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$C-$cosine and mixed C_0-cosine families of bounded linear operators on non-Archimedean Banach spaces

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Abstract

In this paper, we introduce and check some properties of $C-$cosine and mixed C_0-cosine families of bounded linear operators on non-Archimedean Banach spaces. We show some results for $C-$cosine and mixed C_0-cosine families of bounded linear operators on non-Archimedean Banach spaces. In contrast with the classical setting, the parameter of mixed C_0-cosine family of bounded linear operators belongs to a clopen ball Ω_r of the ground field $\mathbb{K}$. Examples are given to support our work.

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1 Introduction and Preliminaries

In the classical functional analysis, the Cauchy equations $f(x+y)+f(x-y)=2f(x)f(y)$ and $f(x+y)=f(x)f(y)$ can be generalized as the form $f(x+y)=H(f(x),f(y))$, where H is a scalar-valued function of two variables which stimulated S. Harsinder to discover and study the mixed semigroups of linear operators on Archimedean Banach spaces ([8]). The classical C_0-cosine family has been studied by M. Sova, H. O. Fattorini, M. Kostić, for more details, we refer to [7], [10] and [13]. Moreover, the mixed $C-$cosine family of linear operators studied by M. Mosallanezhada, M. Janfada, for more details, we refer to [11]. Recently, A. El Amrani et al. introduced the notions of C_0-groups, $C-$groups, mixed C_0-groups and cosine families of bounded linear operators on non-Archimedean Banach spaces for more details, we refer to [1], [2], [5] and [6]. Throughout this paper, X is a non-Archimedean (n.a) Banach space over a (n.a) non trivially complete valued field $\mathbb{K}$ with valuation $|\cdot|$, $B(X)$ denotes the set of all bounded linear operators from X into X, $\mathbb{Q}_p$ is the field of p-adic numbers ($p \geq 2$ being a prime) equipped with p-adic valuation $|.|_p$, $\mathbb{Z}_p$ denotes the ring of p-adic integers, which is the unit ball of $\mathbb{Q}_p$ centered at zero. For more details and related issues, we refer to [3], [4], [9], [12] and [14]. We denote the completion of algebraic closure of $\mathbb{Q}_p$ under the p-adic absolute value $|\cdot|_p$ by

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$\mathbb{C}_p$ (see [9], p.45). Remember that a free Banach space X is a non-Archimedean Banach space for which there exists a family $(e_i)_{i\in\mathbb{N}}$ in $X\backslash\{0\}$ such that every element $x \in X$ can be written in the form of a convergent sum $x = \sum\limits_{i\in\mathbb{N}} x_i e_i$, $x_i \in \mathbb{K}$ and $\|x\| = \sup\limits_{i\in\mathbb{N}} |x_i|\|e_i\|$. The family $(e_i)_{i\in\mathbb{N}}$ is called an orthogonal basis. In a free Banach space X, each bounded linear operator A on X can be written in a unique fashion as a pointwise convergent series, that is, there exists an infinite matrix $(a_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ with coefficients in $\mathbb{K}$ such that

$$A = \sum_{i,j\in\mathbb{N}} a_{ij} e_j^{'} \otimes e_i, \; and \; \forall j \in \mathbb{N}, \;\; \lim_{i\to\infty} |a_{ij}|\|e_i\| = 0,$$

where $(\forall j \in \mathbb{N})$ $e_j^{'}(x) = x_j$ $\left(e_j^{'} \text{ is the linear form associated with } e_j\right)$.
Moreover, for each $j \in \mathbb{N}$, $Ae_j = \sum\limits_{i\in\mathbb{N}} a_{ij} e_i$ and its norm is defined by

$$\| A \| = \sup_{i,j} \frac{|a_{ij}|\|e_i\|}{\|e_j\|}.$$

For more details, we refer to [3, 4]. Now, as in [6], take $r > 0$, Ω_r is the open ball of $\mathbb{K}$ centred at 0 with radius $r > 0$, that is $\Omega_r = \{k \in \mathbb{K} : |k| < r\}$. In the non-Archimedean context, the family $\{C(t),\ t \in \Omega_r\}$, $C : \Omega_r \to B(X)$, is called cosine family of bounded linear operators on X if

$$\text{for all } t,\ s \in \Omega_r,\ C(s + t) + C(s - t) = 2C(s)C(t)$$

and $C(0) = I$, where I is the identity operator on X. The cosine family of bounded linear operators has been extensively studied by A. El Amrani, A. Blali, J. Ettayb, and, M. Babahmed. For more details, we refer to [6]. Let $\mathbb{K} = \mathbb{Q}_p$ and A is a bounded linear operator on a free Banach space X satisfying $\|A\| < r = p^{\frac{-1}{p-1}}$, then the function defined by

$$\text{for all } t \in \Omega_{\frac{-1}{p-1}},\ f(t) = \left(\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} A^n\right) u_0,$$

for a fixed $u_0 \in X$, is the solution to homogeneous p-adic second order differential equation given by

$$\frac{d^2u(t)}{dt^2} = Au(t),\ u(0) = u_0.$$

The aim of this work is to introduce the mixed C_0-cosine family of bounded linear operators on a non-Archimedean Banach space and study some of its properties.

Definition 1.1. [6] Let $r > 0$ be a real number. A function $C : \Omega_r \longrightarrow B(X)$ is called a C_0 or strongly continuous operator cosine function on X if

(i) $C(0) = I$,

(ii) For every $t,\ s \in \Omega_r,\ C(t+s)+C(t-s)=2C(t)C(s)$,

(iii) For each $x \in X,\ t \longrightarrow C(t)x$ is continuous on Ω_r.

