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Mild positive solutions for iterative Caputo-Hadamard fractional differential equations

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Abstract. The existence of mild positive solutions for an iterative Caputo-Hadamard fractional differential equation is proved using the Schauder fixed point theorem. In addition, the uniqueness and continuous dependence of mild positive solutions are investigated and some new results are obtained. Finally, two examples are given to illustrate our obtained results.

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1. Introduction

Iterative and fractional differential equations have many applications in the automatic control problems, the physical problems, the mechanical problems, the biological problems, the disease transmission problems, the problems of long-range planning in economics, and in many other areas of science and technology. In particular, problems concerning the existence, uniqueness and stability of solutions for iterative and fractional differential equations have received the attention of many authors, see [1, 2, 3, 4, 5, 6, 7, 9, 8, 10, 11, 13] and the references therein.

Zhao and Liu [13] established the existence of periodic solutions for the following iterative differential equation

$$x'(t) = \sum_{k=1}^{N} c_k(t)\, x^{[k]}(t),$$

where c_k, $k = 1, ..., N$ are periodic continuous functions.

In [7], Ibrahim studied the existence and uniqueness of solutions for the following iterative fractional differential equation

$$\begin{cases} {}^{C}D_{0^+}^{\alpha} x(t) = f(t, x(x(t))),\ t \in [0, T], \\ x(0) = x_0, \end{cases}$$

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where $x_0 \in [0,T]$, ${}^{C}D^{\alpha}_{0+}$ is the Caputo fractional derivative of order $\alpha \in (0,1)$ and $f : [0,T] \times [0,T] \to [0,T]$ is a continuous function.

Prasad, Khuddush and Leela [11] investigated the existence, uniqueness and Ulam stability of solutions for the following iterative fractional differential equation

$$\begin{cases} {}^{C}D^{\alpha}_{0+} x(t) = f(t, x(t), x(x(t))), \ t \in [0,1], \\ x(0) = A, \ x(1) = B, \end{cases}$$

where $0 \leq A \leq B \leq 1$, $\alpha \in (1,2)$ and $f : [0,T] \times [0,T] \times [0,T] \to \mathbb{R}$ is a continuous function.

Iterative fractional differential equations involving Caputo and Riemann-Liouville fractional derivatives have been investigated by several researchers. However, the literature on iterative Hadamard fractional differential equations is not yet as rich.

In this paper, inspired and motivated by the works mentioned above and the references therein, we concentrate on the existence, uniqueness and continuous dependence of mild positive solutions for the following nonlinear iterative fractional differential equation

$$\text{(1.1)} \qquad \begin{cases} D^{\alpha}_{1+} x(t) = f\left(x^{[0]}(t), x^{[1]}(t), x^{[2]}(t), ..., x^{[N]}(t)\right), \ t \in [1,T], \\ x(1) = 1, \ x'(1) = 0, \end{cases}$$

where $x^{[0]}(t) = t$, $x^{[1]}(t) = x(t)$, $x^{[2]}(t) = x(x(t))$, ..., $x^{[N]}(t) = x^{[N-1]}(x(t))$ are the iterates of the state $x(t)$, D^{α}_{1+} is the Caputo-Hadamard fractional derivative of order $\alpha \in (1,2)$ and f is a positive continuous function. To prove the existence of mild positive solutions, we transform (1.1) into an integral equation and then use the Schauder fixed point theorem. After then, we show the uniqueness and continuous dependence of mild positive solutions. Finally, we provide two example to illustrate our obtained results.

2. Preliminaries

Let $C([1,T], \mathbb{R})$ be the Banach space of all real-valued continuous functions defined on the compact interval $[1,T]$, endowed with the norm

$$\|x\| = \sup_{t \in [1,T]} |x(t)|.$$

For $1 < L \leq T$ and $M > 0$, we define the set

$$C(L,M) = \{x \in C([1,T], \mathbb{R}), \ 1 \leq x \leq L, \\ |x(t_2) - x(t_1)| \leq M |t_2 - t_1|, \ \forall t_1, t_2 \in [1,T]\}.$$

which is a bounded closed convex subset of $C([1,T], \mathbb{R})$.

