Novi Sad J. Math. Vol. 53, No. 2, 2023, 121-130
https://doi.org/10.30755/NSJOM.12394

On a connection between fuzzy subgroups and F-inverse covers of inverse monoids

Elton Pasku[1]

Abstract. We define two categories, the category $\mathfrak{FG}$ of fuzzy subgroups, and the category $\mathfrak{FC}$ of F-inverse covers of inverse monoids, and prove that there is a full and faithful embedding of $\mathfrak{FG}$ into $\mathfrak{FC}$. As a by-product of this embedding we get that the level subgroups of a given fuzzy subgroup can be realized as the $\mathcal{H}$-classes of a Clifford monoid that is canonically constructed from the fuzzy subgroup. This connection we find between fuzzy subgroups and inverse monoids is new and unexplored before and shows that, at least from a categorical viewpoint, fuzzy subgroups belong to the standard mathematics as much as they do to the fuzzy one.

AMS Mathematics Subject Classification (2010): 03E72, 20N25, 20M18.

Key words and phrases: Fuzzy subgroup, inverse monoid, F-inverse cover, dual premorphism.

1. Introduction and preliminaries

The theory of fuzzy sets originates with the article [11] of Zadeh and has aimed since than to help other branches of mathematics that study ambiguity or uncertainty. Along with fuzzy sets, fuzzy analogues have been developed, in particular the theory of fuzzy groups which started with the paper [8] of Rosenfeld. Given a set X, a fuzzy subset A of X is a function $A : X \to [0,1]$. For every $x \in X$, the value $A(x)$ represents the degree of membership of x in A. This is what makes A look like an uncertain set. On the other hand, the definition of fuzzy groups is a bit more complex and is given below.

Definition 1.1. Let G be a group. A fuzzy subgroup of G is a map $\mu : G \to [0,1]$ such that:

(i) for all $x, y \in G$, $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, and

(ii) for all $x \in G$, $\mu(x^{-1}) \geq \mu(x)$.

It turns out that for all $x \in G$, $\mu(x) = \mu(x^{-1})$, and $\mu(e) \geq \mu(x)$ where e is the unit of G. There is no restriction if we replace $[0,1]$ in this definition by $\mu([0,1])$, and in this way the definition may be restated as follows.

Definition 1.2. A fuzzy subgroup of G is a triple (G, μ, U) where G is a group, $U \subseteq [0,1]$ and $\mu : G \to U$ is a surjective map satisfying the properties:

[1]Universiteti i Tiranës, Fakulteti i Shkencave Natyrore, Departamenti i Matematikës, Tiranë, Albania e-mail: elton.pasku@fshn.edu.al

(1) for all $x, y \in G$, $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, and

(2) for all $x \in G$, $\mu(x^{-1}) = \mu(x)$.

We will revisit this definition in a while in order to show that a fuzzy subgroup of G can be interpreted as a dual premorphism from G onto a certain inverse monoid, and it is this interpretation that will lead us to the proof of our main theorem. Before doing that, let us first recall a few basic concepts from semigroup theory. We assume throughout that every semigroup $(S, \cdot)$ is a monoid, thus having a unit element denoted by 1. This is not a real restriction, since every semigroup S can be embedded into a monoid by simply adjoining a unit 1 to S and defining $1 \cdot x = x = x \cdot 1$ for every $x \in S$. Given a monoid $(S, \cdot)$ there are always defined Green's relations $\mathcal{L}, \mathcal{R}$ and $\mathcal{H}$ in S by setting for every $a, b \in S$,

$$(a, b) \in \mathcal{L} \Leftrightarrow Sa \cup \{a\} = Sb \cup \{b\},$$
$$(a, b) \in \mathcal{R} \Leftrightarrow aS \cup \{a\} = bS \cup \{b\}.$$

