X$ is surjective, then w is the unique solution of (5.9).

Proof. Without loss of generality, we may assume that $u = \theta$. In this case, (5.2) can be written as follows:

$$\begin{aligned} N_2(S(x+\Theta(x)y) - S(x) - y, t) \geq \min\{&N_2(y, \phi(t)), N_2(S(x), \phi(t)),\\ &N_2(S(x+\Theta(x)y), \phi(t)), \qquad t \geq 0. \end{aligned} \tag{5.11}$$

By the condition (1) and (5.10), we have $x_n \in Dom(S)$ for $n = 0, 1, 2, \cdots$. Replacing x and y by x_n and $-S(x_n)$, $n = 0, 1, 2, \cdots$, in (5.11), respectively, we have

$$\begin{aligned} N_2(S(x_{n+1}), t) &\geq \min\{N_2(S(x_n), \phi(t)), N_2(S(x_n), \phi(t)), N_2(S(x_{n+1}), \phi(t))\}\\ &= \min\{N_2(S(x_n), \phi(t)), N_2(S(x_{n+1}), \phi(t))\}, \qquad t \geq 0. \end{aligned}$$

By the decreasing property of N_2, we have

$$N_2(S(x_{n+1}), t) \geq N_2(S(x_n), \phi(t)) \geq \cdots \geq N_2(S(x_0), \phi^{n+1}(t)) \tag{5.12}$$

for all $t \geq 0$. In view of the assumption (3), (5.10) and (5.12), we have

$$\begin{aligned} N_1(x_n - x_{n+1}, t) &= N_1(\Theta(x_n)(S(x_n)), t)\\ &\geq N_2(S(x_n), t/M)\\ &\;\;\vdots\\ &\geq N_2(S(x_0), \phi^n(t/M)) \end{aligned}$$

for all $t \geq 0$. By the same method as in the proof of Theorem 5.4, we can prove that $\{x_n\}$ is a τ_1-Cauchy sequence in X. Since (X, N_1, T) is τ_1-complete, let $x_n \xrightarrow{\tau_1} w$. Hence, from the condition (Φ) and (5.12), we have $S(x_n) \xrightarrow{\tau_1} \theta$. Therefore, by the closedness of S, we have $w \in Dom(S)$ and $S(w) = \theta$.

Next, we prove the uniqueness of solution of the operator equation $u = S(x)$. In fact, if $z \in Dom(S)$ and $S(z) = \theta$, by the surjectivity of $\Theta(w)$, there exists a point $y \in Y$ such that $z - w = \Theta(w)y$. Since $S(w) = S(z) = \theta$ and $N_2(\theta, t) = 1$, from (5.11), we have

$$\begin{aligned} N_2(y,t) &\geq N_2(S(z) - S(w) - y, t) \geq \min\{N_2(y, \phi(t)), N_2(S(w), \phi(t)), N_2(S(z), \phi(t)) \\ &= N_2(y, \phi(t)), \qquad t \geq 0, \end{aligned}$$

which implies that

$$N_2(y,t) \geq N_2(y, \phi(t)) \geq \cdots \geq N_2(y, \phi^n(t))$$

for all $t \geq 0$ and $n = 1, 2, \cdots$. Letting $n \to \infty$, from the condition (Φ) we have $N_2(y,t) = 1$ for all $t > 0$. This means that $y = \theta$, i.e., $z = w$. This completes the proof. ■

Now, using Theorems 5.4 and 5.5, we obtain two fixed point theorems for set-valued and single-valued mappings:

Theorem 5.6. *Let (X, N, T) be a τ-complete N.A. fuzzy normed space and T be a t-norm of H-type. Let $Q : X \to \Omega_X$ satisfy the following condition:*

$$\text{(5.13)} \quad N(Qx, Qy, t) \geq \min\{N(x - y, \phi(t)), N(x - Qx, \phi(t)), N(y - Qy, \phi(t))\}$$

for all $t \geq 0$ and $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ satisfies the condition (Φ). Suppose further that, for any $A, B \in \Omega_X$ and $a \in A$, there exists a point $b \in B$ such that

$$N(a - b, t) \geq N(A, B, t), \quad t \geq 0.$$

Then there exists a point $w \in X$ such that $w \in Qw$, i.e., w is a fixed point of Q.

Proof. Putting $S(x) = x - Qx$ and $\Theta(x) = I_X$, where I_X is the identity mapping on X, the mappings S and Θ satisfy all the hypotheses of Theorem 5.4. Therefore, there exists a point $w \in X$ such that $\theta \in S(w) = w - Qw$, which means that w is a fixed point of Q. This completes the proof. ■

Theorem 5.7. *Let (X, N, T) be a τ-complete N.A. fuzzy normed space and T be a t-norm of H-type. Let $Q : X \to X$ satisfy the following condition:*

$$\text{(5.14)} \quad N(Qx - Qy, t) \geq \min\{N(x - y, \phi(t)), N(x - Qx, \phi(t)), N(y - Qy, \phi(t))\}$$

for all $t \geq 0$ and $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ satisfies the condition (Φ). Then there exists a point $w \in X$ such that $w = Qw$, that is, w is the unique fixed point of Q and, for any $x_0 \in X$, the iterative sequence $\{x_n\}$ in X converges to w in the topology τ, where $x_n = Qx_{n-1}$, $n = 2, 3, 4, \cdots$.

