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# Finslerian hypersurfaces of a Finsler space with deformed randers $(\alpha, \beta)$- metric

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**Abstract.** In the present paper, we have studied the Finslerian hypersurfaces of a Finsler Space with deformed Randers $(\alpha, \beta)$-metric. Further, we have endeavored to prove the conditions under which these hypersurfaces of a special Finsler space with deformed Randers $(\alpha, \beta)$ metric will be a hyperplane of the first, second and third kinds.

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## 1. Introduction

Finsler metric $L(\alpha, \beta)$ of two variables is known as $(\alpha, \beta)$-metric, where $L$ is positively homogeneous function of degree one in an $n$-dimensional differentiable manifold $M^n$ and the space $F^n = \{M^n, L\}$ is Finsler space with $(\alpha, \beta)$-metric. The interesting and important examples of an $(\alpha, \beta)$-metric are Randers metric $\alpha + \beta$, Kropina metric $\frac{\alpha^2}{\beta}$ and Matsumoto metric $\frac{\alpha^2}{(\alpha-\beta)}$ [7, 9]. Matsumoto [7] studied these metrics in detail and obtained various interesting geometrical properties which shows the importance of these metrics in the development of Finsler geometry.

In 2018, Tripathi and Chaubey [12] considered two different deformed special $(\alpha, \beta)$-metrics of degree two in which one was formed as the combination of Randers and Riemannian metric and determined the nonholonomic frame due to this metric. The nonholonomic frame is to be considering if the metric defined by:

**Definition 1.1.** A space $F^n = \{M^n, F(x, y)\}$ is said to be Finsler space with $(\alpha,\beta)$-metric, if there exists a 2-homogeneous function $L$ of two variables such that the Finsler metric $F : TM \to \mathbb{R}$ is given by

$$F^2(x, y) = L\{\alpha(x, y), \beta(x, y)\},$$

where $\alpha^2(x, y) = a_{ij}(x)y^i y^j, \alpha$ is a Riemannian metric on the manifold $M^n$ and $\beta(x, y) = b_i(x)y^i$ is a 1-form on the manifold $M^n$.

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The special $(\alpha, \beta)$-metric [12] which is expressed in the form

$$L = (\alpha + \beta)\alpha = \alpha^2 + \alpha\beta, \tag{1.1}$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form with $||\beta_x||_\alpha < b_0$, $\forall\, x \in M^n$ is known as deformed Randers ($\alpha$,$\beta$)- metric.

The concept of Finslerian hypersurface was first introduced by Matsumoto [6] in 1985 and where he obtained the three types of hypersurfaces that were called hyperplane of the first, second and third kinds. Moreover, many authors [1, 2, 3, 4, 5, 8, 10, 11] studied the Finslerian hypersurfaces for the Finsler space equipped with 1-degree homogeneous Finsler metric and obtained the conditions under which it will be a hyperplane of the first, second and third kinds.

In the present paper, we have considered the deformed Randers $(\alpha, \beta)$-metric which forms a nonholonomic frame of $(\alpha, \beta)$-metric [12] and a 2-degree homogeneous function written in equation (1.1). Further, we have examined the conditions under these hypersurfaces of a special Finsler space with deformed Randers $(\alpha, \beta)$ metric will be a hyperplane of the first, second and third kinds.

## 2. Finsler spaces $F^n$ with deformed Randers ($\alpha$,$\beta$)-metric

In the present paper we consider an $n$-dimensional Finsler space $F^n = \{M^n, L(\alpha, \beta)\}$, that is equipped with deformed Randers $(\alpha, \beta)$-metric which is given by the equation (1.1).

Differentiating equation (1.1) partially with respect to $\alpha$ and $\beta$, we have

$$L_\alpha = 2\alpha + \beta, \quad L_\beta = \alpha, \quad L_{\alpha\alpha} = 2,$$

$$L_{\beta\beta} = 0 \quad and \quad L_{\alpha\beta} = 1,$$

where $L_\alpha = \frac{\partial L}{\partial \alpha}$, $L_\beta = \frac{\partial L}{\partial \beta}$, $L_{\alpha\alpha} = \frac{\partial L\alpha}{\partial \alpha}$, $L_{\beta\beta} = \frac{\partial L_\beta}{\partial \beta}$, $L_{\alpha\beta} = \frac{\partial L_\alpha}{\partial \beta}$.