and $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. There is a special class of monoids which are of particular interest, the so called Clifford monoids. They are those monoids which satisfy the properties: (1) each $\mathcal{H}$-class contains a unique idempotent, (2) the idempotents are central. It turns out that in Clifford monoids the relations $\mathcal{L}, \mathcal{R}, \mathcal{H}$ coincide and that every $\mathcal{H}$-class forms a group. Clifford monoids are in fact a special case of a very much studied class of monoids, the so called inverse monoids. An inverse monoid is a monoid M such that for every $x \in M$ there is a unique $x^{-1} \in M$, called the inverse of x, such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. An obvious example of an inverse monoid is a lower semilattice with a greatest element where the multiplication is the usual meet operation. Every inverse monoid M comes equipped with a natural partial order $\preceq$ defined by: $x \preceq y$ if and only if there is an idempotent $e \in M$ such that $x = ye$. Every inverse monoid M has a smallest group congruence which is denoted by σ and is characterized by $(x, y) \in \sigma$ if and only if there is an idempotent $e \in M$ such that $xe = ye$. The name minimum group congruence comes from the fact that S/σ is a group with the following universal property. If γ is another group congruence in S, then there is a unique homomorphism $\xi : S/\sigma \to S/\gamma$ such that the following diagram commutes

$$\begin{tikzcd} S \arrow[r, "\sigma^\sharp"] \arrow[d, "\gamma^\sharp"'] & S/\sigma \arrow[dl, "\xi"] \\ S/\gamma & \end{tikzcd}$$

where $\sigma^\sharp$ and $\gamma^\sharp$ are the canonical epimorphisms. An elementary but important consequence of this property is that every homomorphism $\varphi : S \to T$ of inverse monoids induces a unique homomorphism $\varphi^\sigma : S/\sigma \to T/\sigma$ such that the following diagram is commutative

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "\sigma^\sharp"'] & T \arrow[d, "\sigma^\sharp"] \\ S/\sigma \arrow[r, "\varphi^\sigma"'] & T/\sigma. \end{tikzcd}$$

An explicit description of φ^σ is given in [10]. An inverse monoid satisfying the property that each σ-class contains a greatest element with respect to the natural partial order is called an F-inverse monoid. An F-inverse monoid F is called an F-inverse cover of an inverse monoid M over the group F/σ if there exists a surjective idempotent separating homomorphism $F \to M$, that is a surjective homomorphism of monoids $F \to M$ whose restriction on the semilattice of idempotents of F is also a surjection onto the semilattice of idempotents of M. There are a number of important results concerning F-inverse monoids which are related with covers and expansions of inverse monoids. The reader can find useful material in papers [1], [4] and [10]. Regarding F-inverse covers of inverse monoids, they are closely related with dual premorphisms between inverse monoids. The respective definition reads as follows.

Definition 1.3. (see [1]) A dual premorphism $\psi : M \to N$ between inverse monoids is a map such that:

(i) $\psi(xy) \succeq \psi(x)\psi(y)$ for all $x, y \in M$, and

(ii) $\psi(x^{-1}) = (\psi(x))^{-1}$ for all $x \in M$.

The following is Theorem VII.6.11 of [7] and gives a relationship between dual premorphisms and F-inverse covers.

Theorem 1.4. *Let H be a group and M an inverse monoid. If $\psi : H \to M$ is a dual premorphism such that for every $u \in M$, there is an $h \in H$ with $u \preceq \psi(h)$, then*

$$F = \{(u,h) \in M \times H | u \preceq \psi(h)\},$$

is an F-inverse cover of M over H. Conversely, every F-inverse cover of M over H can be so constructed (up to isomorphism).

Now we are ready to show that Definition 1.2 of a fuzzy subgroup as a triple, matches with Definition 1.3 in the special case when the domain of the premorphism is a group and the codomain is a subsemilattice of $[0,1]$ with a unit element, although at a first glance it seems that the two are unrelated since, unlike a premorphism as a map with codomain an inverse monoid, in Definition 1.2, the map μ has codomain U which is not given an inverse monoid structure. Luckily, we can overcome this difficulty very easily. The fact that for all $x \in G$, $\mu(x) \leq \mu(e)$ says exactly that $\sup(U) = \mu(e)$ and that this belongs to U. Also U is clearly a poset, where the order is the one inherited by the usual order in $[0,1]$, and as explained above, U has a greatest element. We remark here the general fact that each poset $U \subseteq [0,1]$ which has a greatest element α can be regarded as an inverse monoid with multiplication $\wedge$ defined by