We assume that the positive function f is globally Lipschitz in x_k, that is, there exist positive constants c_1, c_2, ..., c_N such that

$$\text{(2.1)} \qquad |f(t, x_1, x_2, ..., x_N) - f(t, y_1, y_2, ..., y_N)| \leq \sum_{k=1}^{N} c_k |x_k - y_k|.$$

Also, we introduce the following positive constants

$$\lambda = \sup_{t\in[1,T]} \{f(t,0,0,...,0)\},$$
$$\zeta = \lambda + L\sum_{k=1}^{N} c_k \sum_{j=0}^{k-1} M^j.$$

Definition 2.1 ([9]). The Hadamard fractional integral of order $\alpha > 0$ of a function $x : [1,\infty) \longrightarrow \mathbb{R}$ is given by

$$I_{1^+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} x(s) \frac{ds}{s},$$

where Γ is the gamma function.

Definition 2.2 ([9]). The Caputo-Hadamard fractional derivative of order $\alpha > 0$ of a function $x : [1,\infty) \longrightarrow \mathbb{R}$ is given by

$$D_{1^+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \delta^n(x)(s) \frac{ds}{s},$$

where $\delta^n = \left(t\frac{d}{dt}\right)^n$ and $n = [\alpha] + 1$.

Lemma 2.3 ([9]). *Suppose that* $x \in C^{n-1}([1,+\infty),\mathbb{R})$ *and* $x^{(n)}$ *exists almost everywhere on any bounded interval of* $[1,\infty)$. *Then*

$$\left(I_{1^+}^{\alpha} D_{1^+}^{\alpha} x\right)(t) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(1)}{k!} (\log t)^k.$$

In particular, when $\alpha \in (1,2)$, $\left(I_{1^+}^{\alpha} D_{1^+}^{\alpha} x\right)(t) = x(t) - x(1) - x'(1)\log t$.

Definition 2.4. A function $x \in C([1,T],\mathbb{R})$ is said to be a mild solution of the problem (1.1) if x satisfies the corresponding integral equation of (1.1).

From Lemma 2.3, we deduce the following lemma.

Lemma 2.5. *Let* $x \in C([1,T],\mathbb{R})$ *is a mild solution of (1.1) if* x *satisfies* (2.2)

$$x(t) = 1 + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f\left(x^{[0]}(s), x^{[1]}(s), x^{[2]}(s), ..., x^{[N]}(s)\right) \frac{ds}{s},$$

for $t \in [1,T]$.

Lemma 2.6 ([13]). *For* $\varphi, \psi \in C(L,M)$, *we get*

$$\left\|\varphi^{[m]} - \psi^{[m]}\right\| \le \sum_{j=0}^{m-1} M^j \|\varphi - \psi\|, \; m = 1,2,...$$

Theorem 2.7 (Schauder fixed point theorem [12]). *Let* $\mathbb{M}$ *be a nonempty convex compact subset of a Banach space* $(\mathbb{B}, \|.\|)$. *If* $\mathcal{A} : \mathbb{M} \to \mathbb{M}$ *is a continuous operator, then* $\mathcal{A}$ *has a fixed point.*

3. Main results

In this section, we use Theorem 2.7 to prove the existence of mild positive solutions for (1.1). Moreover, we will introduce the sufficient conditions of the uniqueness and continuous dependence of mild positive solutions of (1.1).

To transform (2.2) to apply the Schauder fixed point theorem, we define the mapping $\mathcal{A} : C(L,M) \to C([1,T],\mathbb{R})$ by

(3.1)
$$(\mathcal{A}\varphi)(t) = 1+\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),...,\varphi^{[N]}(s)\right)\frac{ds}{s},$$

for $t \in [1,T]$. Since $C(L,M)$ is a uniformly bounded, equicontinuous and closed subset of the space $C([1,T],\mathbb{R})$, then $C(L,M)$ is a compact subset. So, to show that the mapping $\mathcal{A}$ has a fixed point, we will prove that the mapping $\mathcal{A}$ is well defined, $\mathcal{A}(C(L,M)) \subset C(L,M)$ and $\mathcal{A}$ is a continuous mapping.

Lemma 3.1. *Assume that (2.1) holds. If*

$$1+\frac{\zeta(\log T)^{\alpha}}{\Gamma(\alpha+1)} \leq L, \tag{3.2}$$

and

$$\frac{\zeta(\log T)^{\alpha-1}}{\Gamma(\alpha)} \leq M, \tag{3.3}$$

then, $\mathcal{A} : C(L,M) \to C([1,T],\mathbb{R})$ *is well defined and* $\mathcal{A}(C(L,M)) \subset C(L,M)$.