Proof. Putting $S(x) = x - Qx$ and $\Theta(x) = I_X$, the mappings S and Θ satisfy all the hypotheses of Theorem 5.5. Therefore, there exists a point $w \in X$ such that $\theta = S(w) = w - Qw$, i.e., w is a fixed point of Q. This completes the proof. ■

Remark 5.8. In Theorem 5.6, if we assume that $T(t,t) \geq t$ for all $t \in [0,1]$, then by Remark 5.3, (5.13) can be weakened as follows:

$$N(Qx - Qy, t) \geq \min\{N(x-y, \phi(t)), N(x - Qx, \phi(t)), N(y - Qy, \phi(t)), N(y - Qx, \phi(t)), N(x - Qy, \phi(t))\}$$

for all $t \geq 0$.

As an application, in the sequel we use some result stated above to show the existence and uniqueness of the solutions for nonlinear Volterra integral equations on a kind of particular fuzzy normed space.

In what follows, let $[0,a]$ be a fixed real interval ($0 < a < \infty$) and $(X, \|.\|_X)$ a real Banach space. We denote by $C([0,a];X)$ the Banach space of all X-valued continuous functions defined on $[0,a]$ with the norm defined by

$$\|x\|_C = \sup_{0 \leq t \leq a} \|x(t)\|_X, \qquad x(t) \in C([0,a];X). \tag{5.15}$$

As well as the norm $\|.\|_C$, the space $C([0,a];X)$ can be endowed with another norm $\|.\|_*$ which is defined as follows:

$$\|x\|_* = \sup_{0 \leq t \leq a} (e^{-Lt} \|x(t)\|_X), \tag{5.16}$$

where L is any positive number. It is clear that the norm $\|x\|_*$ is equivalent to the norm $\|.\|_C$.

We also denote by $(C([0,a];X), N, \min)$ the fuzzy normed space, where N is the fuzzy norm defined by

$$N(x,t) = \begin{cases} 0, & \text{if } t \leq \|x\|_*; \\ 1, & \text{if } t > \|x\|_*. \end{cases}$$

where $x(s), y(s) \in C([0,a];X), t \in \mathbb{R}$.

Now we study the existence and uniqueness of solutions of the following kind of nonlinear Volterra integral equations:

$$x(t) = y(t) + \int_0^t K(t,s,x(s))\, ds, \qquad 0 \leq t \leq t, \tag{5.17}$$

where $y(t) \in C([0,a];X)$ is any given function.

Example 5.9. Let $(X, \|.\|_X), C([0,a];X)$ and $(C([0,a];X), N, \min)$ be the same as stated as above. Suppose the following conditions are satisfied:

(i) $K(t,s,x(s)) \in C(([0,a] \times [0,a] \times C([0,a];X)); X)$ and

$$\|K\|_C = \sup_{t,s \in [0,a], x \in X} \|K(s,t,x)\|_X < \infty,$$

(ii) there exists $m \in \mathbb{Z}^+$ and a constant $\alpha \in (0,1)$ such that

$$N(S^m x - S^m y, t) = \min_{p,q \in \{x,y,S^m x, S^m y\}} N\left(p - q, \frac{t}{\alpha}\right)$$

for all $x, y \in C([0,a];X)$ and $t \in \mathbb{R}^+$, where the mappings S and T^m are defined as follows:

$$(Sx)(t) = y(t) + \int_0^t K(t,s,x(s))\,ds,$$
$$(S^m x)(t) = y(t) + \int_0^t K(t,s,S^{m-1}x(s))\,ds,$$

(iii) for any $x(t) \in C([0,a];X)$, the set $\{S^n x(t)\}_{n=0}^{\infty}$ is bounded.

Then for any $x_0(t) \in C([0,a];X)$, the sequence $\{S^n x_0(t)\}_{n=0}^{\infty}$ converges in the norm $\|.\|_C$ to a solution $x_*(t) \in C([0,a];X)$ of the equation (5.17).

6. Conclusion

Chang [4] defined contractor and contractor couple in probabilistic normed spaces. In this paper, we have extended the notion of contractor and contractor couple to fuzzy normed spaces. With the help of contractor couple we have proved the existence theorem of solutions for set-valued nonlinear operator equations in fuzzy normed spaces. We have applied our existence theorem to prove a new fixed point theorem in fuzzy normed spaces. Several queries are raised by this work. The first of those, the examination of the conditions that allow one to simply apply the existence theorem and fixed point theorem, we have discussed, which are mostly stated as purely mathematical results. The second question is which of our theorems can give constructive proofs. Other queries will be posed and and indeed all are under investigation and can be thought of elsewhere.

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