In the Finsler space $F^n = \{M^n, L(\alpha, \beta)\}$ the normalized element of support $l_i = \dot{\partial}_i L$ and angular metric tensor $h_{ij}$ are given by [7]

$$l_i = \alpha^{-1} L_\alpha Y_i + L_\beta b_i, \tag{2.1}$$

and

$$h_{ij} = p a_{ij} + q_0 b_i b_j + q_{-1}(b_i Y_j + b_j Y_i) + q_{-2} Y_i Y_j, \tag{2.2}$$

where $Y_i = a_{ij} y^j$. For the fundamental metric (1.1) the above constants are

$$p = 2\alpha^2, \quad q_0 = 0, \quad q_{-1} = \alpha, \quad q_{-2} = 0. \tag{2.3}$$

The fundamental metric tensor $g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$ and its reciprocal tensor $g^{ij}$ for $L = L(\alpha, \beta)$ are given by

$$g_{ij} = pa_{ij} + p_0 b_i b_j + p_{-1}(b_i Y_j + b_j Y_i) + p_{-2} Y_i Y_j, \tag{2.4}$$

where

$$\begin{cases} p_0 = q_0 + L_\beta^2 = \alpha^2, \\ p_{-1} = q_{-1} + L^{-1} p L_\beta = 3\alpha, \\ p_{-2} = q_{-2} + p^2 L^{-2} = 4. \end{cases} \tag{2.5}$$

The reciprocal tensor $g^{ij}$ of $g_{ij}$ is given by

$$g^{ij} = p^{-1} a^{ij} - s_0 b^i b^j - s_{-1}(b^i y^j + b^j y^i) - s_{-2} y^i y^j, \tag{2.6}$$

where $b^i = a^{ij} b_j, \quad b^2 = a_{ij} b^i b^j$ and

$$\begin{cases} s_0 = \frac{1}{\tau p}\{pp_0 + (p_0 p_{-2} - p_{-1}^2)\alpha^2\}, \\ s_{-1} = \frac{1}{\tau p}\{pp_{-1} + (p_0 p_{-2} - p_{-1}^2)\beta\}, \\ s_{-2} = \frac{1}{\tau p}\{pp_{-2} + (p_0 p_{-2} - p_{-1}^2) b^2\}, \\ \tau = p(p + p_0 b^2 + p_{-1}\beta) + (p_0 p_{-2} - p_{-1}^2)(\alpha^2 b^2 - \beta^2). \end{cases} \tag{2.7}$$

The hv-torsion tensor $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ is given by

$$2pC_{ijk} = p_{-1}(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \gamma_1 m_i m_j m_k, \tag{2.8}$$

where

$$\gamma_1 = p\frac{\partial p_0}{\partial \beta} - 3p_{-1} q_0, \qquad m_i = b_i - \alpha^{-2}\beta Y_i. \tag{2.9}$$

Here $m_i$ is a non-vanishing covariant vector orthogonal to the element of support $y^i$. Thus we have

**Proposition 2.1.** *The normalised supporting element* $l_i$ *and angular metric tensor* $h_{ij}$ *of an n-dimensional Finsler space* $F^n$ *equipped with a deformed Randers* $(\alpha, \beta)$ *metric* $L$ *are given by (2.1) and (2.2), respectively.*

**Proposition 2.2.** *The fundamental metric tensor* $g_{ij}$ *and its reciprocal tensor* $g^{ij}$ *of an n-dimensional Finsler space* $F^n$ *equipped with a deformed Randers* $(\alpha, \beta)$ *metric* $L$ *are given by (2.4) and (2.6), respectively.*

**Proposition 2.3.** *The Cartan hv-torsion tensor of an n-dimensional Finsler space* $F^n$ *equipped with a deformed Randers* $(\alpha, \beta)$ *metric* $L$ *is given by (2.8).*

Let $\{^{\,i}_{jk}\}$ be the component of Christoffel symbols for the associated Riemannian space $R^n$ and $\nabla_k$ be the covariant derivative with respect to $x^k$ relative to Christoffel symbol. Now we define

$$2E_{ij} = b_{ij} + b_{ji}, \qquad 2F_{ij} = b_{ij} - b_{ji}, \tag{2.10}$$

where $b_{ij} = \bigtriangledown_j b_i$.