Proof. Let $\mathcal{A}$ be defined by (3.1). Clearly, $\mathcal{A}$ is well defined. Next, for $\varphi \in C(L,M)$, we get

$$\begin{aligned}&|(\mathcal{A}\varphi)(t)|\\&\leq 1+\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\left|f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),...,\varphi^{[N]}(s)\right)\right|\frac{ds}{s}.\end{aligned}$$

But

$$\begin{aligned}&\left|f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),...,\varphi^{[N]}(s)\right)\right|\\&=\left|f\left(s,\varphi^{[1]}(s),\varphi^{[2]}(s),...,\varphi^{[N]}(s)\right)-f(s,0,0,...,0)+f(s,0,0,...,0)\right|\\&\leq\left|f\left(s,\varphi^{[1]}(s),\varphi^{[2]}(s),...,\varphi^{[N]}(s)\right)-f(s,0,0,...,0)\right|+|f(s,0,0,...,0)|\\&\leq\lambda+\sum_{k=1}^{N}c_k\sum_{j=0}^{k-1}M^j\|\varphi\|\\&\leq\lambda+L\sum_{k=1}^{N}c_k\sum_{j=0}^{k-1}M^j=\zeta,\end{aligned}$$

then

$$|(\mathcal{A}\varphi)(t)| \leq 1+\frac{\zeta}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s} \leq 1+\frac{\zeta(\log T)^{\alpha}}{\Gamma(\alpha+1)} \leq L.$$

From (3.2), we get

$$1 \leq (\mathcal{A}\varphi)(t) \leq |(\mathcal{A}\varphi)(t)| \leq L.$$

Let $t_1, t_2 \in [1,T]$ with $t_1 < t_2$, we obtain

$$\begin{aligned}
&|(\mathcal{A}\varphi)(t_1)-(\mathcal{A}\varphi)(t_2)| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left|\left(\log\frac{t_1}{s}\right)^{\alpha-1}-\left(\log\frac{t_2}{s}\right)^{\alpha-1}\right| \\
&\times\left|f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),...,\varphi^{[N]}(s)\right)\right|\frac{ds}{s} \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\left|f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),...,\varphi^{[N]}(s)\right)\right|\frac{ds}{s} \\
&\leq \frac{\zeta}{\Gamma(\alpha)}\left(\int_1^{t_1}\left(\left(\log\frac{t_2}{s}\right)^{\alpha-1}-\left(\log\frac{t_1}{s}\right)^{\alpha-1}\right)\frac{ds}{s}+\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\frac{ds}{s}\right) \\
&\leq \frac{\zeta}{\Gamma(\alpha+1)}\left((\log t_2)^{\alpha}-(\log t_1)^{\alpha}\right) \\
&\leq \frac{\zeta(\log T)^{\alpha-1}}{\Gamma(\alpha)}|t_2-t_1|.
\end{aligned}$$

By using (3.3), we have

$$|(\mathcal{A}\varphi)(t_1)-(\mathcal{A}\varphi)(t_2)| \leq M|t_2-t_1|.$$

Therefore, $\mathcal{A}\varphi \in C(L,M)$ for all $\varphi \in C(L,M)$. So, we conclude that

$$\mathcal{A}(C(L,M)) \subset C(L,M).$$

□

Lemma 3.2. *Assume that (2.1) holds. Then the mapping $\mathcal{A} : C(L,M) \to C([1,T],\mathbb{R})$ given by (3.1) is continuous.*

Proof. Let $\varphi, \psi \in C(L,M)$, we get

$$\begin{aligned}
&|(\mathcal{A}\varphi)(t)-(\mathcal{A}\psi)(t)| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left|f\left(\varphi^{[0]}(s),\varphi^{[1]}(s),\varphi^{[2]}(s),...,\varphi^{[N]}(s)\right)\right. \\
&\left.-f\left(\psi^{[0]}(s),\psi^{[1]}(s),\psi^{[2]}(s),...,\psi^{[N]}(s)\right)\right|\frac{ds}{s}.
\end{aligned}$$

By using (2.1), we have

$$|(\mathcal{A}\varphi)(t)-(\mathcal{A}\psi)(t)| \leq \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\sum_{k=1}^{N} c_k \left\|\varphi^{[k]}-\psi^{[k]}\right\| \frac{ds}{s}.$$

So, from Lemma 2.6, we obtain

$$\begin{aligned}
&|(\mathcal{A}\varphi)(t)-(\mathcal{A}\psi)(t)| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\sum_{k=1}^{N} c_k \sum_{j=0}^{k-1} M^j \|\varphi-\psi\| \frac{ds}{s} \\
&\leq \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\sum_{k=1}^{N} c_k \sum_{j=0}^{k-1} M^j \|\varphi-\psi\|,
\end{aligned}$$

then

$$\|\mathcal{A}\varphi-\mathcal{A}\psi\| \leq \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\sum_{k=1}^{N} c_k \sum_{j=0}^{k-1} M^j \|\varphi-\psi\|.$$