$$u \wedge v = \min\{u, v\}.$$

The unit of $(U, \wedge)$ is clearly the greatest element α and so $(U, \wedge)$ becomes a lower semilattice with a unit element. This fact will be used several times in our future proofs without further mention. Now we explain how a fuzzy subgroup (G, μ, U) as in Definition 1.2 can be interpreted as a dual premorphism from the group G onto the inverse monoid $(U, \wedge)$ and conversely. First, we observe that the order $\leq$ in U coincides with the natural order $\preceq$ in the inverse monoid $(U, \wedge)$. Indeed, for every $x, y \in U$,

$$\begin{aligned} x \leq y &\Leftrightarrow x \wedge x = y \wedge x \\ &\Leftrightarrow x \preceq y \qquad\qquad \text{(since } x \text{ is an idempotent of } (U, \wedge)\text{)}. \end{aligned}$$

Secondly, as a result of this observation we see that the map $\mu : G \to U$ which satisfies (1) and (2) of definition 1.2 is nothing but a dual premorphism from the group G onto the inverse monoid $(U, \wedge)$, and conversely. Indeed, it is obvious that if μ satisfies (1) of Definition 1.2, then it satisfies (i) of Definition 1.3, and conversely. Also, if μ satisfies (2) of Definition 1.2, then for all $x \in G$,

$$\mu(x^{-1}) = \mu(x) = (\mu(x))^{-1},$$

where the second equality holds true since in $(U, \wedge)$ every element coincides with its own inverse. This proves that μ satisfies (ii) of Definition 1.3. Conversely, any map $\mu : G \to U$ that satisfies (ii) of Definition 1.3, satisfies also (2) of Definition 1.2 since

$$\mu(x^{-1}) = (\mu(x))^{-1} = \mu(x),$$

for every $x \in G$.

Finally, for everything unexplained here on semigroups in general and inverse semigroups in particular, we refer the reader to the monographs [2], [3] and [7]. While for basics on fuzzy sets and fuzzy groups we refer the reader to [6], [8] and [11]. The book of Mac Lane [5] contains the necessary material on categories and functors.

2. The definitions of $\mathfrak{FG}$ and $\mathfrak{FC}$

We define the category of fuzzy subgroups $\mathfrak{FG}$ in the following way. The objects of $\mathfrak{FG}$ are triples (G, μ, U) as defined in Definition 1.2, and if (G, μ_1, U) and (H, μ_2, V) are two such triples, a morphism from (G, μ_1, U) to (H, μ_2, V) is a pair (f, λ) where $f : G \to H$ is a group homomorphism, and $\lambda : U \to V$ is an order preserving map with the property that $\lambda(\sup(U)) = \sup(V)$, and, for all $x \in G$

$$\mu_2 f(x) = \lambda \mu_1(x).$$

The unit morphism on an object (G, μ, U) is defined to be the pair $(1_G, 1_U)$. Now if (K, μ_3, W) is another object from $\mathfrak{FG}$, and $(g, \lambda') : (H, \mu_2, V) \to$

(K, μ_3, W) is another a morphism, we define the composition $(g, \lambda') \circ (f, \lambda)$ as the pair $(gf, \lambda'\lambda)$. We will show that this pair is indeed a morphism from (G, μ_1, U) to (K, μ_3, W). For every $x \in G$,

$$\begin{aligned} \mu_3 gf(x) &= \lambda' \mu_2 f(x) \\ &= \lambda' \lambda \mu_1(x). \end{aligned}$$

Also $\lambda'\lambda$ is order preserving, and $\lambda'\lambda(\sup(U)) = \lambda'(\sup(V)) = \sup(W)$. The properties that $\mathfrak{FG}$ should satisfy to be a category are straightforward. In the introduction we remarked that in the definition of an object (G, μ, U) we can regard U as an inverse monoid with multiplication $\wedge$ and μ as a premorphism. But then, the definition of a morphism in $\mathfrak{FG}$ between any two such objects would involve mixed concepts, because on the one hand we have homomorphisms of groups, and on the other hand, order preserving maps between posets. This is not an inconsistency since for any two posets U and V with respective greatest elements α and β, it holds true that every order preserving map $\lambda : U \to V$ which sends α to β, is in fact a homomorphism between monoids $(U, \wedge)$ and $(V, \wedge)$. Indeed, let $u, v \in U$ such that $u \leq v$, then

$$\begin{aligned} \lambda(u \wedge v) &= \lambda(u) && (\text{since } u \leq v) \\ &= \lambda(u) \wedge \lambda(v) && (\text{since } \lambda(u) \leq \lambda(v)). \end{aligned}$$