A cosine family of bounded linear operators $(C(t))_{t\in\Omega_r}$ is uniformly continuous if $\lim\limits_{t\to 0}\|C(t)-I\|=0$.
The linear operator A defined by

$$D(A)=\{x\in X:\lim_{t\to 0}2\frac{C(t)x-x}{t^2}\text{ exists}\}$$

and

$$\text{for each } x\in D(A), Ax=\lim_{t\to 0}2\frac{C(t)x-x}{t^2}$$

is called the infinitesimal generator of cosine family $(C(t))_{t\in\Omega_r}$.

2 Main results

Recall that k is the residue class field of $\mathbb{K}$. Throughout this paper, we assume that $\mathbb{K}$ is a complete non-Archimedean valued field of characteristic zero with $char(k)=p$ $\Big(p$ is a prime integer number$\Big)$. We begin with the following definition.

Definition 2.1. Let $r>0$ and $C\in B(X)$ be invertible. A one parameter family $(C(t))_{t\in\Omega_r}$ of bounded linear operators from X into X is called a $C-$cosine family if

(i) $C(0)=C$;

(ii) For every $t,\ s\in\Omega_r,\ C\Big(C(t+s)+C(t-s)\Big)=2C(t)C(s)$;

(iii) For each $x\in X$, $t\longrightarrow C(t)x$ is continuous on Ω_r.

The linear operator A defined by

$$D(A)=\{x\in X:\lim_{t\to 0}2\frac{C(t)x-Cx}{t^2}\text{ exists}\},$$

and

$$\text{for each } x\in D(A),\ Ax=C^{-1}\lim_{t\to 0}2\frac{C(t)x-Cx}{t^2},$$

is called the infinitesimal generator of $(C(t))_{t\in\Omega_r}$.

We have the following remark.

Remark 2.2. Generally in Definition 2.1, if $C\in B(X)$ is just injective (not invertible), $D(A)=\{x\in X:\lim\limits_{t\to 0}2\dfrac{C(t)x-Cx}{t^2}$ exists in the range of $C\}$.

We start with the following statements.

Lemma 2.3. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $(C(t))_{t\in\Omega_r}$ be a $C-$cosine family on X, then for each $t \in \Omega_r$, $CC(2t) = 2C(t)^2 - C^2$.*

Proof. Obvious. □

Remark 2.4. Suppose that $\mathbb{K} = \mathbb{Q}_p$. From Lemma 2.3, if $p \neq 2$, we have for all $t \in \Omega_r$, $C(\frac{t}{2})^2 = \frac{CC(t)+C^2}{2}$.

Lemma 2.5. *Let $(C(t))_{t\in\Omega_r}$ be a $C-$cosine family on X, then:*

(i) For every $t \in \Omega_r$, $C(-t) = C(t)$,

(ii) For each $t,\ s \in \Omega_r$, $C(t)C(s) = C(s)C(t)$.

Proof. (i) It suffices to take $t = 0$ in (ii) of Definition 2.1.

(ii) For each $t,\ s \in \Omega_r$, we have:

$$\begin{aligned} 2C(t)C(s) &= C\Big(C(t-s) + C(t+s)\Big) \\ &= C\Big(C(s-t) + C(s+t)\Big) \\ &= 2C(s)C(t). \end{aligned}$$

Then for all $t,\ s \in \Omega_r$, $C(t)C(s) = C(s)C(t)$.

□

Remark 2.6. Let $(C(t))_{t\in\Omega_r}$ be a C_0-cosine family with infinitesimal generator A, and let $C \in B(X)$ be invertible such that for all $t \in \Omega_r$, $CC(t) = C(t)C$. Define for each $t \in \Omega_r$ the family of linear operators $S(t) = C(t)C$. Then $(S(t))_{t\in\Omega_r}$ is a $C-$cosine family of infinitesimal generator A. In this sense, Definition 2.1 generalizes Definition 1.1 of C_0- cosine family.

We continue with the following example.

Example 2.7. Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $A, C \in B(X)$ such that C is invertible, $AC = CA$ and $\|A\| < r$ with $r = p^{\frac{-1}{p-1}}$. Then for all $t \in \Omega_r$, $C(t) = \sum_{n\in\mathbb{N}} \frac{t^{2n}}{(2n)!} CA^n$, in particular if $C = (I - A)^{-1}$, then $(C(t))_{t\in\Omega_r}$ is a $C-$cosine family of bounded linear operators of infinitesimal generator A on X. It is easy to see that

(i) $C(0) = C$.

(ii) For all $t, s \in \Omega_r, 2C(t)C(s) = C\Big(C(s+t) + C(s-t)\Big)$.

(iii) For all $x \in X$, $C(\cdot)x : \Omega_r \to X$ is continuous on Ω_r.

(iv) For all $x \in D(A)$, $2C^{-1}\Big(\lim_{t\to 0} \frac{C(t)x - Cx}{t^2}\Big) = Ax$.

We have the following proposition.

