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On the sensitivity analysis of spread options using Malliavin calculus

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Abstract. In this paper we derive tractable formulae for price sensitivities of two-dimensional spread options using Malliavin calculus. In particular, we consider spread options with asset dynamics driven by geometric Brownian motion and stochastic volatility models. Unlike the fast Fourier transform approach, the Malliavin calculus approach does not require the joint characteristic function of underlying assets to be known and is applicable to spread options with discontinuous payoff functions. The results obtained reveal that the Malliavin calculus approach gives the price sensitivities in terms of the expectation of spread option payoff functional multiplied with some random variables (Malliavin weights) which are independent of the payoff functional. The results show the flexibility of Mallavin calculus approach when applied to spread options.

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1. Introduction

A spread option is an option whose payoff is based on the difference (i.e. the spread) between two or more underlying assets. Spread options are common in several markets such as the fixed-income, currency, commodity, and energy markets [1, 21]. For instance, in the fixed income markets, a popular product in the United States of America is the Note Against Bond (NOB) spread in which a yield curve is created between the 30-year bond futures contract (long position) and the 10-year US Treasury note futures (short position). In the commodity markets, spread options are based on the difference between the prices of the same commodity at two different locations or at two different points in time, as well as between the prices of different grades of the same commodity [4]. In the energy markets, crack spread options and spark spread options are prevalent. The crack spread is based on the differential between the price of crude oil and refined petroleum products. The spread represents the refinement margin made by the oil refinery by “cracking” the crude oil into a refined petroleum product [9]. The spark spread refers to differences between the price of electricity and

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the price of fuel, and are are widely used by power plant operators to optimize their revenue streams [11]. Spread options are sometimes traded on exchanges, but most often as over-the-counter transactions.

Spread options are popular because they are designed to mitigate adverse movements between two market variables [20, 21]. For instance, the crack spread are used by refineries to hedge against price fluctuations, mitigate risk, or secure a profit margin on the production output [21]. The spark spread represents the margin of the power plant, which takes fuel to run its generator to produce electricity. However, spread options are complex contracts as they take two underlying assets as the reference point in their payoff and they are harder to value compared to plain vanilla call and puts due to the 2-dimensionality [6, 21]. Nonetheless, these options represent interesting alternatives for those seeking coverage of positions in several assets.

The sensitivity analysis is carried over parameters appearing in the model for the price dynamics. Price sensitivities are derivatives (in the classical sense) of the price of spread options with respect to parameters of the model. For example, *Delta*, denoted by Δ_i, $i = 1, 2$, is the derivative of the price of spread option with respect to the initial price of the underlying asset. *Gamma*, denoted by Γ_i, $i = 1, 2$, is the second derivative of the price of spread option with respect to the initial price of the underlying assets. *Vega*, denoted by $\mathcal{V}_i$, $i = 1, 2$, is the derivative of the price of spread option with respect to the volatility of the underlying assets [16]. For spread options, two values are obtained for each which are associated with the price of the underlying assets under consideration.

The derivation of price sensitivities for spread options is a challenging task due to lack of closed formulae for their prices. Analytical methods applicable to log-normal models that involve linear approximations of the nonlinear exercise boundary has been used. Kirk [13] presented an analytical approximation which is a generalization of Margrabe's formula [15] for an exchange option, that is, a spread option with zero-strike price. Kirk's formula performs well in practice. Carmona and Durrleman [4] and later Li et al. [14] derive a number of lower and upper bounds for the spread option price that combine to produce accurate analytical approximation formulas in log-normal asset models. These results were used to approximate values of price sensitivities via direct differentiation. Alfeus and Schlögl [1] calculated the change in the spread option value with respect to change in volatility parameters of each asset assuming that the model (joint) characteristic function is known, on price obtained using fast Fourier transform approach. Hurd and Zhou [11] and Dempster and Hong [7] also explored the use of fast Fourier transform in three different models for spread options on two stocks, namely the geometric Brownian motion, stochastic volatility model and the variance gamma model. A formula for calculating the *vega* (sensitivity to volatility) was presented using direct differentiation.

Over the past two decades, a fairly large and growing literature have been developed around the computation of price sensitivities using Malliavin calculus. Fournié et al. [10] introduce the application of Malliavin calculus to the computation of price sensitivities on markets driven by the Brownian motion only. Their work was further extended by several authors. El-Khatib and Privault [8] use Malliavin calculus on Poisson space to derive price sensitivities in a market driven by the Poisson processes. Davis and Johansson [5] utilise Malliavin calculus to calculate price sensitivities in a jump diffusion setting assuming a separability condition. Petrou [19] derived price sensitivities using Malliavin calculus for markets driven by square integrable Lévy processes. Both Malliavin calculus and Fourier transform were used by Benth et al. [3] to compute price sensitivities within a jump-diffusion framework. Mhlanga [17] makes use of Malliavin calculus to compute price sensitivities where small jumps for Lévy processes were approximated by a Brownian motion. Yilmaz [22] uses Malliavin calculus approach to compute price sensitivities for underlying asset and interest rate involving stochastic volatility and stochastic interest rate models, respectively. Kawai and Kohatsu-Higa [12] use Malliavin calculus to obtain expressions for price sensitivities for an asset price dynamics model defined with time-changed Brownian motion. In all these references the Malliavin calculus approach was not applied to spread options. The Malliavin calculus has a derivative operator and its adjoint coincides with the Itô stochastic integral on adapted processes, which provides a natural way to make explicit computations of weight functions.

The purpose of this paper is to derive formulae for price sensitivities of two-dimensional spread options using Malliavin calculus. The paper focuses on asset dynamics that are driven by geometric Brownian motion and stochastic volatility models. The general Lévy models are left for further studies. The fast Fourier transform, which has been applied in most papers dealing with spread options, requires that the joint characteristic function of the underlying assets be known in advance. In practice, the joint characteristic function may not be available, for example, in spread options of Asian-type, necessitating the need for alternative approaches. The Malliavin calculus adopted in the present paper does not require the joint characteristic function to be known and is applicable to spread options with discontinuous payoff functions. This demonstrates the flexibility of Malliavin calculus approach.

The contribution of this paper is to provide tractable formulae for price sensitivities of spread options in the context of Malliavin calculus. In particular, we provide formulae for the *Delta*, *Gamma* and *Vega* with respect to each of the underlying asset prices. In passing, we generalize the calculation of price sensitivities of spread options proposed in Carmona and Durrleman [4], Alfeus and Schlögl [1], Hurd and Zhou [11], and Li et al. [14]. We also discuss the localised Malliavin calculus which improves the Malliavin calculus. The localised Malliavin calculus approach uses Malliavin calculus approach only around the point of discontinuity, and direct methods outside. Our results reveal that the Malliavin calculus approach gives the price sensitivities in terms of the expectation of spread option payoff functional multiplied with some Malliavin weights which are independent of the payoff functional. This is consistent with results in Fournié et al. [10]. Our results gain importance in view of the application of Malliavin calculus for the computation of price sensitivities of spread options in mathematical finance.

The paper is structured as follows. In Section 2, we describe the spread option pricing problem. Section 3 is devoted to a discussion on Malliavin calculus. We provide an overview of necessary tools needed in our proofs. Section 4 specifies the market setting considered in this paper. In Section 5, we present theoretical formulae for computing the price sensitivities. Examples are provided in Section 6. The spread option with stochastic volatility is presented in Section 7. Section 8 deals with the localization of Malliavin calculus when computing price sensitives of spread options. Section 9 concludes this paper.

2. Mathematical setup

For a fixed T, we consider a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{0\leq t\leq T}, P)$ defined in the usual sense. Consider a spread option of European call type between two stock price processes $S_1 = (S_1(t))_{0\leq t\leq T}$ and $S_2 = (S_2(t))_{0\leq t\leq T}$ with maturity time T and exercise price K. Its payoff at time T is given by

$$(S_2(T) - S_1(T) - K)^+$$

where $(\cdot)^+ = \max\{\cdot, 0\}$. At maturity, if the spread $S_2(T) - S_1(T)$ is greater than the exercise price K, the option holder exercises the option and gains the difference between the spread and the strike price. If the spread is less than 0, the option holder does not exercise the option, and the payoff is 0. The price of the spread option u at time $t = 0$ is expressed by the risk-neutral expectation

$$u = \mathbb{E}[e^{-rT}\,(S_2(T) - S_1(T) - K)^+] \tag{2.1}$$

where r is the risk-free interest rate, which is here assumed to be constant. Define $\Phi(S_1(T), S_2(T)) := (S_2(T) - S_1(T) - K)^+$. Then (2.1) can be expressed as follows

$$u = \mathbb{E}[e^{-rT}\Phi(S_1(T), S_2(T))]. \tag{2.2}$$

Equation (2.2) shows the price of the spread option with payoff function being a function of terminal values of two assets $S_1(T)$ and $S_2(T)$.