In addition to that, the fact that $\lambda(\alpha) = \beta$ says that λ is a homomorphism of monoids. The converse is also true, that is, any homomorphism of monoids $\lambda : (U, \wedge) \to (V, \wedge)$, is an order preserving map since for every $u, v \in [0, \alpha]$ such that $u \leq v$,

$$\begin{aligned} \lambda(u) &= \lambda(u \wedge v) && (\text{since } u \leq v) \\ &= \lambda(u) \wedge \lambda(v) && (\text{since } \lambda \text{ is a homomorphism}), \end{aligned}$$

which implies that $\lambda(u) \leq \lambda(v)$. The condition that $\lambda(\alpha) = \beta$ follows from the fact that λ is a monoid homomorphism. Finally, we remark that λ maps the greatest element α of the single σ-class of $(U, \wedge)$ to the greatest element β of the single σ-class of $(V, \wedge)$.

The definition of the category $\mathfrak{FC}$ of F-inverse covers of inverse monoids is an extension of the definition of the category $\mathfrak{F}$ of F-inverse semigroups made in [10]. The objects of $\mathfrak{FC}$ are triples (T, M, φ) where T is an F-inverse monoid, M is an inverse monoid, and φ is a homomorphism of monoids which is surjective and idempotent separating. We say that T is an F-inverse cover of M over T/σ. If now (T', M', φ') is another triple as above, then a morphism from (T, M, φ) to (T', M', φ') is a pair (f_*, λ) with $f_* : T \to T'$ and $\lambda : M \to M'$ monoid morphisms which map the greatest element of every σ-class onto the greatest element of some σ-class, and that satisfy the commutativity condition $\varphi' f_* = \lambda \varphi$. The identity morphism on the object (T, M, φ) is defined as the pair $(1_T, 1_M)$ which clearly satisfies the above commutativity condition. The composition of morphisms is defined in the following fashion. If $(f_*, \lambda) : (T, M, \varphi) \to (T', M', \varphi')$ and $(f'_*, \lambda') : (T', M', \varphi') \to (T'', M'', \varphi'')$ are two morphisms, then their composition is defined to be the pair $(f'_* f_*, \lambda'\lambda)$. This is indeed a morphism from (T, M, φ) to (T'', M'', φ'') since

$$\begin{aligned}\varphi''(f'_* f_*) &= (\varphi'' f'_*) f_* \\ &= (\lambda'\varphi') f_* = \lambda'(\varphi' f_*) \\ &= \lambda'(\lambda\varphi) = (\lambda'\lambda)\varphi,\end{aligned}$$

and that both compositions $f'_* f_*$ and $\lambda'\lambda$ map the greatest element of a σ-class onto the greatest element of some σ-class since their respective components do so. Finally, it is easy to see that $\mathfrak{FC}$ is indeed a category.

3. The embedding

Looking back to the definition of an object (G, μ, U) from $\mathfrak{FG}$, but with U regarded now as an inverse monoid $(U, \wedge)$, we have already observed that the map $\mu : G \to U$ is nothing but a dual premorphism between inverse monoids. We also note that μ satisfies the extra property that for every $u \in U$, there exists $x \in G$ such that $u \leq \mu(x)$. One such x is for instance the unit of G. It follows from Theorem 1.4, that there is an F-inverse cover of U over G which we write with the long notation $\mathfrak{C}(G, \mu, U)$. More explicitly,