Proposition 2.8. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $(C(t))_{t\in\Omega_r}$ be a C_1-cosine family with infinitesimal generator A and $C_2 \in B(X)$ be invertible such that for all $t \in \Omega_r, C_2C(t) = C(t)C_2$, then $(C_2C(t))_{t\in\Omega_r}$ is a C_1C_2-cosine family on X.*

Proof. For each $t \in \Omega_r$, $S(t) = C_2C(t)$. Then $(S(t))_{t\in\Omega_r}$ is a C_1C_2-cosine family on X. In fact,

(i) $S(0) = C_2C(0) = C_1C_2$,

(ii) For all $s, t \in \Omega_r$,

$$\begin{aligned} S(s)S(t) &= C_2C(s)C_2C(t) \\ &= 2C(s)C(t)C_2^2 \\ &= C_1\Big(C(s+t)+C(s-t)\Big)C_2^2 \\ &= C_1C_2^2\Big(C(s+t)+C(s-t)\Big) \\ &= C_1C_2\Big(S(s+t)+S(s-t)\Big). \end{aligned}$$

(iii) Since for all $x \in X$, $C(\cdot)x : \Omega_r \to X$ is continuous and $C_2 \in B(X)$, $S(\cdot)x : \Omega_r \to X$ is continuous for all $x \in X$. Thus, $(S(t))_{t\in\Omega_r}$ is a C_1C_2-cosine family of bounded linear operators on X. ☐

Recall that $\mathbb{C}_p^+ = \{a \in \mathbb{C}_p : |1-a| < 1\}$. For each $a \in \mathbb{C}_p^+$ where $p \neq 2$, the element

$$\sqrt{a} = a^{\frac{1}{2}} = \sum_{n\in\mathbb{N}} \binom{\frac{1}{2}}{n}(a-1)^n \tag{2.1}$$

is the unique positive square root of a. For more details see [12, Section 49, page 143].

Example 2.9. Assume that $\mathbb{K} = \mathbb{C}_p$ with $p \neq 2$ and $r = p^{\frac{-1}{p-1}}$. Let X be a free non-Archimedean Banach space over $\mathbb{C}_p$ and $(e_i)_{i\in\mathbb{N}}$ a base of X. Define for each $t \in \Omega_r$, $x = \sum_{i\in\mathbb{N}} x_ie_i \in X$,

$$C(t)x = \sum_{i\in\mathbb{N}}(1-\alpha_i)ch(t\sqrt{\mu_i})x_ie_i,$$

where $(\alpha_i)_{i\in\mathbb{N}} \subset \Omega_r$, fixed $(\mu_i)_{i\in\mathbb{N}} \subset \mathbb{C}_p^+$. It is easy to check that the family $(C(t))_{t\in\Omega_r}$ is well defined on X.

Proposition 2.10. *The operators defined above form a $C-$cosine family of bounded linear operators, whose infinitesimal generator is the bounded diagonal operator A defined by $Ax = \sum_{i\in\mathbb{N}} \sqrt{\mu_i}x_ie_i$ for each $x = \sum_{i\in\mathbb{N}} x_ie_i \in X$.*

Proof. Let X be a free non-Archimedean Banach space over $\mathbb{C}_p$ and $(e_i)_{i\in\mathbb{N}}$ a base of X. Define for each $t \in \Omega_r$, $i \in \mathbb{N}$,

$$C(t)e_i = (1-\alpha_i)ch(t\sqrt{\mu_i})e_i \overset{def}{=} \left(\sum_{n\in\mathbb{N}} \frac{(1-\alpha_i)\mu_i^n t^{2n}}{(2n)!}\right)e_i,$$

where $(\alpha_i)_{i\in\mathbb{N}} \subset \Omega_r$, $(\mu_i)_{i\in\mathbb{N}} \subset \mathbb{C}_p^+$. From for all $i \in \mathbb{N}$, $t\mu_i \in \Omega_r$, we have for all $t \in \Omega_r,\ x \in X,\ \|C(t)x\| \leq \sup_{i\in\mathbb{N}} \Big|(1-\alpha_i)ch(t\sqrt{\mu_i})\Big|_p \|x\| < \infty$, then $\Big(\forall t \in \Omega_r\Big)$ $\|C(t)\|$ is finite. Hence the family $(C(t))_{t\in\Omega_r}$ is well defined on X. Set for all $i \in \mathbb{N}$, $Ce_i = (1-\alpha_i)e_i$, hence C is an invertible diagonal operator and also, it is easy to see that

(i) $C(0) = C$;

(ii) For all $t, s \in \Omega_r, 2C(t)C(s) = C\Big(C(s+t) + C(s-t)\Big)$;

(iii) For all $x \in X$, $C(\cdot)x : \Omega_r \to X$ is continuous on Ω_r.

Thus $(C(t))_{t\in\Omega_r}$ is a $C-$cosine family of bounded linear operators on X. Let B be the infinitesimal generator of $(C(t))_{t\in\Omega_r}$. It remains to show that $A = B$. Let us show that $D(B) = X\Big(= D(A)\Big)$. Clearly, for each $t \in \Omega_r^*$, and $i \in \mathbb{N}$,

$$2\frac{C(t)e_i - Ce_i}{t^2} = 2C\Big(\frac{ch(t\sqrt{\mu_i}) - 1}{t^2}\Big)e_i.$$

Thus, for all $t \in \Omega_r^*$ and for all $i \in \mathbb{N}$,

$$2C^{-1}\Big(\frac{C(t)e_i - Ce_i}{t^2}\Big) = 2\Big(\frac{ch(t\sqrt{\mu_i}) - 1}{t^2}\Big)e_i.$$