3. A Primer on Malliavin calculus

In this section we recall some of the basic properties of Malliavin calculus as highlighted in Fournié et al. [10] and Mhlanga [16]. We refer to Nualart [18] for a detailed exposition on Malliavin calculus.
For $h(\cdot) \in H = L^2([0,T], \mathbb{R}^d)$, denote by $W(h)$ the Wiener stochastic integral $\int_0^T h(t)dW_t$. Let $\mathcal{S}$ denote the class of random variables of the form

$$F = f(W(h_1), \dots, W(h_n))$$

where $f \in C_p^\infty(\mathbb{R}^n)$, $(h_1, ..., h_n) \in H^n$ and $n \geq 1$. For $F \in \mathcal{S}$, we define the Malliavin derivative $DF = (D_tF)_{t\in[0,T]}$ of F as the H-valued random variable given by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), ..., W(h_n))h_i(t). \tag{3.1}$$

For any $p \geq 1$, the Malliavin derivative operator D is closable from $L^p(\Omega)$ to $L^p(\Omega; H)$. Its domain is denoted by $\mathbb{D}^{1,p}$ with respect to the norm

$$\| F \|_{1,p} = (\mathbb{E}[| F |^p] + \mathbb{E}[\| DF \|_H^p])^{\frac{1}{p}} .$$

We also introduce δ, the Skorohod integral, defined as the adjoint operator of D which is a linear operator on $L^2([0,T] \times \Omega, \mathbb{R}^d)$ with values in $L^2(\Omega)$ and we denote by $\mathrm{Dom}(\delta)$ its domain.

We now state the basic properties of the Malliavin derivative and the Skorohod integral. The next proposition is the chain rule for the Malliavin derive operator [18, Proposition 1.2.3, p. 28].

Proposition 3.1 (Chain rule property). *Fix $p \geq 1$. For $\varphi \in C_b^1(\mathbb{R}^n, \mathbb{R})$ and $F = (F_1, ..., F_n)$ a random vector whose components belong to $\mathbb{D}^{1,p}$, $\varphi(F) \in \mathbb{D}^{1,p}$ and for $t \in [0,T]$, one has*

$$D_t \varphi(F) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(F) D_t F_i. \tag{3.2}$$

The next proposition shows that, in the case of Markov diffusion process, the Malliavin derivative operator is closely related to the derivative of the process with respect to the initial condition [10, Property P2].

Proposition 3.2. *Let $\{X_t,\ t \geq 0\}$ be an $\mathbb{R}^n$ valued Itô process whose dynamics are governed by the stochastic differential equation*

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \tag{3.3}$$

where b and σ are supposed to be continuously differentiable functionals with bounded derivatives and $\sigma(x)$ is a nonsingular matrix for all $x \in \mathbb{R}^n$. Let $\{Y_t,\ t \geq 0\}$ be the associated first variation process given by the stochastic differential equation

$$dY_t = b'(X_t) Y_t dt + \sum_{i=1}^{n} \sigma_i'(X_t) Y_t dW_t^i, \qquad Y_0 = I_n, \tag{3.4}$$

where I_n is the identity matrix of $\mathbb{R}^n$, primes denote derivatives and σ_i is the i-th column vector of σ. The process $\{X_t,\ t \geq 0\}$ belongs to $\mathbb{D}^{1,2}$ and its Malliavin derivative is given by

$$D_r X_t = Y_t Y_r^{-1} \sigma(X_r) 1_{\{r \leq t\}}, \quad r \geq 0 \ \ a.s., \tag{3.5}$$

which is equivalent to

$$Y_t = D_r X_t \sigma^{-1}(X_r) Y_r 1_{\{r \leq t\}} \qquad a.s. \tag{3.6}$$

The adjoint operator δ associated with the Malliavin derivative operator D exists since the Malliavin derivative operator D is densely defined. The next proposition gives the relationship between the Malliavin derivative operator and the adjoint operator [18, Definition 1.3.1, p. 36]

Proposition 3.3 (Integration by parts formula)**.** *If u belongs to Dom(δ), then $\delta(u) = \int_0^T u_t \delta W_t$ is the element of $L^2(\Omega)$ characterised by the integration by parts formula*

$$\forall F \in \mathbb{D}^{1,2} \quad \mathbb{E}[F\delta(u)] = \mathbb{E}\left[\int_0^T D_t F \cdot u(t)dt\right]. \tag{3.7}$$

An important property of the adjoint operator δ is that its domain Dom(δ) contains all adapted stochastic processes which belong to $L^2([0,T]\times\Omega,\mathbb{R}^d)$. For such processes, the adjoint operator δ coincides with the stochastic integral [18, Proposition 1.3.11, p. 44]

Proposition 3.4. *If u is an adapted process belonging to $L^2([0,T]\times\Omega,\mathbb{R}^d)$ then the Skorohod integral and the Itô integral coincide, that is,*

$$\delta(u) = \int_0^T u(t)dW(t). \tag{3.8}$$

Moreover, if the random variable F is $\mathcal{F}_T$-adapted and belongs to $\mathbb{D}^{1,2}$ then for any $u \in$ Dom(δ), the random variable Fu is Skorohod integrable. This yields the following proposition [18, Proposition 1.3.3, p. 39]

Proposition 3.5. *If $F \in \mathbb{D}^{1,2}$ and $u \in$ Dom(δ) such that $\mathbb{E}[F^2 \int_0^T u_t^2 dt] < \infty$, one has*

$$\delta(Fu) = F\delta(u) - \int_0^T D_t F \cdot u_t dt \tag{3.9}$$

whenever the right hand side belongs to $L^2(\Omega)$. In particular, if u is in addition adapted, one simply has

$$\delta(Fu) = F\int_0^T u_t dW_t - \int_0^T D_t F_t \cdot u_t dt. \tag{3.10}$$

4. Asset Dynamics

To compute price sensitivities of (2.2) with respect to model parameters of the two underlying assets we must specify the risk-neutral dynamics of the two underlying assets $S_1(t)$ and $S_2(t)$. We consider the underlying assets $S_1(t)$ and $S_2(t)$ under the risk-neutral measure to be given by the two-dimensional system of Itô stochastic differential equations of the form

$$\begin{aligned} dS_1(t) &= S_1(t)[\mu_1(S_1(t),S_2(t))dt + \sigma_1(S_1(t),S_2(t))dW_1(t)] \quad S_1(0) = x_1, \\ dS_2(t) &= S_2(t)[\mu_2(S_1(t),S_2(t))dt + \sigma_2(S_1(t),S_2(t))dW_2(t)] \quad S_2(0) = x_2, \end{aligned} \tag{4.1}$$

where $W_1(t)$ and $W_2(t)$ are correlated standard Brownian motions with correlation coefficient $\rho \in (-1,1)$. The coefficients μ_1, μ_2, σ_1 and σ_2 are assumed to

satisfy the usual conditions to ensure the existence and uniqueness of a strong solution of (4.1).
Given an arbitrary W_1, there exists $\widetilde{W}_2$ which is independent of W_1 and W_2. Then we can express W_2 as follows $W_2(t) = \rho W_1(t) + \sqrt{1-\rho^2}\widetilde{W}_2(t)$. We also express W_1 as $W_1(t) = \widetilde{W}_1(t)$. We can rewrite (4.1) as

$$\begin{aligned} dS_1(t) &= S_1(t)[\mu_1(S_1(t), S_2(t))dt + \sigma_1(S_1(t), S_2(t))d\widetilde{W}_1(t)] \\ dS_2(t) &= S_2(t)[\mu_2(S_1(t), S_2(t))dt + \rho\sigma_2(S_1(t), S_2(t))d\widetilde{W}_1(t) \\ &\qquad + \sigma_2(S_1(t), S_2(t))\sqrt{1-\rho^2}d\widetilde{W}_2(t)]. \end{aligned} \tag{4.2}$$