Let $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, \Gamma^{i}_{jk})$ be the Cartan connection of $F^n$. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \{^{\,i}_{jk}\}$ of the special Finsler space $F^n$ is given by

$$(2.11)\quad \begin{cases} D^i_{jk} = B^i E_{jk} + F^i_k B_j + F^i_j B_k + B^i_j b_{0k} + B^i_k b_{0j} \\ \qquad -b_{0m} g^{im} B_{jk} - C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} \\ \qquad +\lambda^s (C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}), \end{cases}$$

where

$$(2.12)\quad \begin{cases} B_k = p_0 b_k + p_{-1} Y_k, \\ B^i = g^{ij} B_j, \\ F^k_i = g^{kj} F_{ji}, \\ B_{ij} = \frac{\{p_{-1}(a_{ij} - \alpha^{-2} Y_i Y_j) + \frac{\partial p_0}{\partial \beta} m_i m_j\}}{2}, \\ B^k_i = g^{kj} B_{ji}, \\ A^m_k = B^m_k E_{00} + B^m E_{k0} + B_k F^m_0 + B_0 F^m_k, \\ \lambda^m = B^m E_{00} + 2 B_0 F^m_0, \\ B_0 = B_i y^i. \end{cases}$$

Here $'0'$ denotes contraction with $y^i$ except the quantities $p_0, q_0$ and $s_o$. Thus, we have

**Proposition 2.4.** *The difference tensor $D^i_{jk}$ of the Cartan connection $C\Gamma$ for the n-dimensional Finsler space $F^n$ is equipped with a deformed Randers $(\alpha, \beta)$ metric $L$ is given by (2.12).*

## 3. Induced Cartan connection

Let $F^{(n-1)}$ be a hypersurface of $F^n$ given by the equation $x^i = x^i(u^\alpha)$ where $\{\alpha = 1, 2, 3, ..., (n-1)\}$. The element of support $y^i$ of $F^n$ is to be taken as tangential to $F^{(n-1)}$, that is [6],

$$(3.1)\qquad y^i = B^i_\alpha(u) v^\alpha.$$

The metric tensor $g_{\alpha\beta}$ and hv-tensor $C_{\alpha\beta\gamma}$ of $F^{(n-1)}$ are given by

$$g_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta, \qquad C_{\alpha\beta\gamma} = C_{ijk} B^i_\alpha B^j_\beta B^k_\gamma,$$

and at each point $(u^\alpha)$ of $F^{(n-1)}$, a unit normal vector $N^i(u, v)$ is defined by

$$g_{ij}\{x(u,v), y(u,v)\} B^i_\alpha N^j = 0, \qquad g_{ij}\{x(u,v), y(u,v)\} N^i N^j = 1.$$

The angular metric tensor $h_{\alpha\beta}$ of the hypersurface is given by

$$(3.2)\qquad h_{\alpha\beta} = h_{ij} B^i_\alpha B^j_\beta, \qquad h_{ij} B^i_\alpha N^j = 0, \qquad h_{ij} N^i N^j = 1.$$

The relations between the components of $(B^\alpha_i, N_i)$ and its inverse $(B^i_\alpha, N^i)$ is given by

$$B_i^\alpha = g^{\alpha\beta} g_{ij} B_\beta^j, \quad B_\alpha^i B_i^\beta = \delta_\alpha^\beta, \quad B_i^\alpha N^i = 0, \quad B_\alpha^i N_i = 0.$$

$$N_i = g_{ij} N^j, \quad B_i^k = g^{kj} B_{ji}, \quad B_\alpha^i B_j^\alpha + N^i N_j = \delta_j^i.$$