It follows, for all $x = \sum_{i\in\mathbb{N}} x_i e_i \in X$, $t \in \Omega_r^*$ we have

$$|x_i|_p \left\|2C^{-1}\Big(\frac{C(t)e_i - Ce_i}{t^2}\Big)\right\| \leq M|x_i|_p\|e_i\| \to 0 \ as\ i \to \infty. \tag{2.2}$$

Thus, $D(B) = \Big\{x = (x_i)_{i\in\mathbb{N}} : \lim_{i\to\infty} |x_i|_p \Big\|C^{-1}\Big(\frac{C(t)e_i - Ce_i}{t^2}\Big)\Big\| = 0\Big\}$. To complete the proof, it suffices to prove that

$$\Big(\forall i \in \mathbb{N}\Big) \lim_{t\to 0} \Big\|Ae_i - 2C^{-1}\Big(\frac{C(t)e_i - Ce_i}{t^2}\Big)\Big\| = 0.$$

Since $\lim_{t\to 0} 2\Big(\frac{ch(t\sqrt{\mu_i}) - 1}{t^2}\Big) = \mu_i$, then $A = B$ is the infinitesimal generator of the C-cosine family $(C(t))_{t\in\Omega_r}$. □

Definition 2.11. Let X be a non-Archimedean Banach space over $\mathbb{K}$, and $(C(t))_{t\in\Omega_r}$ be a $C-$cosine family of bounded linear operators on X. Then $(C(t))_{t\in\Omega_r}$ is said to be uniformly $C-$cosine family on X if

$$\lim_{t\to 0}\|C(t)-C\|=0.$$

We have the following theorem.

Theorem 2.12. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $A\in B(X)$ such that $\|A\|<r=p^{\frac{-1}{p-1}}$. Then A is the infinitesimal generator of a uniformly $C-$cosine family of bounded linear operators $(C(t))_{t\in\Omega_r}$.*

Proof. Suppose that A is a bounded linear operator on X with $\|A\|<r=p^{\frac{-1}{p-1}}$ and set, for all $t\in\Omega_r$,

$$C(t)=\sum_{n\in\mathbb{N}}\frac{(I-A)t^{2n}A^n}{(2n)!}. \tag{2.3}$$

Clearly, the series given by (2.3) converges in norm and defines a family of bounded linear operators on X by $|t|\|A\|<r$. Furthermore,

(i) $C(0)=I-A$, $\Big($ from $\|A\|<r<1$, we have $I-A$ is invertible$\Big)$.

(ii) The same as in Proposition 2.10.

(iii) It is easy to check that for all $x\in X$, $S(\cdot)x:\Omega_r\to X$ is continuous on Ω_r.

Thus $(C(t))_{t\in\Omega_r}$ is a $C-$cosine family of bounded linear operators on X where $C=I-A$. By a simple calculation, we obtain that $\lim\limits_{t\to 0}\|C(t)-C\|=0$ and for all $t\in\Omega_r^*$, $2C^{-1}\Big(\frac{C(t)-C}{t^2}\Big)=2\sum\limits_{n=0}^{\infty}\frac{t^{2n}A^{n+1}}{(2(n+1))!}$. Hence, for all $t\in\Omega_r^*$,

$$\begin{aligned}\Big\|2C^{-1}\Big(\frac{C(t)-C}{t^2}\Big)-A\Big\| &= \Big\|2\sum_{n=1}^{\infty}\frac{t^{2n}A^{n+1}}{(2(n+1))!}\Big\|\\ &\le \|2A\|\|\xi_t\|\\ &< \|\xi_t\|,\end{aligned}$$

where $\xi_t=\sum\limits_{n=1}^{\infty}\frac{t^{2n}A^n}{(2(n+1))!}$ converges to zero as $t\to 0$. Consequently,

$$\lim_{t\to 0}\Big\|2C^{-1}\Big(\frac{C(t)-C}{t^2}\Big)-A\Big\|=0. \tag{2.4}$$

Hence, $(C(t))_{t\in\Omega_r}$ given above is an uniformly $C-$cosine family of bounded linear operators whose infinitesimal generator is A. $\square$

Definition 2.13. Let $(C(t))_{t\in\Omega_r}$ be a $C-$cosine family of bounded linear operators with the infintesimal generator A, $(C(t))_{t\in\Omega_r}$ is said to be $C-$cosine family of contractions if for all $t \in \Omega_r$, $\|C(t)\| \leq 1$.

Example 2.14. Assume that $\mathbb{K} = \mathbb{C}_p$, with $p \neq 2$, let $A \in B(X)$ such that $\|A\| < r \left(r = p^{\frac{-1}{p-1}}\right)$. Set, for all $t \in \Omega_r$, $C(t) = (I - A)\sum_{n\in\mathbb{N}} \frac{t^{2n}A^n}{(2n)!}$, then $(C(t))_{t\in\Omega_r}$ is a $C-$cosine family of bounded linear operators with the infinitesimal geneartor A. Hence, for all $t \in \Omega_r$,

$$\begin{aligned} \|C(t)\| &= \Big\|(I - A)\sum_{n\in\mathbb{N}} \frac{t^{2n}A^n}{(2n)!}\Big\| \\ &\leq \|(I - A)\|\Big\|\sum_{n\in\mathbb{N}} \frac{t^{2n}A^n}{(2n)!}\Big\| \\ &\leq 1. \end{aligned}$$

Consequently, $(C(t))_{t\in\Omega_r}$ is a $C-$cosine family of contractions on X.