Setting $S(t) = (S_1(t), S_2(t))^*$ and $\mathbb{W} = (W_1(t), W_2(t))^*$ ($(\cdot)^*$ denote the transpose of $(\cdot)$) and a two-dimensional notation we can write (4.2) as

$$dS(t) = \beta(S(t))dt + a(S(t))d\mathbb{W}(t) \tag{4.3}$$

where

$$\beta(S(t)) = \begin{pmatrix} \mu_1(S(t))S_1(t) \\ \mu_2(S(t))S_2(t) \end{pmatrix}$$

and

$$a(S(t)) = \begin{pmatrix} \sigma_1(S(t))S_1(t) & 0 \\ \rho\sigma_2(S(t))S_2(t) & \sqrt{1-\rho^2}\sigma_2(S(t))S_2(t) \end{pmatrix}.$$

We assume that β and a are both at least twice continuously differentiable functions with bounded derivatives and that $a(x)$ is a nonsingular matrix. To ensure that (4.3) has a unique strong solution we further assume that β and a satisfy the Lipschitz and polynomial growth conditions.
The first variation process $\{Y(t),\ 0 \le t \le T\}$ associated to $\{S(t),\ 0 \le t \le T\}$ given in (4.3) is defined by the stochastic differential equation

$$dY(t) = \beta'(S(t))Y(t)dt + \sum_{i=1}^{2} a_i'(S(t))Y(t)dW_i(t), \quad Y(0) = I_2, \tag{4.4}$$

where I_2 is the 2×2 identity matrix of $\mathbb{R}^2$, the primes denote derivatives and a_i is the i-th column matrix of a.

5. Computation of price sensitivities

Following Fournié et al. [10], we assume that the diffusion matrix a satisfies the uniform elliptic condition:

$$\exists \epsilon > 0 \quad (a(x)\xi)^*(a(x)\xi) \ge \epsilon \mid \xi \mid^2 \quad \text{for all } \ x, \xi \in \mathbb{R}^n, \ \text{ with } \ \xi \ne 0. \tag{5.1}$$

In the computation of Greeks via Malliavin calculus, a weight function which is independent of the payoff function is obtained. To obtain a valid computation result, one has to guarantee that the Malliavin weights do not degenerate with probability one. To avoid this degeneracy we introduce the set Υ_n (see [10]) defined by

$$\Upsilon_n = \{\alpha \in L^2([0,T]) \mid \int_0^{t_i} \alpha(t)dt = 1 \text{ for all } i = 1, \dots, n\}. \tag{5.2}$$

We need the following lemma [16].

Lemma 5.1. *If* $(Y(t)Y^{-1}(r)\alpha(r)) \in L^2([0,T]\times\Omega)$ *for all* $r, t \in [0,T]$*, then* $S(t)$ *is Malliavin differentiable and the Malliavin derivative of* $X(t)$ *can be written as follows:*

$$D_r S(t) = Y(t)Y^{-1}(r)a(S(r))\mathbf{1}_{r\le t}, \quad r \ge 0, \quad a.s. \tag{5.3}$$

which is equivalent to

$$Y(t) = \int_0^T D_r S(t)\alpha(r)a^{-1}(S(r))Y(r)dr \qquad \forall \alpha \in \Upsilon_n. \tag{5.4}$$

We consider a square integrable payoff function, $\Phi = \Phi(S_1(T), S_2(T))$, that is continuously differentiable with bounded derivatives. Precisely, Φ satisfies

$$\mathbb{E}[\Phi^2(S_1(T), S_2(T)] < \infty.$$

From the arbitrage theory, the price of the spread option can be expressed in terms of the expectation as in (2.2). We have the following results.

Proposition 5.2. *Suppose that the functions* β *and* a *in (4.3) are continuously differentiable with bounded derivatives, and the diffusion matrix* a *satisfies the uniform ellipticity condition (5.1). In addition, the spread payoff function* Φ *is square integrable and continuously differentiable with bounded derivatives. Then for all* $\alpha \in \Upsilon_n$*, we have*

$$\Delta_1 = \mathbb{E}[e^{-rT}\Phi(S_1(T), S_2(T))\pi^{\Delta_1}], \tag{5.5}$$

where the Malliavin weight π^{Δ_1} *is*

$$\pi^{\Delta_1} = \int_0^T \alpha(t)(a^{-1}(S(t))Y(t))^* d\mathbb{W}(t). \tag{5.6}$$

Proof. First assume $\Phi \in C_c^\infty(\mathbb{R}^2, \mathbb{R})$. We have

$$\begin{aligned} \Delta_1 &= \frac{\partial}{\partial x_1}\mathbb{E}[e^{-rT}\Phi(S_1(T), S_2(T))] = \mathbb{E}[e^{-rT}\frac{\partial}{\partial x_1}\Phi(S_1(T), S_2(T))] \\ &= \mathbb{E}[e^{-rT}\Phi'(S_1(T), S_2(T))\frac{\partial S_1(T)}{\partial x_1}] = \mathbb{E}[e^{-rT}\Phi'(S_1(T), S_2(T))Y_1(T)], \end{aligned}$$

where the interchange of the derivative and the expectation is justified by the dominated convergence theorem. In fact, as $\varepsilon \to 0$

$$\frac{\Phi(S_1(T)(1+\frac{\varepsilon}{x_1}), S_2(T)) - \Phi(S_1(T), S_2(T))}{\varepsilon} \to \Phi'(S_1(T), S_2(T))Y_1(T) \quad \text{a.s}$$

and by the Taylor theorem

$$\begin{aligned}&\left|\frac{\mathbb{E}[\Phi(S_1(T)(1+\frac{\varepsilon}{x_1}),S_2(T))-\Phi(S_1(T),S_2(T))]}{\varepsilon}\right|\\&\leq\int_0^1\mathbb{E}\left[\left|\Phi'(S_1(T)(1+\frac{\delta\varepsilon}{x_1}),S_2(T))\left(\frac{S_1(T)(1+\frac{\delta\varepsilon}{x_1})}{x_1}\right)\right|\right]d\delta\end{aligned}$$

which is clearly uniformly bounded in ε. This proves that

$$\Delta_1=\mathbb{E}[e^{-rT}\Phi'(S_1(T),S_2(T))Y_1(T)].$$

From Lemma 5.1 we have

$$\Delta_1=\mathbb{E}\left[\int_0^T e^{-rT}\Phi'(S_1(T),S_2(T))D_rS_1(t)\alpha(t)a^{-1}(S(t))Y_1(t)dt\right].$$

An application of Proposition 3.1 and Proposition 3.3, and the fact that the Skorohod integral coincides with the Itô stochastic integral (Proposition 3.4) yields

$$\begin{aligned}\Delta_1&=\mathbb{E}\left[\int_0^T e^{-rT}D_s\Phi(S_1(T),S_2(T))\alpha(t)a^{-1}(S(t))Y_1(t)dt\right]\\&=\mathbb{E}\left[e^{-rT}\Phi(S_1(T),S_2(T))\int_0^T\alpha(t)\left(a^{-1}(S(t))Y(t)\right)^*dW(t)\right].\end{aligned}$$

This is the desired result for $\Phi\in C_c^\infty(\mathbb{R}^2,\mathbb{R})$.
Now consider the general case, when $\Phi\in L^2(\mathbb{R}^2,\mathbb{R})$. Since the set $C_c^\infty(\mathbb{R}^2,\mathbb{R})$ of infinitely differentiable functions with compact support is dense in $L^2(\mathbb{R}^2,\mathbb{R})$, we can always find a sequence $(\Phi_n)_n\in C_c^\infty(\mathbb{R}^2,\mathbb{R})$ of infinitely differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}$ with compact support such that

$$\lim_{n\to+\infty}\mathbb{E}[|e^{-rT}\Phi_n(S_1(T),S_2(T))-e^{-rT}\Phi(S_1(T),S_2(T))|^2]=0.$$