$$\mathfrak{C}(G, \mu, U) = \{(u, x) \in U \times G | u \leq \mu(x)\}$$

is an inverse monoid whose idempotents turn out to be all the pairs $(u, 1)$, where 1 is the unit of G, in particular the unit element is $(\mu(1), 1)$. The natural order has a simple description: $(u, x) \preceq (v, y)$ if and only if $y = x$ and $u \leq v$. The σ-class of an element (u, x) consists of all the elements (v, x) with $v \leq \mu(x)$ and its greatest element with respect to the natural order is $(\mu(x), x)$. Finally note that the projection in the first coordinate $\varphi : \mathfrak{C}(G, \mu, U) \to U$, $(u, x) \mapsto u$ is a surjective homomorphism and idempotent separating. We call the triple $(\mathfrak{C}(G, \mu, U), \varphi, U)$ the F-inverse cover associated with the fuzzy subgroup (G, μ, U). The monoid $\mathfrak{C}(G, \mu, U)$ seems to be useful in connecting inverse semigroups with fuzzy subgroups. An argument which goes in favor of this is that the $\mathcal{H}$-classes of $\mathfrak{C}(G, \mu, U)$ correspond in a way that will be made precise below, to the so called level subsets of (G, μ, U). Level subsets are defined in [9] as follows. Given a fuzzy subgroup (G, μ, U) and $u \in U$, then the level subset μ_u of the fuzzy subset μ is defined by

$$\mu_u = \{h \in G | \mu(h) \geq u\}.$$

It is proved in Theorem 2.1 of [9] that such subsets are in fact subgroups of G. Before we see the connection they have with the $\mathcal{H}$-classes of $\mathfrak{C}(G, \mu, U)$, we note that $\mathfrak{C}(G, \mu, U)$ is a Clifford monoid. Indeed, it is an inverse monoid by its definition, and its idempotents are central. To see the latter, let $(u, 1)$ be an idempotent, and (v, h) an arbitrary element, then

$$(u, 1)(v, h) = (u \wedge v, h) = (v \wedge u, h) = (v, h)(u, 1).$$

To see what an $\mathcal{H}$-class looks like, we recall first that the relations $\mathcal{H}$ and $\mathcal{R}$ coincide in Clifford semigroups. Let now $(v,h) \in \mathfrak{C}(G,\mu,U)$ be such that $(v,h)\mathcal{R}(u,1)$ where $(u,1)$ is some idempotent. There are $(w,a),(w',b) \in \mathfrak{C}(G,\mu,U)$ such that

$$(v \wedge w, ha) = (v,h)(w,a) = (u,1)$$

and

$$(u \wedge w', b) = (u,1)(w',b) = (v,h),$$

which together imply that $u = v$. Therefore, if $(v,h) \in H_{(u,1)}$, then necessarily $v = u$. Conversely, any $(u,h) \in \mathfrak{C}(G,\mu,U)$ is $\mathcal{R}$ (hence $\mathcal{H}$)-equivalent with $(u,1)$. Indeed, this follows easily from the fact that $(u,h^{-1}) \in \mathfrak{C}(G,\mu,U)$ and since the following hold true

$$(u,h)(u,h^{-1}) = (u,1) \text{ and } (u,1)(u,h) = (u,h).$$

All we said means that for any fixed $u \in U$, $H_{(u,1)} = \{(u,h) | u \leq \mu(h)\}$ and this forms a subgroup of $\mathfrak{C}(G,\mu,U)$. Now we show that each level subgroup μ_u is in fact isomorphic to $H_{(u,1)}$. Indeed, the map

$$\phi : H_{(u,1)} \to \mu_u \text{ such that } (u,h) \mapsto h,$$

is clearly bijective and a homomorphism.

Now we prove our main result.

Theorem 3.1. *There is a full and faithful embedding of the category $\mathfrak{FG}$ of fuzzy subgroups into the category $\mathfrak{FC}$ of F-inverse covers of inverse monoids.*

Proof. Define $\Omega : \mathfrak{FG} \to \mathfrak{FC}$ on objects by sending each fuzzy subgroup (G,μ,U) to its corresponding F-inverse cover $(\mathfrak{C}(G,\mu,U),\varphi,U)$. Further, for each morphism $(f,\lambda) : (G,\mu_1,U) \to (H,\mu_2,V)$ in $\mathfrak{FG}$, if the corresponding F-inverse covers are respectively $(\mathfrak{C}(G,\mu_1,U),\varphi_1,U)$ and $(\mathfrak{C}(H,\mu_2,V),\varphi_2,V)$, then we define

$$f_* : (\mathfrak{C}(G,\mu_1,U) \to \mathfrak{C}(H,\mu_2,V)$$

by setting

$$f_*(u,x) = (\lambda(u), f(x)).$$

This map is correct since $\lambda(u) \leq \mu_2(f(x))$. Indeed, from the definition of the morphism (f,λ), we see that

$$\begin{aligned} \mu_2 f(x) &= \lambda\mu_1(x) \\ &\geq \lambda(u) && \text{(since } \mu_1(x) \geq u). \end{aligned}$$