We have the following theorem.

Theorem 2.15. *Let $(C(t))_{t\in\Omega_r}$ be a $C-$cosine family satisfying: there exists $M > 0$ such that for each $t \in \Omega_r$, $\|C(t)\| \leq M$, and let A be its infinitesimal generator. Then, for every $x \in D(A)$, $t \in \Omega_r$, $C(t)x \in D(A)$, and $AC(t)x = C(t)Ax$.*

Proof. Let $x \in D(A)$ and let $t \in \Omega_r^*$ *and* $s \in \Omega_r$. Using Definition 2.1, and the boundedness of $C(t)$ and (ii) of Lemma 2.5, it easily follows that:
(2.5)

$$2\frac{C(t)C(s)x - CC(s)x}{t^2} = C(s)\left(2\frac{C(t)x - Cx}{t^2}\right) \to C(s)CAx = CC(s)Ax$$

as $t \longrightarrow 0$. Consequently, $C(s)x \in D(A)$ and $AC(s)x = C(s)Ax$. □

As an illustration, we will discuss the solvability of some second order linear homogeneous p-adic differential equations.

Remark 2.16. Let X be a non-Archimedean Banach space over $\mathbb{Q}_p$, let $A \in B(X)$ such that $\|A\| < r = p^{\frac{-1}{p-1}}$, the function $u(t) = C(t)x = \sum_{n\in\mathbb{N}} \frac{t^{2n}}{(2n)!}(I - A)A^n x$, for some $x \in D(A)$, is the solution to the homogeneous p-adic differential equation given by

$$\frac{d^2u(t)}{dt^2} = Au(t),\ t \in \Omega_r,\ u(0) = (I - A)x,\ u^{'}(0) = 0,$$

where $A : D(A) \subset X \to X$ is the infinitesimal generator of the $C-$cosine family $(C(t))_{t\in\Omega_r}$, and $u : \Omega_r \to D(A)$ is an $X-$ valued function.

We have the following definition.

Definition 2.17. [6] Let X and Y two non-Archimedean Banach spaces over a non-Archimedean valued field $\mathbb{K}$. For all $T \in B(X)$ and $S \in B(Y)$, the operator $T \oplus S$ is defined on the Banach space $X \oplus Y = \{(x, y) : x \in X, y \in Y\} = \{x \oplus y \ : x \in X, \ y \in Y\}$ endowed with the non-Archimedean norm $\|x \oplus y\| = \max(\|x\|, \ \|y\|)$, by

$$(\forall x \oplus y \in X \oplus Y) \ \ (T \oplus S)(x \oplus y) = Tx \oplus Sy = (Tx, Sy).$$

We continue by stating the following theorem.

Theorem 2.18. *Let $(C(t))_{t\in\Omega_r}$ be a $C-$cosine family of infinitesimal generator A on X. Set, for all $t \in \Omega_r$, $S(t) = C(t) \oplus I$. Then the following statements hold:*

(i) $(S(t))_{t\in\Omega_r}$ *is a $C \oplus I-$cosine family on $X \oplus X$.*

(ii) The generator of $(S(t))_{t\in\Omega_r}$ is the operator T defined on $D(T) = D(A)\oplus X$ by:

$$\textit{for all } x \in D(A), y \in X \quad T(x \oplus y) = Ax \oplus 0.$$

Proof. (i) Since $(C(t))_{t\in\Omega_r}$ is a $C-$cosine family of infinitesimal generator A on X, then

$$S(0) = C(0) \oplus I = C \oplus I.$$

Let $x \oplus y \in X \oplus X$ and $t, \ s \in \Omega_r$, we have:

$$\begin{aligned}
2S(t)S(s)(x \oplus y) &= 2S(t)(C(s) \oplus I)(x \oplus y) \\
&= 2(C(t) \oplus I)(C(s)x \oplus y) \\
&= 2C(t)C(s)x \oplus 2y \\
&= C\Big(C(t-s)(x) + C(t+s)(x)\Big) \oplus 2y \\
&= CC(t-s)x \oplus y + CC(t+s)x \oplus y \\
&= (C \oplus I)S(t-s)(x \oplus y) + (C \oplus I)S(t+s)(x \oplus y) \\
&= (C \oplus I)(S(t-s) + S(t+s))(x \oplus y).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\lim_{t\to 0} \|S(t)(x \oplus y) - (C \oplus I)(x \oplus y)\| &= \lim_{t\to 0} \|(C(t)x - Cx) \oplus 0\| \\
&= \lim_{t\to 0} \max\left(\|C(t)x - Cx\|, 0\right) \\
&= \lim_{t\to 0} \|C(t)x - Cx\| \\
&= 0.
\end{aligned}$$

Therefore $(S(t))_{t\in\Omega_r}$ is a $C \oplus I-$cosine family on $X \oplus X$.

(ii) Let $x \in D(A)$ and $y \in X$, we have

$$\begin{aligned}
2\lim_{t\to 0} \frac{S\left(t\right)\left(x \oplus y\right) - (C \oplus I)x \oplus y}{t^2} &= 2\lim_{t\to 0} \frac{C\left(t\right)\left(x\right) \oplus y - Cx \oplus y}{t^2} \\
&= 2\lim_{t\to 0} \frac{\left(C\left(t\right)\left(x\right) - Cx\right) \oplus 0}{t^2} \\
&= CAx \oplus 0 = (C \oplus I)(Ax \oplus 0).
\end{aligned}$$

Thus, for all $x \in D(A)$, $y \in X$ we have

$$2(C \oplus I)^{-1}\left(\lim_{t\to 0} \frac{S\,(t)\,(x \oplus y) - (C \oplus I)x \oplus y}{t^2}\right) = Ax \oplus 0.$$

Then $D(T) = D(A) \oplus X$ and $T(x \oplus y) = A(x) \oplus 0$, for all $x \in D(A)$.