Hence, by the Cauchy-Schwarz inequality with $\mathbb{E}[|\pi^{\Delta_1}|^2]<\infty$ we have that for each x_1

$$\begin{aligned}&\left|\mathbb{E}\left[e^{-rT}\Phi_n(S_1(T),S_2(T))\pi^{\Delta_1}-e^{-rT}\Phi(S_1(T),S_2(T))\pi^{\Delta_1}\right]\right|\\&\leq\left(\mathbb{E}\left[\left|e^{-rT}\Phi_n(S_1(T),S_2(T))-e^{-rT}\Phi(S_1(T),S_2(T))\right|^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\left|\pi^{\Delta_1}\right|^2\right]\right)^{\frac{1}{2}}\\&\to 0\end{aligned}$$

as $n\to\infty$. Then we obtain that

$$\begin{aligned}&\mathbb{E}\left[e^{-rT}\Phi_n(S_1(T)(1+\frac{\varepsilon}{x_1}),S_2(T))\right]-\mathbb{E}\left[e^{-rT}\Phi_n(S_1(T),S_2(T))\right]\\&=\int_0^\varepsilon\mathbb{E}\left[e^{-rT}\Phi_n(S_1(T)(1+\frac{h}{x_1}),S_2(T))\pi^{\Delta_1}\right]dh\qquad(5.7)\end{aligned}$$

Then, by taking limits as n tends to ∞, we obtain that $\mathbb{E}[e^{-rT}\Phi(S_1(T)(1+\frac{\varepsilon}{x_1}),S_2(T))]$ is continuous in x_1. In a similar fashion, we can prove that $\mathbb{E}[e^{-rT}\Phi(S_1(T),S_2(T))\pi^{\Delta_1}]$ is continuous in x_1. Finally, by taking limits as n tends to ∞ in (5.7) and dividing by ε, we obtain that $\mathbb{E}[e^{-rT}\Phi(S_1(T),S_2(T))]$ is differentiable with respect to x_1 and the desired formula holds. This completes the proof. □

Proposition 5.3. *Suppose that the functions β and a in (4.3) are continuously differentiable with bounded derivatives, and the diffusion matrix a satisfies the uniform ellipticity condition (5.1). In addition, the spread payoff function Φ is square integrable and continuously differentiable with bounded derivatives. Then for all $\alpha\in\Upsilon_n$, we have*

$$\Delta_2=\mathbb{E}[e^{-rT}\Phi(S_1(T),S_2(T))\pi^{\Delta_2}], \tag{5.8}$$

where the Malliavin weight π^{Δ_2} is

$$\pi^{\Delta_2}=\int_0^T\alpha(t)(a^{-1}(S(t))Y(t))^*d\mathbb{W}(t). \tag{5.9}$$

Proof. The proof follows the same arguments as the proof of Proposition 5.2. □

Proposition 5.4. *Suppose that the functions β and a in (4.3) are continuously differentiable with bounded derivatives, and the diffusion matrix a satisfies the uniform ellipticity condition (5.1). In addition, the spread payoff function Φ is square integrable and continuously differentiable with bounded derivatives. Then for all $\alpha\in\Upsilon_n$, we have*

$$\Gamma_1=\mathbb{E}[e^{-rT}\Phi(S_1(T),S_2(T))\pi^{\Gamma_1}], \tag{5.10}$$

where the Malliavin weight π^{Γ_1} is

$$\pi^{\Gamma_1}=\left(\pi^{\Delta_1}\right)^2-\frac{1}{x_1}\pi^{\Delta_1}-\int_0^T\alpha(t)\left(a^{-1}(S(t))Y_1(t)\right)^2dt. \tag{5.11}$$

Proof. Using Proposition 5.2, it suffices to show the result with $\Phi\in C_c^\infty(\mathbb{R}^2,\mathbb{R})$. Define $G:=x_1\pi^{\Delta_1}$ so that $\pi^{\Delta_1}=\frac{1}{x_1}G$. Then $\frac{\partial\pi^{\Delta_1}}{\partial x_1}=-\frac{1}{x_1^2}G=-\frac{1}{x_1}\pi^{\Delta_1}$. We have

$$\begin{aligned}\Gamma_1&=\frac{\partial^2}{\partial x_1^2}\mathbb{E}[e^{-rT}\Phi(S_1(T),S_2(T))]=\frac{\partial}{\partial x_1}\mathbb{E}[e^{-rT}\Phi(S_1(T),S_2(T))\pi^{\Delta_1}]\\&=\mathbb{E}[e^{-rT}\Phi'(S_1(T),S_2(T))Y_1(T)\pi^{\Delta_1}]-\frac{1}{x_1}\mathbb{E}[e^{-rT}\Phi(S_1(T),S_2(T))\pi^{\Delta_1}].\end{aligned} \tag{5.12}$$

where the interchange of the derivative and the expectation are justified by the dominated convergence theorem. For the first term in (5.12) we use similar arguments as in the proof of Proposition 5.2:

$$\begin{aligned}
&\mathbb{E}[e^{-rT}\Phi'(S_1(T),S_2(T))Y_1(T)\pi^{\Delta_1}]\\
&=\mathbb{E}\left[\int_0^T e^{-rT}D_t\Phi(S_1(T),S_2(T))\alpha(t)(a^{-1}(S(t))Y_1(t))\pi^{\Delta_1}\right]\\
&=\mathbb{E}\left[e^{-rT}\Phi(S_1(T),S_2(T))\delta\left(\alpha(\cdot)(a^{-1}(X(\cdot))Y_1(\cdot))\pi^{\Delta_1}\right)\right].
\end{aligned}$$

Finally, applying the the integration by parts formula (Proposition 3.5) with $D_t\pi^{\Delta_1}=\alpha(t)(a^{-1}(S(t))Y(t))$, we have

$$\begin{aligned}
\delta\left(\alpha(\cdot)(a^{-1}(X(\cdot))Y_1(\cdot))\pi^{\Delta_1}\right) &= \pi^{\Delta_1}\int_0^T\alpha(t)(a^{-1}(S(t))Y_1(t))dW_1(t)\\
&\quad-\int_0^T\alpha(t)\left(a^{-1}(S(t))Y_1(t)\right)^2dt\\
&= \left(\pi^{\Delta_1}\right)^2-\int_0^T\alpha(t)\left(a^{-1}(S(t))Y_1(t)\right)^2dt. \qquad (5.13)
\end{aligned}$$

Combining (5.12) with (5.13) we get the desired result for $\Phi\in C_c^\infty(\mathbb{R}^2;\mathbb{R})$. Now consider the general case, when $\Phi\in L^2(\mathbb{R}^2,\mathbb{R})$. Since the set $C_c^\infty(\mathbb{R}^2,\mathbb{R})$ of continuously differentiable functions with compact support is dense in $L^2(\mathbb{R}^2,\mathbb{R})$, we can always find a sequence $(\Phi_n)_n\in C_c^\infty(\mathbb{R}^2,\mathbb{R})$ of continuously differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}$ with compact support such that

$$\lim_{n\to+\infty}\mathbb{E}[|e^{-rT}\Phi_n(S_1(T),S_2(T))-e^{-rT}\Phi(S_1(T),S_2(T))|^2]=0. \qquad (5.14)$$

Let us define

$$f(x_1,x_2):=\mathbb{E}[e^{-rT}\Phi(S_1(T),S_2(T))\pi^{\Gamma_1}]$$

where π^{Γ_1} is given by (5.11). By applying the obtained result for Φ_n as well as the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left|\frac{\partial^2}{\partial x_1^2}\mathbb{E}[e^{-rT}\Phi_n(S_1(T),S_2(T))]-f(x_1,x_2)\right|\\
&=\left|\mathbb{E}[e^{-rT}(\Phi_n(S_1(T),S_2(T))-\Phi(S_1(T),S_2(T)))\pi^{\Gamma_1}]\right|\\
&\le\left(\mathbb{E}[e^{-2rT}(\Phi_n(S_1(T),S_2(T))-\Phi(S_1(T),S_2(T)))^2]\right)^{\frac{1}{2}}\left(\mathbb{E}[\left|\pi^{\Gamma_1}\right|^2]\right)^{\frac{1}{2}} \qquad (5.15)
\end{aligned}$$