The induced Cartan connection $IC\Gamma = (\Gamma_{\beta\gamma}^{*\alpha}, G_\beta^\alpha, C_{\beta\gamma}^\alpha)$ of $F^{(n-1)}$ from the Cartan's connection $C\Gamma = (\Gamma_{jk}^{*i}, \Gamma_{0k}^{*i}, C_{jk}^{*i})$ is given by [6],

$$\Gamma_{\beta\gamma}^{*\alpha} = B_i^\alpha (B_{\beta\gamma}^i + \Gamma_{jk}^{*i} B_\beta^j B_\gamma^k) + M_\beta^\alpha H_\gamma,$$

$$G_\beta^\alpha = B_i^\alpha (B_{0\beta}^i + \Gamma_{0j}^{*i} B_\beta^j), \quad C_{\beta\gamma}^\alpha = B_i^\alpha C_{jk}^i B_\beta^j B_\gamma^k,$$

where

$$M_{\beta\gamma} = N_i C_{jk}^i B_\beta^j B_\gamma^k, \quad M_\beta^\alpha = g^{\alpha\gamma} M_{\beta\gamma}, \quad H_\beta = N_i (B_{0\beta}^i + \Gamma_{oj}^{*i} B_\beta^j) \quad \text{and}$$

$$B_{\beta\gamma}^i = \frac{\partial B_\beta^i}{\partial u^\gamma}, \qquad B_{0\beta}^i = B_{\alpha\beta}^i v^\alpha.$$

The quantities $M_{\beta\gamma}$ and $H_\beta$ are called the second fundamental v-tensor and normal curvature vector, respectively [6]. The second fundamental h-tensor $H_{\beta\gamma}$ is defined as [6]

$$H_{\beta\gamma} = N_i (B_{\beta\gamma}^i + \Gamma_{jk}^{*i} B_\beta^j B_\gamma^k) + M_\beta H_\gamma, \tag{3.3}$$

where

$$M_\beta = N_i C_{jk}^i B_\beta^j N^k. \tag{3.4}$$

The relative h and v-covariant derivatives of projection factor $B_\alpha^i$ with respect to $IC\Gamma$ are given by

$$B_{\alpha|\beta}^i = H_{\alpha\beta} N^i, \qquad B_\alpha^i|_\beta = M_{\alpha\beta} N^i.$$

It is obvious from the equation (3.3) that $H_{\beta\gamma}$ is generally not symmetric and

$$H_{\beta\gamma} - H_{\gamma\beta} = M_\beta H_\gamma - M_\gamma H_\beta. \tag{3.5}$$

From above equation, we get

$$H_{0\gamma} = H_\gamma, \quad H_{\gamma 0} = H_\gamma + M_\gamma H_0. \tag{3.6}$$

We shall use the following lemmas which are due to Matsumoto [6] in the next section

**Lemma 3.1.** *The normal curvature $H_0 = H_\beta v^\beta$ vanishes if and only if the normal curvature vector $H_\beta$ vanishes.*

**Lemma 3.2.** *A hypersurface $F^{(n-1)}$ is a hyperplane of the first kind with respect to connection $C\Gamma$ if and only if $H_\alpha = 0$.*

**Lemma 3.3.** *A hypersurface $F^{(n-1)}$ is a hyperplane of the second kind with respect to connection $C\Gamma$ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.*

**Lemma 3.4.** *A hypersurface $F^{(n-1)}$ is a hyperplane of the third kind with respect to connection $C\Gamma$ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = M_{\alpha\beta} = 0$.*

## 4. Hypersurface $F^{(n-1)}(c)$ of a Finsler space with deformed Randers $(\alpha,\beta)$-metric

In this section, we have assumed that the vector field $b_i(x) = \frac{\partial b}{\partial x^i}$ is a gradient of some scalar function $b(x)$ in a deformed Randers $(\alpha,\beta)$ metric $L(\alpha,\beta) = \alpha^2 + \alpha\beta$. Later, we considered a hypersurface $F^{(n-1)}(c)$ given by the equation $b(x) = c$ [10].