□

Definition 2.19. Let $r > 0$ be a real number. A family $(S(t))_{t\in\Omega_r}$ of bounded linear operators is said to satisfy p-adic $H-$generalized cosine family of bounded linear operators on X if

$$\text{for all } t,\ s \in \Omega_r,\ S(s+t) + S(s-t) = H\Big(S(s),\ S(t)\Big),$$

where $H:\ B(X) \times B(X) \to B(X)$ is a function.

Remark 2.20. If $H\Big(S(s),\ S(t)\Big) = 2S(s)S(t)$, with $S(0) = I$, then $(S(t))_{t\in\Omega_r}$ is a cosine family of bounded linear operators on X.

We have the following definition.

Definition 2.21. Let $r > 0$ be a real number. A family $(S(t))_{t\in\Omega_r}$ of bounded linear operators is said to be $H - C_0-$cosine family or generalized C_0-cosine family of bounded linear operators on X if

(1) $S(0) = I$; where I is the identity operator of X.

(2) For all $t,\ s \in \Omega_r$,

$$\begin{aligned} S(s+t) + S(s-t) &= H\Big(S(s),\ S(t)\Big) \\ &= 2S(s)S(t) + 2D\Big(S(s) - C(s)\Big)\Big(S(t) - C(t)\Big), \end{aligned}$$

where $(C(t))_{t\in\Omega_r}$ is a C_0-cosine family of bounded linear operators with the infinitesimal generator A_0 and $D \in B(X)$.

(3) For each $x \in X$, $S(\cdot)x : \Omega_r \longrightarrow X$ is continuous on Ω_r.

The linear operator A defined by

$$D(A) = \{x \in X : 2\lim_{t\to 0} \frac{S(t)x - x}{t^2} \text{ exists}\}$$

and

$$\text{for each } x \in D(A),\ Ax = 2\lim_{t\to 0} \frac{S(t)x - x}{t^2},$$

is called the infinitesimal generator of the $H - C_0-$cosine family $(S(t))_{t\in\Omega_r}$.

Remark 2.22. Let $(S(t))_{t\in\Omega_r}$ be a generalized C_0-cosine family on X, if $D = 0$, then $(S(t))_{t\in\Omega_r}$ is a C_0- cosine family of linear operators on X.

From Definition 2.21, when $D = \alpha I$ for $\alpha \in \mathbb{K}$, we have the following definition.

Definition 2.23. Let $r > 0$ be a real number. A family $(S(t))_{t\in\Omega_r}$ is said to be a mixed C_0-cosine family or a mixed strongly continuous cosine family of bounded linear operators on X if

(1) $S(0) = I$; where I is the identity operator of X.

(2) For all $t,\ s \in \Omega_r$,

$$\begin{aligned} S(s+t) + S(s-t) &= H\Big(S(s),\ S(t)\Big) \\ &= 2S(s)S(t) + 2\alpha\Big(S(s) - C(s)\Big)\Big(S(t) - C(t)\Big), \end{aligned}$$

where $(C(t))_{t\in\Omega_r}$ is a C_0-cosine family of bounded linear operators with the infinitesimal generator A_0 and $\alpha \in \mathbb{K}$.

(3) For each $x \in\ X,\ S(\cdot)x : \Omega_r \longrightarrow\ X$ is continuous on Ω_r.

The linear operator A defined by

$$D(A) = \{x \in X : 2\lim_{t\to 0} \frac{S(t)x - x}{t^2} \text{ exists}\}$$

and

$$\text{for each } x \in D(A),\ Ax = 2\lim_{t\to 0} \frac{S(t)x - x}{t^2},$$

is called the infinitesimal generator of the $H - C_0-$cosine family $(S(t))_{t\in\Omega_r}$.

2.1 Question

Can we characterize the infinitesimal generator of mixed C_0-cosine family of bounded linear operators on infinite dimensional non-Archimedean Banach space?

Remark 2.24. Let $(S(t))_{t\in\Omega_r}$ be a mixed C_0-cosine family on X, if $\alpha = 0$, then $(S(t))_{t\in\Omega_r}$ is a C_0- cosine family of linear operators on X.

Example 2.25. Assume that $\mathbb{K} = \mathbb{C}_p$ with $p \neq 2$ and $r = p^{\frac{-1}{p-1}}$, let X be a non-Archimedean Banach space over $\mathbb{C}_p$, and let $A \in B(X)$ such that $\|A\| < r$. Put

$$\text{for all } t \in \Omega_r,\ S(t) = ch(tA) + tAsh(tA),$$

where $ch(tA) = \sum_{n\in\mathbb{N}} \frac{t^{2n}}{(2n)!}A^{2n}$ and $sh(tA) = \sum_{n\in\mathbb{N}} \frac{t^{2n+1}}{(2n+1)!}A^{2n+1}$. It is easy to see that the following statements hold:

(1) If $\alpha = -1$, then $\{S(t)\}_{t\in\Omega_r}$ is a mixed C_0-cosine family with $C(t) = ch(tA)$.