As noted before, the first expression on the right hand side of (5.15) converges uniformly on compacts to zero. As a continuous function in x_1, the second expression on the right hand side of (5.15) is bounded on any compact set. It therefore follows that

$$\frac{\partial^2}{\partial x_1^2}\mathbb{E}[e^{-rT}\Phi_n(S_1(T),S_2(T))]\to f(x_1,x_2) \qquad (5.16)$$

uniformly on compacts (in x_1). From (5.14) and (5.16), we conclude that $\mathbb{E}[e^{-rT}\Phi(S_1(T), S_2(T))]$ is twice differentiable with respect to x_1 and the derivative is given by

$$\frac{\partial^2}{\partial x_1^2}\mathbb{E}[e^{-rT}\Phi(S_1(T), S_2(T))] = f(x_1, x_2) = \mathbb{E}[e^{-rT}\Phi(S_1(T), S_2(T))\pi^{\Gamma_1}],$$

which complete the proof. □

Proposition 5.5. *Suppose that the functions β and a in (4.3) are continuously differentiable with bounded derivatives, and the diffusion matrix a satisfies the uniform ellipticity condition (5.1). In addition, the spread payoff function Φ is square integrable and continuously differentiable with bounded derivatives. Then for all $\alpha \in \Upsilon_n$, we have*

$$\Gamma_2 = \mathbb{E}[e^{-rT}\Phi(S_1(T), S_2(T))\pi^{\Gamma_2}], \tag{5.17}$$

where the Malliavin weight π^{Γ_2} is

$$\pi^{\Gamma_2} = \left(\pi^{\Delta_2}\right)^2 - \frac{1}{x_2}\pi^{\Delta_2} - \int_0^T \alpha(t)\left(a^{-1}(S(t))Y_2(t)\right)^2 dt. \tag{5.18}$$

Proof. The proof follows the same arguments as in the proof of Proposition 5.4. □

Next we consider the derivative of the price of the spread option with respect to the volatilities σ_i, $i = 1, 2$. This computation is not as straightforward as in the computations of Δ_i and Γ_i, for $i = 1, 2$. Here, instead of computing the derivative of the price of the spread option with respect to the associated volatilities, we consider adding a perturbation term ε to the volatility term and then observe the effect of the perturbation on the price of the spread option. To avoid degeneracy, we introduce the set of deterministic functions

$$\widetilde{\Upsilon}_n = \{\widetilde{\alpha} \in L^2([0,T]) : \int_{t_{i-1}}^{t_i} \widetilde{\alpha}(t)dt, \quad \forall i = 1, ..., n.\}$$

Let $\gamma : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$ be a direction function for the volatility term such that $\varepsilon \in [-1, 1]$, γ and $\sigma + \varepsilon\gamma$ are continuously differentiable with bounded derivatives and verify Lipschitz conditions such that the following uniform elliptic condition is satisfied:

$$\exists \epsilon > 0 \quad \xi^*(a + \varepsilon\gamma)^*(x)(a + \varepsilon\gamma)(x)\xi \geq \epsilon \parallel \xi \parallel^2, \quad \forall \ \xi, x \in \mathbb{R}^2, \ \ \xi \neq 0. \tag{5.19}$$

As in Fournié et al. [10], we consider the perturbed process $(S^\varepsilon(t))_{t\in[0,T]}$ as a solution of the following stochastic differential equation

$$dS^\varepsilon(t) = \beta(S^\varepsilon(t))dt + [a(S^\varepsilon(t)) + \varepsilon\gamma(S^\varepsilon(t))]d\mathbb{W}(t), \quad S^\varepsilon(0) = x. \tag{5.20}$$

We also relate to this perturbed process the perturbed price of the spread option u^ε defined by

$$u^\varepsilon = \mathbb{E}[e^{-rT}\Phi(S_1^\varepsilon(T), S_2^\varepsilon(T))]. \tag{5.21}$$

We also introduce a variation process with respect to ε, which is the derivative of $X^\varepsilon(t)$ with respect to the parameter ε ($Z^\varepsilon(t) := \frac{\partial X^\varepsilon(t)}{\partial \varepsilon}$):

$$\begin{aligned} dZ^\varepsilon(t) &= \beta'(S^\varepsilon(t))Z^\varepsilon(t)dt + \sum_{i=1}^{2}(a_i'(S^\varepsilon(t)) + \varepsilon\gamma_i'(S^\varepsilon(t)))Z^\varepsilon(t)dW_i(t) \\ &\quad + \gamma(S^\varepsilon(t))d\mathbb{W}(t), \\ Z^\varepsilon(0) &= 0_{2\times 2} \end{aligned} \tag{5.22}$$

where $0_{2\times 2}$ is the zero column vector in $\mathbb{R}^2$ and γ_i' denotes the derivative of the i-th column.

Proposition 5.6. *Assume that the uniformly elliptic condition (5.19) holds and for* $B(t_i) = Y^{-1}(t_i)Z(t_i) = Y^{-1}(t_i)Z^{\varepsilon=0}(t_i)$, $i = 1, 2$, *there exists* $a^{-1}(X)YB \in Dom(\delta)$. *Then, for any square integrable spread option payoff function,* Φ, *with continuously differentiable and bounded derivatives,*

$$\frac{\partial}{\partial\varepsilon}u^\varepsilon \mid_{\varepsilon=0} = \mathbb{E}[e^{-rT}\Phi(S_1(T), S_2(T))\delta\left(a^{-1}(S(\cdot))Y(\cdot)\widetilde{B}(\cdot)\right)] \tag{5.23}$$

holds. Here

$$\widetilde{B}(t) = \sum_{i=1}^{2}\widetilde{\alpha}(t)(B(t_i) - B(t_{i-1}))1_{\{t\in[t_{i-1},t_i]\}},$$

for $t_0 = 0$ *and* $\widetilde{\alpha} \in \tilde{\Upsilon}_n$. *Moreover, if* B *is Malliavin differentiable, then*

$$\begin{aligned} \delta\left(a^{-1}(S(\cdot)Y(\cdot)\widetilde{B}(\cdot)\right) &= \sum_{i=1}^{2}\left\{B^*(t_i)\int_{t_{i-1}}^{t_i}\widetilde{\alpha}(t)(a^{-1}(S(t))Y(t))^*d\mathbb{W}(t)\right. \\ &\quad -\int_{t_{i-1}}^{t_i}\widetilde{\alpha}(t)\,Tr((D_tB(t_i))a^{-1}(S(t))Y(t))dt \\ &\quad \left.-\int_{t_{i-1}}^{t_i}\widetilde{\alpha}(t)(a^{-1}(S(t))Y(t)B(t_{i-1}))^*d\mathbb{W}(t)\right\}. \end{aligned}$$

Proof. The proof follows the same line of argument as the proof of Proposition 3.1.5 in [16]. □

6. Examples

We consider the risk-neutral price dynamics given by the following systems of stochastic differential equations

$$\begin{aligned} dS_1(t) &= S_1(t)[(r - q_1)dt + \sigma_1 dW_1(t)], \quad S_1(0) = x_1, \\ dS_2(t) &= S_2(t)[(r - q_2)dt + \sigma_2 dW_2(t)], \quad S_2(0) = x_2, \end{aligned} \tag{6.1}$$

where q_1 and q_2 are the instantaneous dividend yields, σ_1 and σ_2 are positive constants volatilities, $S_1(t)$ and $S_2(t)$ are prices of two assets at time t, and

$W_1(t)$ and $W_2(t)$ are two standard Brownian motions with correlation parameter $\rho \in (-1, 1)$. For all $t \in [0, T]$ we define

$$\widetilde{W}_2(t) := \frac{1}{\sqrt{1-\rho^2}} \left(W_2(t) - \rho W_1(t) \right) \quad \text{and} \quad W_1(t) = \widetilde{W}_1(t).$$

The process $\{\widetilde{W}_2(t) \quad 0 \leq t \leq T\}$ is a Brownian motion which is independent of $W_1(t)$ and $W_2(t)$. Then the system of stochastic differential equations (6.1) can be rewritten in matrix form

$$dS(t) = \beta(S(t))dt + a(S(t))d\mathbb{W}(t), \quad X(0) = (x_1, x_2)$$

where

$$\beta(S(t)) = \begin{pmatrix} (r-q_1)S_1(t) \\ (r-q_2)S_2(t) \end{pmatrix}$$

and

$$a(S(t)) = \begin{pmatrix} \sigma_1 S_1(t) & 0 \\ \rho\sigma_2 S_2(t) & \sigma_2\sqrt{1-\rho^2}S_2(t) \end{pmatrix}.$$