From the parametric equation $x^i = x^i(u^\alpha)$ of $F^{(n-1)}(c)$, we get

$$\frac{\partial b(x)}{\partial u^\alpha} = 0, \qquad \frac{\partial b(x)}{\partial x^i}\frac{\partial x^i}{\partial u^\alpha} = 0 \qquad \text{and} \qquad b_i B^i_\alpha = 0\ .$$

The above equation shows that $b_i(x)$ is covariant component of a normal vector field of hypersurface $F^{(n-1)}(c)$. Further, we have

$$b_i B^i_\alpha = 0, \quad \textit{and} \quad b_i y^i = 0 \quad \textit{i.e} \quad \beta = 0, \tag{4.1}$$

and the induced metric $L(u,v)$ of $F^{(n-1)}(c)$ is given by

$$L(u,v) = a_{\alpha\beta} v^\alpha v^\beta, \quad a_{\alpha\beta} = a_{ij} B^i_\alpha B^j_\beta. \tag{4.2}$$

Writing $\beta = 0$ in the equations (2.3), (2.5) and (2.7) we have

$$\begin{cases} p = 2\alpha^2, \quad q_0 = 0, \\ q_{-1} = \alpha, \quad q_{-2} = 0, \\ p_0 = \alpha^2, \quad p_{-1} = 3\alpha, \\ p_{-2} = 4, \quad \tau = -10\alpha^8(2+b^2)b^2, \\ s_0 = \frac{3}{20\alpha^6(2+b^2)b^2}, \\ s_{-1} = -\frac{3}{10\alpha^7(2+b^2)b^2}, \\ s_{-2} = \frac{5b^2-8}{20\alpha^8 b^2(2+b^2)}. \end{cases} \tag{4.3}$$

From equation (2.6) we have,

$$\begin{cases} g^{ij} = \frac{1}{2\alpha^2}a^{ij} - \frac{3}{20\alpha^2 b^2(2+b^2)}b^i b^j + \frac{3}{10\alpha^7 b^2(2+b^2)}(b^i y^j + b^j y^i) \\ \qquad -\frac{5b^2-8}{20\alpha^8 b^2(2+b^2)}y^i y^j. \end{cases} \tag{4.4}$$

Thus along $F^{(n-1)}(c)$, equations (4.3) and (4.4) lead to

$$g^{ij}b_i b_j = \frac{b^2(10b^2+17)}{20\alpha^2(2+b^2)}.$$

So, we get

$$b_i(x(u)) = \frac{b}{2\alpha}\sqrt{\frac{10b^2-17}{5(2+b^2)}}N_i, \tag{4.5}$$

where $b^2 = a^{ij}b_i b_j$, and $b$ is the length of the vector $b^i$.

Again from the equations (4.4) and (4.5), we get

$$b^i = a^{ij}b_j = \frac{20\alpha^2(2+b^2)}{(10b^2+17)}N^i - \frac{6}{\alpha^5(10b^2+17)}y^i. \tag{4.6}$$

Thus, we have

**Theorem 4.1.** *The induced Riemannian metric in a Finslerian hypersurface $F^{(n-1)}(c)$ of a Finsler space with deformed Randers $(\alpha,\beta)$ metric $L=\alpha^2+\alpha\beta$ is given by (4.2) and the scalar function $b(x)$ is given by (4.5) and (4.6).*

Now the angular metric tensor $h_{ij}$ and metric tensor $g_{ij}$ of $F^n$ are given by

$$\text{(4.7)}\qquad \begin{cases} h_{ij}=2\alpha^2 a_{ij}+\alpha(b_iY_j+b_jY_i),\\ g_{ij}=2\alpha^2 a_{ij}+\alpha^2 b_ib_j+3\alpha(b_iY_j+b_jY_i). \end{cases}$$

If $h^{(a)}_{\alpha\beta}$ denotes the angular metric tensor of the Riemannian $a_{ij}(x)$ along Finslerian hypersurfaces $F^{(n-1)}(c)$, then from equations (4.1), (4.7) and (3.2), we have $h_{\alpha\beta}=h^{(a)}_{\alpha\beta}$.

Thus along the Finslerian hypersurface $F^{(n-1)}(c)$, $\quad \frac{\partial p_0}{\partial\beta}=0$.