Hence, the mapping $\mathcal{A}$ is continuous. □

Theorem 3.3. *Assume that conditions (2.1), (3.2) and (3.3) hold. Then (1.1) has a mild positive solution* $x \in C(L, M)$*.*

Proof. By Lemmas 3.1 and 3.2, all the hypotheses of the Schauder fixed point theorem are satisfied. Thus, there exists a fixed point $x \in C(L, M)$ such that $x = \mathcal{A}x$. Hence, by using Lemma 2.5, the problem (1.1) has a mild positive solution $x \in C(L, M)$. □

Theorem 3.4. *Assume that the hypotheses of Theorem 3.3 are satisfied. If*

$$\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\sum_{k=1}^{N} c_k \sum_{j=0}^{k-1} M^j < 1, \tag{3.4}$$

then (1.1) has a unique mild positive solution $x \in C(L, M)$*.*

Proof. From Theorem 3.3, it follows that (1.1) has at least one mild positive solution in $C(L, M)$. Hence, we need only to prove that the mapping $\mathcal{A}$ defined in (3.1) is a contraction on $C(L, M)$. For any $\varphi, \psi \in C(L, M)$, we have

$$\begin{aligned}
&|(\mathcal{A}\varphi)(t)-(\mathcal{A}\psi)(t)| \\
&\leq \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\sum_{k=1}^{N} c_k \sum_{j=0}^{k-1} M^j \|\varphi-\psi\|.
\end{aligned}$$

Thus

$$\|\mathcal{A}\varphi-\mathcal{A}\psi\| \leq \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\sum_{k=1}^{N} c_k \sum_{j=0}^{k-1} M^j \|\varphi-\psi\|.$$

Hence, $\mathcal{A}$ is a contraction mapping by (3.4). Therefore, by the contraction mapping principle, we conclude that the problem (1.1) has a unique mild positive solution $x \in C(L, M)$. $\square$

Theorem 3.5. *Suppose that the hypotheses of Theorem 3.4 are satisfied. Then, the unique mild positive solution of (1.1) depends continuously on the function f.*

Proof. Let $f_1, f_2 : [1,T] \times \mathbb{R}^N \to [0,+\infty)$ two continuous functions. By Theorem 3.4, there exist two unique corresponding functions $x_1, x_2 \in C(L, M)$ such that

$$x_1(t) = 1 + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f_1\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), ..., x_1^{[N]}(s)\right) \frac{ds}{s},$$

and

$$x_2(t) = 1 + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f_2\left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s), ..., x_2^{[N]}(s)\right) \frac{ds}{s}.$$

So, we obtain

$$\begin{aligned}
&|x_2(t) - x_1(t)| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left| f_2\left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s), ..., x_2^{[N]}(s)\right) \right. \\
&\left. - f_1\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), ..., x_1^{[N]}(s)\right) \right| \frac{ds}{s}.
\end{aligned}$$

But

$$\begin{aligned}
&\left| f_2\left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s), ..., x_2^{[N]}(s)\right) \right. \\
&\left. - f_1\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), ..., x_1^{[N]}(s)\right) \right| \\
&= \left| f_2\left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s), ..., x_2^{[N]}(s)\right) \right. \\
&- f_2\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), ..., x_1^{[N]}(s)\right) \\
&+ f_2\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), ..., x_1^{[N]}(s)\right) \\
&\left. - f_1\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), ..., x_1^{[N]}(s)\right) \right|.
\end{aligned}$$

By using (2.1) and Lemma 2.6, we get

$$\begin{aligned}
&\left| f_2\left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s), ..., x_2^{[N]}(s)\right) \right. \\
&\left. - f_1\left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), ..., x_1^{[N]}(s)\right) \right| \\
&\leq \|f_2 - f_1\| + \sum_{k=1}^{N} c_k \sum_{j=0}^{k-1} M^j \|x_2 - x_1\|.
\end{aligned}$$

Then

$$\|x_2 - x_1\| \leq \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} \|f_2 - f_1\| + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} \sum_{k=1}^{N} c_k \sum_{j=0}^{k-1} M^j \|x_2 - x_1\|.$$