The inverse of a is

$$a^{-1}(S(t)) = \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}S_1(t)S_2(t)} \begin{pmatrix} \sigma_2\sqrt{1-\rho^2}S_2(t) & 0 \\ -\rho\sigma_2 S_2(t) & \sigma_1 S_1(t) \end{pmatrix}.$$

The first variation process is given by

$$dY(t) = \beta'(S(t))Y(t)dt + a_1'(S(t))Y(t)dW_1(t) + a_2'(S(t))Y(t)dW_2(t), \quad Y(0) = I,$$

where

$$\beta'(S(t)) = \begin{pmatrix} r-q_1 & 0 \\ 0 & r-q_2 \end{pmatrix}, \quad a_1'(S(t)) = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \rho\sigma_2 \end{pmatrix},$$

and

$$a_2'(S(t)) = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_2\sqrt{1-\rho^2} \end{pmatrix}.$$

The matrix $(a^{-1}(S(t))Y(t))^*$ has the following form

$$(a^{-1}(S(t))Y(t))^* = \begin{pmatrix} \frac{Y^{11}(t)}{\sigma_1 S_1(t)} & -\frac{\rho Y^{11}(t)}{\sigma_1\sqrt{1-\rho^2}S_1(t)} \\ \frac{Y^{12}(t)}{\sigma_1 S_1(t)} & -\frac{\rho Y^{12}(t)}{\sigma_1 S_1(t)\sqrt{1-\rho^2}} + \frac{Y^{22}(t)}{\sigma_2 S_2(t)\sqrt{1-\rho^2}} \end{pmatrix}.$$

An application of Proposition 5.2 with $Y^{11}(t) = \frac{S_1(t)}{x_1}$, $Y^{22}(t) = \frac{S_2(t)}{x_2}$ and $Y^{21}(t) = 0$ yields

$$\Delta_1 = \mathbb{E}\left[e^{-rT}\Phi(S_1(T), S_2(T)) \frac{\sqrt{1-\rho^2}W_1(T) - \rho W_2(T)}{\sigma_1 x_1 T\sqrt{1-\rho^2}} \right]$$

and an application of Proposition 5.3 yields

$$\Delta_2 = \mathbb{E}\left[e^{-rT}\Phi(S_1(T), S_2(T))\frac{W_2(T)}{\sigma_2 x_2 T\sqrt{1-\rho^2}}\right].$$

For Gamma, an application of Propositions 5.4 - 5.5 yields

$$\Gamma_1 = \mathbb{E}\left[e^{-rT}\Phi(S_1(T), S_2(T))\left\{\left(\pi^{\Delta_1}\right)^2 - \frac{1}{x_1}\left(\pi^{\Delta_1}\right) - \frac{1}{Tx_1^2\sigma_1^2(1-\rho^2)}\right\}\right]$$

and

$$\Gamma_2 = \mathbb{E}\left[e^{-rT}\Phi(S_1(T), S_2(T))\left\{\left(\pi^{\Delta_2}\right)^2 - \frac{1}{x_2}\left(\pi^{\Delta_2}\right) - \frac{1}{Tx_2^2\sigma_2^2(1-\rho^2)}\right\}\right].$$

For *Vega* we perturbed diffusion matrix of the equation for $S_1(t)$ with γ. We note from (5.22) that

$$dZ_1(t) = (r - q_1)Z_1(t)dt + \sigma_1 Z_1(t)dW_1(t) + S_1(t)dW_1(t) \tag{6.2}$$

where we have chosen γ to be

$$\gamma(S(t)) = \begin{pmatrix} S_1(t) & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $S_2(t)$ do not depend on $S_1(t)$ we set $Z_2(t) = 0$. An application of the Itô formula to (6.2) yields the following solution

$$Z_1(t) = S_1(t)(W_1(t) - \sigma_1 t).$$

Since $B(t) = Y^{-1}(t)Z(t)$, we have

$$B(T) = x_1(W_1(T) - \sigma_1 T).$$

The Malliavin derivative of $B(T)$ is calculated as follows

$$D_t B(T) = D_t(x_1(W_1(T) - \sigma_1 T)) = x_1.$$

An application of Proposition 5.6 yields

$$\mathcal{V}_1 = \mathbb{E}[e^{-rT}\Phi(S_T, S_2(T))\pi^{\mathcal{V}_1}]$$

where the Malliavin weight $\pi^{\mathcal{V}_1}$ is given by

$$\pi^{\mathcal{V}_1} = \frac{1}{T}\left\{(W_1(T) - \sigma_1 T)\left(\frac{W_1(T)}{\sigma_1} - \frac{\rho W_2(T)}{\sigma_1\sqrt{1-\rho^2}}\right) - \frac{T}{\sigma_1}\right\}.$$

Following a similar procedure and applying Proposition 5.6 we obtain

$$\mathcal{V}_2 = \mathbb{E}[e^{-rT}\Phi(S_T, S_2(T))\pi^{\mathcal{V}_2}]$$

where the Malliavin weight $\pi^{\mathcal{V}_2}$ is given by

$$\begin{aligned}\pi^{\mathcal{V}_2} = \frac{1}{T}\Big\{&\Big(W_2(T) - \rho\sigma_2 T - \sigma_2\sqrt{1-\rho^2}T\Big)\\ &\times\Bigg(\int_0^T \frac{Y^{12}(t)dW_1(t)}{\sigma_1 S_1(t)} - \frac{\rho}{\sigma_2\sqrt{1-\rho^2}}\int_0^T \frac{Y^{12}(t)dW_2(t)}{S_1(t)}\\ &+\frac{W_2(T)}{\sigma_2\sqrt{1-\rho^2}}\Bigg) - \frac{T}{\sigma_2\sqrt{1-\rho^2}}\Big\}.\end{aligned}$$

7. Application of spread options with stochastic volatility

The stochastic volatility has an important impact on the price of the spread option. The stochastic volatility helps to understand the market evolution completely. We set up the dynamics with stochastic volatility as follows

$$\begin{aligned} dS_1(t) &= S_1(t)[\mu_1(t)dt + \sigma_1\sqrt{V(t)}dW_1(t)], \quad S_1(0) = x_1,\\ dS_2(t) &= S_2(t)[\mu_2(t)dt + \sigma_2\sqrt{V(t)}dW_2(t)], \quad S_2(0) = x_2,\\ dV(t) &= \kappa(1 - V(t))dt + \nu\sqrt{V(t)}dZ(t), \qquad V(0) = v_0, \end{aligned} \tag{7.1}$$

where $S_i(t)$, $i = 1, 2$ denote the asset prices and $V(t)$ represents a volatility factor, κ measures the speed at which $V(t)$ reverts towards 1, ν is the parameter which determines the volatility of the variance process.
We specify that $dW_1(t)dW_2(t) = \rho dt$ and $dW_i(t)dZ(t) = 0$, $i = 1, 2$.
Due to the independence of W_i, $i = 1, 2$ and Z, we have that $S_i(T)$, $i = 1, 2$ given by the integrated variance

$$\overline{V}(T) := \int_0^T V(t)dt,$$

is lognormally distributed. This result can be generalised to two-dimensional case where $S_1(T)$ and $S_2(T)$ given $\overline{V}(T)$ are jointly lognormal. This argument, however, is not pursued in this paper.
Define

$$W_2(t) := \rho\widetilde{W}_1(t) + \sqrt{1-\rho^2}\widetilde{W}_2(t), \quad W_1(t) = \widetilde{W}_1(t), \quad \text{and} \quad Z(t) = \widetilde{W}_3(t).$$

The system of stochastic differential equations (7.1) can be written in matrix form

$$dS(t) = \beta(S(t))dt + a(S(t))d\mathbb{W}(t) \tag{7.2}$$

where $S(t) = (S_1(t), S_2(t), V(t))^*$, $\mathbb{W}(t) = (W_1(t), W_2(t), Z(t))^*$,

$$\beta(S(t)) = \begin{pmatrix} \mu(t)S_1(t) \\ \mu_2(t)S_2(t) \\ \kappa(1 - V(t)) \end{pmatrix},$$

and

$$a(S(t)) = \begin{pmatrix} \sigma_1\sqrt{V(t)}S_1(t) & 0 & 0 \\ \rho\sigma_2\sqrt{V(t)}S_2(t) & \sigma_2\sqrt{1-\rho^2}\sqrt{V(t)}S_2(t) & 0 \\ 0 & 0 & \nu\sqrt{V(t)} \end{pmatrix}.$$