Now, using equations (2.3), (2.9), (4.1) and $\frac{\partial p_0}{\partial\beta}=0$, we get

$$\gamma_1=0,\qquad m_i=b_i.$$

Again using above result the hv-torsion tensor for the Finslerian hypersurfaces $F^{(n-1)}(c)$ of the Finsler space with deformed Randers $(\alpha,\beta)$ metric is given by

$$\text{(4.8)}\qquad C_{ijk}=\frac{3}{4\alpha}(h_{ij}b_k+h_{jk}b_i+h_{ki}b_j).$$

From equations (3.2), (3.3), (3.5), (4.1) and (4.8), we have

$$\text{(4.9)}\qquad M_{\alpha\beta}=\frac{3b}{8\alpha^2}\sqrt{\frac{10b^2-17}{5(2+b^2)}}h_{\alpha\beta}\quad \textit{and}\quad M_\alpha=0.$$

Therefore, equation (3.6) shows that $H_{\alpha\beta}$ is symmetric. Thus, we have

**Theorem 4.2.** *The second fundamental v-tensor in a Finslerian hypersurface $F^{(n-1)}(c)$ of a Finsler space with deformed Randers $(\alpha,\beta)$ metric $L=\alpha^2+\alpha\beta$ is given by (4.9) and the second fundamental h-tensor $H_{\alpha\beta}$ is symmetric.*

Now from (4.1) we have $b_iB^i_\alpha=0$. Then we have

$$b_{i|\beta}B^i_\alpha+b_iB^i_{\alpha|\beta}=0.$$

Therefore, from equation (3.5) and by use of $b_{i|\beta}=b_{i|j}B^j_\beta+b_i|_jN^jH_\beta$, we have

$$\text{(4.10)}\qquad b_{i|j}B^i_\alpha B^j_\beta+b_{i|j}B^i_\alpha N^jH_\beta+b_iH_{\alpha\beta}N^i=0.$$

Since $b_i|_j=-b_hC^h_{ij}$, we get

$$b_{i|j}B^i_\alpha N^j=0.$$

Using above result and equation (4.5) in equation (4.10), we have

$$\frac{b}{2\alpha}\sqrt{\frac{10b^2-17}{5(2+b^2)}}H_{\alpha\beta}+b_{i|j}B^i_\alpha B^j_\beta=0, \tag{4.11}$$

because $b_{i|j}$ is symmetric. Now contracting equation (4.11) with $v^\beta$ and using (3.1), we get

$$\frac{b}{2\alpha}\sqrt{\frac{10b^2-17}{5(2+b^2)}}H_{\alpha}+b_{i|j}B^i_\alpha y^j=0. \tag{4.12}$$

Again contracting equation (4.12) by $v^\alpha$ and by use of equation (3.1), we have

$$\frac{b}{2\alpha}\sqrt{\frac{10b^2-17}{5(2+b^2)}}H_0+b_{i|j}y^i y^j=0. \tag{4.13}$$

From Lemmas 3.1 and 3.2, it is clear that the deformed Randres $(\alpha,\beta)$ metric of the Finsler hypersuface $F^{(n-1)}(c)$ is a hyperplane of the first kind, if and only if $H_0=0$. Thus from equation (4.13) it is obvious that $F^{(n-1)}(c)$ is a hyperplane of the first kind, if and only if $b_{i|j}y^iy^j=0$. This $b_{i|j}$ being the covariant derivative with respect to $C\Gamma$ of $F^n$ defined on $y^i$, $b_{ij}=\bigtriangledown_j b_i$ is the covariant derivative with respect to Riemannian connection $\{^{\,i}_{jk}\}$ and is constructed from $a_{ij}(x)$. Hence $b_{ij}$ does not depend on $y^i$. We shall consider the difference $b_{i|j}-b_{ij}$ where $b_{ij}=\bigtriangledown_j b_i$. The difference tensor $D^i_{jk}=\Gamma^{*i}_{jk}-\{^{\,i}_{jk}\}$ is given by (2.11). Since $b_i$ is a gradient vector, then from (2.10) we have