Also f_* is a monoid homomorphism since if $(u,x),(v,y) \in \mathfrak{C}(G,\mu_1,U)$ such that $u \leq v$, then

$$\begin{aligned} f_*((u,x)(v,y)) &= f_*(u \wedge v, xy) \\ &= f_*(u, xy) \\ &= (\lambda(u), f(xy)) \\ &= (\lambda(u) \wedge \lambda(v), f(x)f(y)) && \text{(since } \lambda(u) \leq \lambda(v)) \\ &= (\lambda(u), f(x))(\lambda(v), f(y)) \\ &= f_*(u,x)f_*(v,y). \end{aligned}$$

Also if α, β are the respective units of U, V, and e_1, e_2 the units of G, H respectively, then
$$f_*(\alpha, e) = (\lambda(\alpha), f(e_1)) = (\beta, e_2),$$
so that f_* preserves the unit element. Further, we see that for every $(u, x) \in \mathfrak{C}(G, \mu_1, U)$,

$$\begin{aligned}\varphi_2 f_*(u, x) &= \varphi_2(\lambda(u), f(x)) \\ &= \lambda(u) \\ &= \lambda\varphi_1(u, x).\end{aligned}$$

Lastly, if $(\mu_1(x), x)$ is the greatest element of its σ-class, then

$$f_*(\mu_1(x), x) = (\lambda(\mu_1(x)), f(x)) = (\mu_2(f(x)), f(x)),$$

where $(\mu_2(f(x)), f(x))$ is the greatest element of its σ-class. Since in addition to what we said, λ is a homomorphism of inverse monoids that maps the greatest element α of the only σ-class of U to the greatest element β of the only σ-class of V, then it follows that the pair $\Omega(f, \lambda) = (f_*, \lambda)$ is a morphism from $(\mathfrak{C}(G, \mu_1, U), \varphi_1, U)$ to $(\mathfrak{C}(H, \mu_2, V), \varphi_2, V)$. Next we show that Ω is functorial. It is obvious that when $(f, \lambda) = (1_G, 1_U)$ is the identity on (G, μ, U), then $\Omega(1_G, 1_U) = id_{(\mathfrak{C}(G,\mu,U),\varphi,U)}$. Let now

$$(f, \lambda) : (G, \mu_1, U) \to (H, \mu_2, V)$$

and
$$(f', \lambda') : (H, \mu_2, V) \to (K, \mu_3, W)$$
be two morphisms in $\mathfrak{FG}$, and

$$(f'f, \lambda'\lambda) : (G, \mu_1, U) \to (K, \mu_3, W)$$

their composition, and we want to prove that

$$\Omega(f'f, \lambda'\lambda) = \Omega(f', \lambda')\Omega(f, \lambda),$$

or equivalently that
$$((f'f)_*, \lambda'\lambda) = (f'_*, \lambda')(f_*, \lambda).$$
This is the same as to prove that $(f'f)_* = f'_* f_*$. The latter is true since for every $(u, x) \in \mathfrak{C}(G, \mu_1, U)$ we have that

$$\begin{aligned}(f'f)_*(u, x) &= ((\lambda'\lambda)(u), (f'f)(x)) \\ &= (\lambda'(\lambda(u)), f'(f(x))) \\ &= f'_*(\lambda(u), f(x)) \\ &= f'_*(f(u, x)).\end{aligned}$$

Next we prove that Ω is faithful. Let (G, μ_1, U) and (H, μ_2, V) be two objects in $\mathfrak{FG}$ and
$$(f, \lambda), (f', \lambda') : (G, \mu_1, U) \to (H, \mu_2, V)$$

be two parallel morphisms, and assume that $\Omega(f,\lambda)=\Omega(f',\lambda')$. Then, from the definition of Ω, $(f_*,\lambda)=(f'_*,\lambda')$, consequently $\lambda=\lambda'$ and $f_*=f'_*$. The second equality implies that for every $x\in G$ and every $(u,x)\in\mathfrak{C}(G,\mu_1,U)$ we have that $f_*(u,x)=f'_*(u,x)$. It follows that $(\lambda(u),f(x))=(\lambda'(u),f'(x))$, consequently $f(x)=f'(x)$. Since the second coordinates of pairs from $\mathfrak{C}(G,\mu_1,U)$ cover the whole of G, then it follows that $f=f'$.