The inverse of a is

$$a^{-1}(S(t)) = \begin{pmatrix} \frac{1}{\sigma_1\sqrt{V(t)}S_1(t)} & 0 & 0 \\ -\frac{\rho}{\sigma_1\sqrt{1-\rho^2}\sqrt{V(t)}S_1(t)} & \frac{1}{\sigma_2\sqrt{1-\rho^2}\sqrt{V(t)}S_2(t)} & 0 \\ 0 & 0 & \frac{1}{\nu\sqrt{V(t)}} \end{pmatrix}.$$

The first variation process is given by

$$\begin{aligned} dY(t) = {} & \beta'(S(t))Y(t)dt + a_1'(S(t))Y(t)dW_1(t) + a_2'(S(t))Y(t)dW_2(t) \\ & + a_3'(S(t))Y(t)dW_3(t) \end{aligned}$$

where

$$\beta'(S(t)) = \begin{pmatrix} \mu_1(t) & 0 & 0 \\ 0 & \mu_2(t) & 0 \\ 0 & 0 & -\kappa \end{pmatrix},$$

$$a_1'(S(t)) = \begin{pmatrix} \sigma_1\sqrt{V(t)} & 0 & \frac{\sigma_1 S_1(t)}{2\sqrt{V(t)}} \\ 0 & \rho\sigma_2\sqrt{V(t)} & \frac{\rho\sigma_2 S_2(t)}{2\sqrt{V(t)}} \\ 0 & 0 & 0 \end{pmatrix},$$

$$a_2'(S(t)) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_2\sqrt{1-\rho^2}\sqrt{V(t)} & \frac{\sigma_2\sqrt{1-\rho^2}S_2(t)}{2\sqrt{V(t)}} \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$a_3'(S(t)) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\nu}{2\sqrt{V(t)}} \end{pmatrix}.$$

The matrix $(a^{-1}(S(t))Y(t))^*$ has the following form

$$\begin{aligned} & (a^{-1}(S(t))Y(t))^* \\ & = \begin{pmatrix} \frac{Y^{11}(t)}{\sigma_1\sqrt{V(t)}S_1(t)} & -\frac{\rho Y^{11}(t)}{\sigma_1\sqrt{1-\rho^2}\sqrt{V(t)}S_1(t)} & 0 \\ \frac{Y^{12}(t)}{\sigma_1\sqrt{V(t)}S_1(t)} & -\frac{\rho Y^{12}(t)}{\sigma_1\sqrt{1-\rho^2}\sqrt{V(t)}S_1(t)} + \frac{Y^{22}(t)}{\sigma_2\sqrt{1-\rho^2}\sqrt{V(t)}S_2(t)} & 0 \\ \frac{Y^{13}(t)}{\sigma_1\sqrt{V(t)}S_1(t)} & -\frac{\rho Y^{13}(t)}{\sigma_1\sqrt{1-\rho^2}\sqrt{V(t)}S_1(t)} & \frac{Y^{33}(t)}{\nu\sqrt{V(t)}} \end{pmatrix}. \end{aligned}$$

This, with the applications of Propositions 5.2 - 5.5, yields the following results.

Proposition 7.1. *Suppose that β and a in (7.2) are continuously differentiable functions with bounded derivatives and that diffusion matrix a satisfies the uniform ellipticity condition (5.1). Then, for any $\alpha \in \Upsilon_n$ and $\Phi(S_1(T), S_2(T))$ square integrable, we have*

1. $$\Delta_1 = \mathbb{E}\left[e^{-rT}\Phi(S_1(T), S_2(T))\pi^{\Delta_1}\right]$$

   where π^{Δ_1} is the Malliavin weight given by

   $$\pi^{\Delta_1} = \frac{1}{\sigma_1 x_1 T}\int_0^T \frac{1}{\sqrt{V(t)}}dW_1(t) - \frac{\rho}{\sigma_1\sqrt{1-\rho^2}x_1 T}\int_0^T \frac{1}{\sqrt{V(t)}}dW_2(t). \tag{7.3}$$

2. $$\Delta_2 = \mathbb{E}\left[e^{-rT}\Phi(S_1(T), S_2(T))\pi^{\Delta_2}\right]$$

   where π^{Δ_2} is the Malliavin weight given by

   $$\begin{aligned}\pi^{\Delta_2} = {} & \frac{1}{\sigma_1 T}\int_0^T \frac{Y^{12}(t)}{\sqrt{V(t)}S_1(t)}dW_1(t) \\ & - \frac{\rho}{\sigma_1\sqrt{1-\rho^2}T}\int_0^T \frac{Y^{12}(t)}{\sqrt{V(t)}S_1(t)}dW_2(t) \\ & + \frac{1}{\sigma_2\sqrt{1-\rho^2}x_2 T}\int_0^T \frac{1}{\sqrt{V(t)}}dW_2(t).\end{aligned} \tag{7.4}$$

Proposition 7.2. *Suppose that β and a in (7.2) are continuously differentiable functions with bounded derivatives and that diffusion matrix a satisfies the uniform ellipticity condition (5.1). Then, for any $\alpha \in \Upsilon_n$ and $\Phi(S_1(T), S_2(T))$ square integrable, we have*

1. $$\Gamma_1 = \mathbb{E}\left[e^{-rT}\Phi(S_1(T), S_2(T))\pi^{\Gamma_1}\right]$$

   where π^{Γ_1} is the Malliavin weight given by

   $$\pi^{\Gamma_1} = \left(\pi^{\Delta_1}\right)^2 - \frac{1}{x_1}\pi^{\Delta_1} - \frac{1}{T\sigma_1^2 x_1^2(1-\rho^2)}\int_0^T \frac{1}{V(t)}dt,$$

   where π^{Δ_1} is given in (7.3).

2. $$\Gamma_2 = \mathbb{E}\left[e^{-rT}\Phi(S_1(T), S_2(T))\pi^{\Gamma_2}\right]$$

   where π^{Γ_2} is the Malliavin weight given by

   $$\pi^{\Gamma_2} = \left(\pi^{\Delta_2}\right)^2 - \frac{1}{x_1}\pi^{\Delta_2} - \frac{1}{T\sigma_2^2 x_2^2(1-\rho^2)}\int_0^T \frac{1}{V(t)}dt,$$

   where π^{Δ_2} is given in (7.4).

To evaluate $\mathcal{V}_1$ we consider the perturbed process given by (5.20) where γ is chosen to be

$$\gamma(S(t)) = \begin{pmatrix} S_1(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where the perturbation is only on the asset price $S_1(t)$. Since $S_2(t)$ and $V(t)$ do not depend on ϵ, we deduce that $Z_1(t) = 0$ and $Z_3(t) = 0$, respectively. From (5.22) we deduce that $Z_1(t)$ satisfies the following stochastic differential equation

$$dZ_1(t) = \mu_1 Z_1(t)dt + \sigma_1\sqrt{V(t)}Z_1(t)dW_1(t) + S_1(t)dW_1(t).$$

An application of Itô formula yields the solution

$$Z_1(t) = S_1(t)\left(W_1(t) - \int_0^t \sigma_1\sqrt{V(s)}ds\right).$$

Since $B(t) = Y^{-1}(t)Z(t)$, one has

$$B(t) = x_1\left(W_1(t) - \int_0^t \sigma_1\sqrt{V(s)}ds\right).$$

An application of Proposition 3.1 and Lemma 5.1 gives

$$\begin{aligned} D_tB(T) &= x_1\left((1,0,0)^* - \sigma_1\int_0^T D_t\sqrt{V(s)}ds\right) \\ &= x_1\left((1,0,0)^* - \frac{\sigma_1}{2}\int_t^T \frac{\sqrt{V(t)}}{\sqrt{V(s)}}\frac{Y^{22}(s)}{Y^{22}(t)}\left(\rho, \sqrt{1-\rho^2}, 0\right)^* ds\right). \end{aligned}$$

Then, one obtains

$$\mathrm{Tr}\left((D_tB(T))a^{-1}(S(t))Y(t)\right) = \frac{1}{\sigma_1\sqrt{V(t)}}.$$

The application of Proposition 5.6 yields the following result.