$$E_{ij}=b_{ij},\qquad F_{ij}=0\qquad and \qquad F^i_j=0.$$

Thus, equation (2.11) reduces to

$$\begin{cases} D^i_{jk}=B^ib_{jk}+B^i_jb_{0k}+B^i_kb_{0j}-b_{0m}g^{im}B_{jk}\\ \qquad -C^i_{jm}A^m_k-C^i_{km}A^m_j+C_{jkm}A^m_sg^{is}\\ \qquad +\lambda^s(C^i_{jm}C^m_{sk}+C^i_{km}C^m_{sj}-C^m_{jk}C^i_{ms}), \end{cases} \tag{4.14}$$

where

$$\begin{cases} B_i=\alpha^2b_i+3\alpha Y_i,\\ B^i=\frac{\alpha^4b^2(10b^2+19)+18}{20\alpha^4b^2(2+b^2)}b^i+\frac{3\alpha^6b^2(10b^2+17)-\alpha^2(5b^2-8)+6b^2}{20\alpha^7b^2(2+b^2)}y^i,\\ \lambda^m=B^mb_{00},\\ B_{ij}=\frac{3}{2\alpha}(a_{ij}-\alpha^{-2}Y_iY_j),\\ B^i_j=-\frac{9}{40\alpha^3b^2(b^2+2)}b^ib_j,\\ A^m_k=B^m_kb_{00}+B^mb_{k0}. \end{cases} \tag{4.15}$$

In view of equations (4.3) and (4.4), the relation in equation (2.12) becomes possible and by virture of equation (4.15) we have $B^i_0=0$, $B_{i0}=0$ and $A^m_0=B^mb_{00}$.

Now, contracting equation (4.14) by $y^k$, we get

$$D^i_{j0} = B^i b_{j0} + B^i_j b_{00} - B^m C^i_{jm} b_{00}\ .$$

Again by contracting the above equation with respect to $y^j$, we have

$$D^i_{00} = B^i b_{00} = \{\tfrac{\alpha^4 b^2(10b^2+19)+18}{20\alpha^4 b^2(2+b^2)} b^i + \tfrac{3\alpha^6 b^2(10b^2+17)-\alpha^2(5b^2-8)+6b^2}{20\alpha^7 b^2(2+b^2)} y^i\} b_{00}\ .$$

Paying attention to equation (4.1), along $F^{(n-1)}(c)$, we get

$$(4.16)\qquad \begin{cases} b_i D^i_{j0} = \frac{\alpha^4 b^2(10b^2+19)+18}{20\alpha^4(b^2+2)} b_{j0} - \frac{9}{40\alpha^3(b^2+2)} b_j b_{00} \\ \qquad -\frac{\alpha^4 b^2(10b^2+19)+18}{20\alpha^4(b^2+2)} b_i b^m C^i_{jm} b_{00}. \end{cases}$$

Contracting (4.16) with $y^j$ implies that

$$(4.17)\qquad b_i D^i_{00} = \frac{\alpha^4 b^2(10b^2+19)+18}{20\alpha^4(2+b^2)}.$$

From equations (3.3), (4.5), (4.6), (4.9) and $M_\alpha = 0$, we have

$$b_i b^m C^i_{jm} B^j_\alpha = b^2 M_\alpha = 0.$$

Thus, the relation of $b_{i|j} = b_{ij} - b_r D^r_{ij}$ the equations (4.16) and (4.17) gives

$$b_{i|j} y^i y^j = b_{00} - b_r D^r_{00} = \tfrac{\alpha^4(40-10b^4+3b^2)-18}{20\alpha^4(b^2+2)} b_{00}\ .$$

Consequently equations (4.12) and (4.13) may be written as

$$(4.18)\qquad \begin{cases} \frac{b}{2\alpha}\sqrt{\frac{10b^2-17}{5(2+b^2)}} H_\alpha + \frac{\alpha^4(40-10b^4+3b^2)-18}{20\alpha^4(b^2+2)} b_{i|j} B^i_\alpha y^j = 0, \\ \frac{b}{2\alpha}\sqrt{\frac{10b^2-17}{5(2+b^2)}} H_0 + \frac{\alpha^4(40-10b^4+3b^2)-18}{20\alpha^4(b^2+2)} b_{00} = 0. \end{cases}$$