(2) If $\alpha = -1$, then for each $t,\ s \in \Omega_r,\ S(s)S(t) = S(t)S(s)$.

We have the following lemma.

Lemma 2.26. *Let $\{S(t)\}_{t\in\Omega_r}$ be an $H-C_0-$cosine family on non-Archimedean Banach space X, then for all $t \in \Omega_r$, $S(-t) = S(t)$.*

Proof. Obvious. □

The following proposition gives a condition for which an $H - C_0-$cosine family commutes.

Proposition 2.27. *Let $(S(t))_{t\in\Omega_r}$ be an $H - C_0-$cosine family on non-Archimedean Banach space X such that $I + D$ is injective and for each $t,\ s \in \Omega_r$, $C(s)S(t) = S(t)C(s)$, then for each $t,\ s \in \Omega_r$, $S(s)S(t) = S(t)S(s)$.*

Proof. Assume that $I + D$ is injective and for each $t,\ s \in \Omega_r$, $C(s)S(t) = S(t)C(s)$, then for each $t,\ s \in \Omega_r$,

$$\begin{aligned} 2S(s)S(t) + 2D\Big(S(s) - C(s)\Big)\Big(S(t) - C(t)\Big) &= S(s + t) + S(s - t) \\ &= S(t + s) + S(t - s) \\ &= 2S(t)S(s) \\ &\quad +2D\Big(S(t) - C(t)\Big) \\ &\quad \times\Big(S(s) - C(s)\Big). \end{aligned}$$

Thus, $(I + D)\Big(S(t)S(s) - S(s)S(t)\Big) = 0$, then for each $t,\ s \in \Omega_r$, $S(s)S(t) = S(t)S(s)$. □

We have the following theorem.

Proposition 2.28. *Let $\{S(t)\}_{t\in\Omega_r}$ be an $H - C_0-$cosine commuting family on non-Archimedean Banach space X of infinitesimal generator A with $\{C(t)\}_{t\in\Omega_r}$, a C_0-cosine family such that for all $t,\ s \in \Omega_r$, $C(s)S(t) = S(t)C(s)$. If $x \in D(A)$, then for all $t \in \Omega_r$, $S(t)x$, $C(t)x \in D(A)$, and $AS(t)x = S(t)Ax$ and $AC(t)x = C(t)Ax$.*

Proof. Let $x \in D(A)$ and let $s \in \Omega_r^*$ *and* $t \in \Omega_r$. It is easy to see that

$$2\left(\frac{S(s)S(t)x - S(t)x}{s^2}\right) = 2S(t)\left(\frac{S(s)x - x}{s^2}\right) \to S(t)Ax \text{ as } s \to 0. \tag{2.6}$$

Consequently, for all $t \in \Omega_r$, $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$.
Let $x \in D(A)$ and let $s \in \Omega_r^*$ *and* $t \in \Omega_r$. Then

$$2\left(\frac{S(s)C(t)x - C(t)x}{s^2}\right) = 2C(t)\left(\frac{S(s)x - x}{s^2}\right) \to C(t)Ax \text{ as } s \to 0. \tag{2.7}$$

Consequently, for all $t \in \Omega_r$, $C(t)x \in D(A)$ and $AC(t)x = C(t)Ax$. □

For $\alpha \in \mathbb{Q}_p\backslash\{-1\}$, set $A_1 = (1+\alpha)A - \alpha A_0$, where A_0 is the infinitesimal generator of the C_0-cosine family $\{C(t)\}_{t\in\Omega_r}$ and A is the infinitesimal generator of $\{S(t)\}_{t\in\Omega_r}$. We have the following theorem.

Theorem 2.29. *Let $\{S(t)\}_{t\in\Omega_r}$ be a mixed C_0-cosine family of inifinitesimal generator A on finite dimensional non-Archimedean Banach space X over $\mathbb{Q}_p$ with $\{C(t)\}_{t\in\Omega_r}$ as a C_0-cosine family of infinitesimal generator A_0 and $\alpha \in \mathbb{Q}_p\backslash\{-1\}$. Set $C_1(t)x = (1+\alpha)S(t)x - \alpha C(t)x$, $x \in X$, then $\{C_1(t)\}_{t\in\Omega_r}$ is a C_0-cosine family of bounded linear operators, whose infinitesimal generator is A_1. Furthermore, for all $x \in X$, and $t \in \Omega_r$,*

$$S(t)x = \frac{1}{1+\alpha}C_1(t)x + \frac{\alpha}{1+\alpha}C(t)x.$$

Proof.

(1) Trivially, $C_1(0)x = (1+\alpha)S(0)x - \alpha C(0)x = x$.

(2) For all $t,\ s \in \Omega_r,\ x \in X$, we have

$$\begin{aligned}
C_1(s+t)x + C_1(s-t)x &= (1+\alpha)\Big(S(s+t) + S(s-t)\Big)x \\
&\quad -\alpha\Big(C(s+t) + C(s-t)\Big)x \\
&= (1+\alpha)\Big(2S(s)S(t) + 2\alpha(S(s) - C(s)) \times \\
&\quad (S(t) - C(t))\Big)x - 2\alpha C(s)C(t)x \\
&= 2(1+\alpha)S(s)S(t)x + 2\alpha(1+\alpha)S(s)S(t)x \\
&\quad -2\alpha(1+\alpha)S(s)C(t)x - 2\alpha(1+\alpha)C(s)S(t)x \\
&\quad +2\alpha(1+\alpha)C(s)C(t)x - 2\alpha C(s)C(t)x \\
&= 2(1+\alpha)^2S(s)S(t)x - 2\alpha(1+\alpha)S(s)C(t)x \\
&- 2\alpha(1+\alpha)C(s)S(t)x + 2\alpha(1+\alpha)C(s)C(t)x \\
&\quad -2\alpha C(s)C(t)x \\
&= 2\Big((1+\alpha)S(s) - \alpha C(s)\Big)\Big(1+\alpha)S(t) - \alpha C(t)\Big)x \\
&= 2C_1(s)C_2(t)x.
\end{aligned}$$