Proposition 7.3. *Suppose that β and a in (7.2) are continuously differentiable functions with bounded derivatives and that diffusion matrix a satisfies the uniform ellipticity condition (5.19). Then, for any $\Phi(S_1(T), S_2(T))$ square integrable with $\widetilde{\alpha}(t) = \frac{1}{T}$, we have*

$$\mathcal{V}_1 = \mathbb{E}[e^{-rT}\Phi(S_1(T), S_2(T))\pi^{\mathcal{V}_1}]$$

where the Malliavin weight $\pi^{\mathcal{V}_1}$ is given by

$$\begin{aligned} \pi^{\mathcal{V}_1} &= \frac{1}{T}\Bigg\{\left(W_1(T) - \int_0^T \sigma_1\sqrt{V(t)}dt\right) \\ &\quad \times \left(\int_0^T \frac{dW_1(t)}{\sigma_1\sqrt{V(t)}} - \frac{\rho}{\sigma_1\sqrt{1-\rho^2}}\int_0^T \frac{dW_2(t)}{\sqrt{V(t)}}\right) - \int_0^T \frac{dt}{\sigma_1\sqrt{V(t)}}\Bigg\}. \end{aligned}$$

A similar procedure used to obtain Proposition 7.3 with perturbation only in the diffusion coefficient of the second asset, $S_1(t)$, together with the application of Proposition 5.6 yields the following result.

Proposition 7.4. *Suppose that β and a in (7.2) are continuously differentiable functions with bounded derivatives and that diffusion matrix a satisfies the uniform ellipticity condition (5.19). Then, for any $\Phi(S_1(T), S_2(T))$ square integrable with $\widetilde{\alpha}(t) = \frac{1}{T}$, we have*

$$\mathcal{V}_2 = \mathbb{E}[e^{-rT}\Phi(S_1(T), S_2(T))\pi^{\mathcal{V}_2}]$$

where the Malliavin weight $\pi^{\mathcal{V}_2}$ is given by

$$\begin{aligned}\pi^{\mathcal{V}_2} &= \frac{1}{T}\Bigg\{\left(W_2(T) - \int_0^T \rho\sigma_2\sqrt{V(t)}dt - \int_0^T \sigma_2\sqrt{1-\rho^2}\sqrt{V(t)}dt\right)\\ &\quad\times\left(\int_0^T \frac{Y^{12}(t)dW_1(t)}{\sigma_1\sqrt{V(t)}S_1(t)} - \frac{\rho}{\sigma_1\sqrt{1-\rho^2}}\int_0^T \frac{Y^{12}(t)dW_2(t)}{\sqrt{V(t)}S_1(t)}\right.\\ &\quad\left.+\frac{1}{\sigma_2\sqrt{1-\rho^2}}\int_0^T \frac{dW_2(t)}{\sqrt{V(t)}}\right) - \int_0^T \frac{dt}{\sigma_2\sqrt{1-\rho^2}\sqrt{V(t)}}\Bigg\}.\end{aligned}$$

8. Localised Malliavin approach for spread options

Following Fournié et al. [10], the variance reduction is achieved by using a localized Malliavin technique. The approach is to localise the Malliavin weights round the strike price K, that is, instead of using the Malliavin calculus approach to derive the Greeks globally, the calculus is only applied locally around the singularities of the payoff function. In our case we localise the payoff function $\Phi(s_1, s_2)$ around $s_2 - s_1 = K$. Direct methods can be used outside the localisation point.

To be precise we introduce the Lipchitz continuous approximation to the Heaviside function:

$$H_a(s_1, s_2) = \begin{cases} 0 & \text{if } s_2 - s_1 < K - a,\\ \frac{s_2 - s_1 - (K-a)}{2a} & \text{if } K - a \le s_2 - s_1 \le K + a,\\ 1 & \text{if } s_2 - s_1 > K + a. \end{cases}$$

$$\begin{aligned} h_a(s_1, s_2) &= \int_{-\infty}^{s1\wedge s_2} H_a(y_1, y_2)dy\\ &= \begin{cases} 0 & \text{if } s_2 - s_1 < K - a,\\ \frac{(s_2 - s_1 - (K-a))^2}{4a} & \text{if } K - a \le s_2 - s_1 \le K + a,\\ s_2 - s_1 - K & \text{if } s_2 - s_1 > K + a. \end{cases}\end{aligned}$$

We observe that $h'_a(s_1, s_2) = H_a(s_1, s_2)$.

Set

$$\Phi_a(s_1, s_2) = \Phi(s_1, s_2) - h_a(s_1, s_2) = (s_2 - s_1 - K)^+ - h_a(s_1, s_2).$$

Notice that $\Phi_a(s_1, s_2)$ vanishes for $s_2 - s_1 < K - a$ and for $s_2 - s_1 \geq K + a$. This means that $\Phi_a(s_1, s_2)$ is a localised version of $\Phi(s_1, s_2)$. We can, therefore, write

$$\Phi(S_1(T), S_2(T)) = \Phi_a(S_1(T), S_2(T)) + h_a(S_1(T), S_2(T)),$$

so that the price of the spread call option is given by

$$u = \mathbb{E}[e^{-rT}\Phi_a(S_1(T), S_2(T))] + \mathbb{E}[e^{-rT}h_a(S_1(T), S_2(T))].$$

We illustrate how the localised Malliavin calculus approach is applied to derive the expression for Δ_1 and Γ_1. We have

$$\begin{aligned}\Delta_1 &= \frac{\partial}{\partial x_1}\mathbb{E}[e^{-rT}\Phi_a(S_1(T), S_2(T))] + \frac{\partial}{\partial x_1}\mathbb{E}[e^{-rT}h_a(S_1(T), S_2(T))] \\ &= \mathbb{E}[e^{-rT}\Phi_a(S_1(T), S_2(T))\pi^{\Delta_1}] + \mathbb{E}[e^{-rT}H_a(S_1(T), S_2(T))Y_1(T)],\end{aligned}$$

with π^{Δ_1} given by (5.6), where the second equality is due to the application of Proposition 5.2 on the first term and direct differentiation on the second term. A similar procedure can be applied to obtain Δ_2.
For Γ_1, we have

$$\begin{aligned}\Gamma_1 &= \frac{\partial^2}{\partial x_1^2}\mathbb{E}[e^{-rT}\Phi_a(S_1(T), S_2(T))] + \frac{\partial^2}{\partial x_1^2}\mathbb{E}[e^{-rT}h_a(S_1(T), S_2(T))] \\ &= \mathbb{E}[e^{-rT}\Phi_a(S_1(T), S_2(T))\pi^{\Gamma_1}] + \frac{1}{x_1^2}\mathbb{E}[e^{-rT}H_a'(S_1(T), S_2(T))S_1^2(T)],\end{aligned}$$

with π^{Γ_1} given by (5.11), where the second equality is due to the application of Proposition 5.4 on the first term and direct differentiation on the second term. A similar procedure can be applied to obtain Γ_2.

9. Concluding remarks

In this paper, tractable formulae for price sensitivities of spread options are presented using Malliavin calculus approach. Precisely, the formulae are presented for asset dynamics driven by geometric Brownian motion and stochastic volatility models of the financial markets. In order to apply the Malliavin calculus approach, the joint characteristic function of the underlying assets is not required. In addition, the Malliavin calculus approach avoids the direct differentiation of the payoff functional. Computing price sensitivities of spread options via the tractable formulae obtained guarantees a convergence rate that is independent of the regularity of the payoff function and dimensionality [2]. The results obtained in this paper form a generalization of the computation of price sensitivities for spread options. The price sensitivities are expressed in terms of the expectation of spread option payoff multiplied with some Malliavin weights which are functions of the underlying asset expressed as stochastic integrals. This is in agreement with results in Fournié et al. [10]. The Malliavin weight functions are independent of the payoff functional, this is suitable for

Monte Carlo methods which can be applied for general spread options, and not specifically for each spread option. The use of localised Malliavin approach helps to reduce the variance when the Monte Carlo methods are applied by localising the Malliavin calculus around the point od discontinuity. Price sensitivities are valuable to investors as well as financial institutions as they are used to find and construct financial risk strategies to hedge against potential sources of the underlying price risk. It remains for future research to consider spread options driven by general Lévy models. It would also be interesting to consider the effect of stochastic correlation in the model.

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