Thus, the condition $H_0 = 0$ is equivalent to $b_{00} = 0$. Using the fact $\beta = b_i y^i = 0$, the condition $b_{00} = 0$ can be written as $b_{ij} y^i y^j = b_i y^i b_j y^j$ for some $c_j(x)$. Thus we can write,

$$(4.19)\qquad 2b_{ij} = b_i c_j + b_j c_i.$$

Now from equations (4.1) and (4.19) we get

$$b_{00} = 0, \quad b_{ij} B^i_\alpha B^j_\beta = 0, \quad b_{ij} B^i_\alpha y^j = 0.$$

Hence from equation (4.18) we get $H_\alpha = 0$, again from equations (4.19) and (4.15) we get $b_{i0} b^i = \frac{c_0 b^2}{2}$, $\lambda^m = 0$, $A^i_j B^j_\beta = 0$ and $B_{ij} B^i_\alpha B^j_\beta = \frac{3}{2\alpha} h_{\alpha\beta}$.

Now with the use of equations (3.3), (4.4), (4.5), (4.6), (4.9) and (4.14) we have

$$(4.20)\qquad b_r D^r_{ij} B^i_\alpha B^j_\beta = -\frac{c_0 b^3}{32\alpha^4}\{\frac{(10b^2-17)}{5(b^2+2)}\}^{\frac{3}{2}} h_{\alpha\beta}.$$

Thus the equation (4.11) reduces to

$$H_{\alpha\beta} + \frac{3c_0 b^2(10b^2 - 17)}{80\alpha^3(b^2+2)} h_{\alpha\beta} = 0. \tag{4.21}$$

Hence the hypersurface $F^{(n-1)}(c)$ is umbilic.

**Theorem 4.3.** *The necessary and sufficient condition for a Finslerian hypersurface $F^{(n-1)}(c)$ of a Finsler space with Randers deformed $(\alpha, \beta)$- metric $L = \alpha^2 + \alpha\beta$ to be a hyperplane of the first kind is (4.19).*

**Corollary 4.4.** *The second fundamental h-tensor in a Finslerian hypersurface $F^{(n-1)}(c)$ of a Finsler space with Randers deformed $(\alpha, \beta)$- metric $L = \alpha^2 + \alpha\beta$ is directly proportional to its angular metric tensor.*

Now from Lemma 3.3, $F^{(n-1)}(c)$ is a hyperplane of the second kind if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$. Thus from (4.20), we get

$$c_0 = c_i(x)y^i = 0.$$

Now, there exists a function $\psi(x)$ such that

$$c_i(x) = \psi(x)b_i(x).$$

Therefore, from equation (4.19), we get

$$2b_{ij} = b_i(x)\psi(x)b_j(x) + b_j(x)\psi(x)b_i(x).$$

This can also be written as

$$b_{ij} = \psi(x)b_i b_j.$$

**Theorem 4.5.** *The necessary and sufficient condition for a Finslerian hypersurface $F^{(n-1)}(c)$ of a Finsler space with Randers deformed $(\alpha, \beta)$- metric $L = \alpha^2 + \alpha\beta$ to be a hyperplane of the second kind is (4.21).*

Again Lemma 3.4, together with (4.9) and $M_\alpha = 0$ shows that $F^{(n-1)}(c)$ is not a hyperplane of the third kind.

**Theorem 4.6.** *The Finslerian hypersurface $F^{(n-1)}(c)$ of a Finsler space with Randers deformed $(\alpha, \beta)$- metric $L = \alpha^2 + \alpha\beta$ is not a hyperplane of the third kind.*

## 5. Conclusion

In the present paper, we have obtained the condition under which a deformed Randers $(\alpha, \beta)$-metric which is defined by equation (1.1) of a 2-degree homogeneous function and this metric also formed a nonholonomic frame of a Finsler space [12]. Further, we have obtained the Theorems 4.3 and 4.5 and they state that the condition under which the Finslerian hypersurfaces of this metric is a hyperplane of the first and the second kind. Again, we have proved in Theorem 4.6 that it is not a hyperplane of the third kind.

Since the noholonomic frame is defined for the Finsler metric of two-degree homogeneous function and this work can also be extended for the study of Finslerian hypersurfaces for those Finsler metric which framed nonholonomic frame for future studies.

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