Moreover, $C_1(0)x = (1+\alpha)x - \alpha x = x$. Thus, $(C_1(t))_{t\in\Omega_r}$ is a cosine family of bounded linear operators on X. Since $(C(t))_{t\in\Omega_r}$ and $(S(t))_{t\in\Omega_r}$ are continuous, then $(C_1(t))_{t\in\Omega_r}$ is continuous. So, $(C_1(t))_{t\in\Omega_r}$ is a C_0- cosine family of bounded linear operators on X.

(3) Now, we show that A_1 is the infinitesimal generator of $\{C_1(t)\}_{t\in\Omega_r}$. For $x \in D(A_1) = D(A) \cap D(A_0)(= X)$. By definition of $D(A)$ and $D(A_0)$, we have

$2\lim_{t\to 0}\Big(\frac{S(t)x-x}{t^2}\Big)=Ax$ and $2\lim_{t\to 0}\Big(\frac{C(t)x-x}{t^2}\Big)=A_0x$. Then,

$$\begin{aligned}2\lim_{t\to 0}\Big(\frac{C_1(t)x-x}{t^2}\Big) &= 2\lim_{t\to 0}\Big(\frac{(1+\alpha)S(t)x-\alpha C(t)x-x}{t^2}\Big)\\ &= 2(1+\alpha)\lim_{t\to 0}\Big(\frac{S(t)x-x}{t^2}\Big)-2\alpha\lim_{t\to 0}\Big(\frac{C(t)x-x}{t^2}\Big)\\ &= (1+\alpha)Ax-\alpha A_0x.\end{aligned}$$

It follows that A_1 is the infinitesimal generator of $(C_1(t))_{t\in\Omega_r}$. □

Proposition 2.30. *Let $(S(t))_{t\in\Omega_r}$ be a mixed C_0 cosine family on non-Archimedean Banach space X over $\mathbb{K}$ with $\alpha\in\mathbb{K}\backslash\{-1\}$ such that for all $t,\ s\in\Omega_r,\ C(s)S(t)=S(t)C(s)$, then for all $t,\ s\in\Omega_r,\ S(s)S(t)=S(t)S(s)$.*

Proof. Assume that for all $t,\ s\in\Omega_r,\ C(s)S(t)=S(t)C(s)$, then for all $t,\ s\in\Omega_r$,

$$\begin{aligned}2S(s)S(t)+2\alpha(S(s)-C(s))(S(t)-C(t)) &= S(s+t)+S(s-t)\\ &= S(t+s)+S(t-s)\\ &= 2S(t)S(s)\\ &\quad +2\alpha(S(t)-C(t))\times\\ &\quad (S(s)-C(s)).\end{aligned}$$

Thus, $(1+\alpha)\Big(S(t)S(s)-S(s)S(t)\Big)=0$. Then, for all $t,\ s\in\Omega_r,\ S(s)S(t)=S(t)S(s)$. □

Let $\{S(t)\}_{t\in\Omega_r}$ be a mixed C_0- cosine family of infinitesimal generator A with $\{C(t)\}_{t\in\Omega_r}$ as a C_0-cosine family of bounded linear operators of infinitesimal generator A_0, with $\alpha\in\mathbb{K}\backslash\{-1\}$. We have the following theorem.

Theorem 2.31. *Let $\{S(t)\}_{t\in\Omega_r}$ be a mixed C_0- cosine family of infinitesimal generator A with $\{C(t)\}_{t\in\Omega_r}$ as a C_0-cosine family with $\alpha\in\mathbb{K}\backslash\{-1\}$ such that for all $t,\ s\in\Omega_r,\ C(s)S(t)=S(t)C(s)$. If $x\in D(A)$, then for all $t\in\Omega_r,\ S(t)x,\ C(t)x\in D(A),\ AS(t)x=S(t)Ax$ and $AC(t)x=C(t)Ax$.*

Proof. Let $x\in D(A)$ and let $s\in\Omega_r^*$ *and* $t\in\Omega_r$. From Proposition 2.30, $S(t)S(s)=S(s)S(t)$ hence,

$$2\left(\frac{S(s)S(t)x-S(t)x}{s^2}\right)=2S(t)\left(\frac{S(s)x-x}{s^2}\right)\to S(t)Ax \text{ as } s\to 0.$$

Consequently, $S(t)Ax\in D(A)$ and $AS(t)x=S(t)Ax$.
Let $x\in D(A)$ and let $s\in\Omega_r^*$ *and* $t\in\Omega_r$. Then,

$$2\left(\frac{S(s)C(t)x-C(t)x}{s^2}\right)=2C(t)\left(\frac{S(s)x-x}{s^2}\right)\to C(t)Ax \text{ as } s\to 0.$$

Consequently, $C(t)x\in D(A)$ and $AC(t)x=C(t)Ax$. □

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