Hence

$$\|x_2 - x_1\| \leq \frac{\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}}{1 - \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} \sum\limits_{k=1}^{N} c_k \sum\limits_{j=0}^{k-1} M^j} \|f_2 - f_1\|,$$

which completes the proof. □

Example 3.6. Let us consider the following nonlinear iterative fractional initial value problem

(3.5)
$$\begin{cases} D_{1^+}^{\frac{5}{3}} x(t) = \frac{1}{5} + \frac{1}{5}\sin t + \frac{1}{8}\sin^2(t)\, x^{[1]}(t) + \frac{1}{12}\cos^2(t)\, x^{[2]}(t), \ t \in [1,e], \\ x(1) = 1, \ x'(1) = 0. \end{cases}$$

Set $T = e$, $\alpha = \frac{5}{3}$ and

$$f(t,x,y) = \frac{1}{5} + \frac{1}{5}\sin t + \frac{1}{8} x \sin^2(t) + \frac{1}{12} y \cos^2(t).$$

So,

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \leq \frac{1}{8}|x_1 - y_1| + \frac{1}{12}|x_2 - y_2|,$$

hence $c_1 = \frac{1}{8}$ and $c_2 = \frac{1}{12}$. If $L = e$ and $M = 4$ in the definition of $C(L,M)$, then f is positive, $\lambda = \sup\limits_{t\in[1,e]} \{f(t,0,0)\} = \frac{2}{5}$ and $\zeta = \frac{2}{5} + e\left(\frac{1}{8} + \frac{4}{12}\right) \simeq 1.646$.

Thus, we get

$$\frac{\zeta(\log T)^\alpha}{\Gamma(\alpha+1)} = \frac{1.646}{\Gamma\left(\frac{8}{3}\right)} \simeq 1.094 \leq L = e,$$

and

$$\frac{\zeta(\log T)^{\alpha-1}}{\Gamma(\alpha)} = \frac{1.646}{\Gamma\left(\frac{5}{3}\right)} \simeq 1.823 \leq M = 4.$$

Also,

$$\frac{(\log T)^\alpha}{\Gamma(\alpha+1)} \sum_{k=1}^{N} c_k \sum_{j=0}^{k-1} M^j = \frac{1}{\Gamma\left(\frac{8}{3}\right)} \left(\frac{1}{8} + \frac{4}{12}\right) \simeq 0.305 < 1.$$

Therefore, the problem (3.5) has a unique mild positive solution which depends continuously on the function f.

Example 3.7. Let us consider the following nonlinear iterative fractional initial value problem

$$(3.6) \qquad \begin{cases} D_{1^+}^{\frac{7}{4}} x(t) = \frac{1}{1+t^2} + \frac{\left|x^{[1]}(t)\right| + \left|x^{[2]}(t)\right|}{7t\left(1 + \left|x^{[1]}(t)\right| + \left|x^{[2]}(t)\right|\right)}, \ t \in [1,e], \\ x(1) = 1, \ x'(1) = 0. \end{cases}$$

Here $T = e$, $\alpha = \frac{7}{4}$ and

$$f(t,x,y) = \frac{1}{1+t^2} + \frac{|x|+|y|}{7t(1+|x|+|y|)}.$$

Clearly, the function f is positive continuous. Also, we have

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \leq \frac{1}{7}|x_1 - y_1| + \frac{1}{7}|x_2 - y_2|.$$

Then $c_1 = c_2 = \frac{1}{7}$. For $L = e$ and $M = 3$ in the definition of $C(L,M)$, we get $\lambda = \sup\limits_{t\in[1,e]} \{f(t,0,0)\} = \frac{1}{2}$ and $\zeta = \frac{1}{2} + e\left(\frac{1}{7} + \frac{3}{7}\right) \simeq 2.053$. So, we obtain

$$\frac{\zeta(\log T)^{\alpha}}{\Gamma(\alpha+1)} = \frac{2.053}{\Gamma\left(\frac{11}{4}\right)} \simeq 1.277 \leq L = e,$$

and

$$\frac{\zeta(\log T)^{\alpha-1}}{\Gamma(\alpha)} = \frac{2.053}{\Gamma\left(\frac{7}{4}\right)} \simeq 2.234 \leq M = 3.$$

Also,

$$\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} \sum_{k=1}^{N} c_k \sum_{j=0}^{k-1} M^j = \frac{1}{\Gamma\left(\frac{11}{4}\right)} \left(\frac{1}{7} + \frac{3}{7}\right) \simeq 0.355 < 1.$$

Hence, the problem (3.6) has a unique mild positive solution which depends continuously on the function f.

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