Finally we prove that Ω is full. Let again (G,μ_1,U) and (H,μ_2,V) be two objects in $\mathfrak{FG}$ and

$$(g,\lambda):(\mathfrak{C}(G,\mu_1,U),\varphi_1,U)\to(\mathfrak{C}(H,\mu_2,V),\varphi_2,V)$$

be a morphism from $\Omega(G,\mu_1,U)$ to $\Omega(H,\mu_2,V)$. We show that g induces a homomorphism $f:G\to H$ such that $g=f_*$ and $\mu_2 f=\lambda\mu_1$. This would prove that $(f,\lambda):(G,\mu_1,U)\to(H,\mu_2,V)$ is a morphism in $\mathfrak{FG}$ such that $(g,\lambda)=\Omega(f,\lambda)$. For every $x\in G$, let $(\mu_1(x),x)$ be the greatest element of its σ-class in $\mathfrak{C}(G,\mu_1,U)$, and let $(\mu_2(x'),x')=g(\mu_1(x),x)$ which is, from the assumption on g, the greatest element of its σ-class in $\mathfrak{C}(H,\mu_2,V)$. It follows that

$$\mu_2(x')=\varphi_2(\mu_2(x'),x')=\varphi_2 g(\mu_1(x),x)=\lambda\varphi_1(\mu_1(x),x)=\lambda\mu_1(x). \tag{3.1}$$

We derive from [10] that g induces a unique homomorphism

$$\tilde{f}:\mathfrak{C}(G,\mu_1,U)/\sigma\to\mathfrak{C}(H,\mu_2,V)/\sigma$$

which maps the σ-class $[(\mu_1(x),x)]$ to the σ-class $[(\mu_2(x'),x')]$, where from above $(\mu_2(x'),x')=g(\mu_1(x),x)$. Considering now the ismorphisms

$$\gamma:G\to\mathfrak{C}(G,\mu_1,U)/\sigma \text{ such that } x\mapsto[(\mu_1(x),x)],$$

and

$$\kappa:H\to\mathfrak{C}(H,\mu_2,V)/\sigma \text{ such that } y\mapsto[(\mu_2(y),y)],$$

we obtain a homomorphism

$$f=\kappa^{-1}\tilde{f}\gamma:G\to H$$

such that $f(x)=x'$ where x' is determined as above. Using (3.1) we see that $\mu_2 f(x)=\lambda\mu_1(x)$, hence (f,λ) is a morphism in $\mathfrak{FG}$ from (G,μ_1,U) to (H,μ_2,V). Now we prove that $(g,\lambda)=\Omega(f,\lambda)=(f_*,\lambda)$ which amounts to saying that $g=f_*$. Before we prove this, we observe that for every $(u,x)\in\mathfrak{C}(G,\mu_1,U)$, the second coordinate of $g(u,x)$ is $x'=f(x)$ as determined above, since g preserves σ-classes, while the first coordinate is

$$\varphi_2 g(u,x)=\lambda\varphi_1(u,x)=\lambda(u).$$

So

$$g(u,x)=(\lambda(u),f(x))=f_*(u,x),$$

consequently, $g=f_*$ as claimed. This completes the proof. $\square$

Acknowledgement

The author wishes to thank the anonymous referee for his (her) valuable comments and suggestions

References

[1] Auinger, K., and Szendrei, M. B. On f inverse covers of inverse monoids. *J. Pure Appl. Algebra. 204*, 3 (2006), 493–506.

[2] Howie, J. M. *Fundamentals of Semigroup Theory.* London Mathematical Society Monographs New Series. Clarendon Press, Oxford, 1995.

[3] Lawson, M. V. *Inverse Semigroup. The Theory of Partial Symmetries.* World Scientific, 1998.

[4] Lawson, M. V., Margolis, S. W., and Steinberg, B. Expansions of inverse semigroups. *J. Aust. Math. Soc. 80* (2006), 205–228.

[5] Mac Lane, S. *Categories for the Working Mathematician*, vol. 5 of *Graduate Texts in Mathematics.* Springer, 1997.

[6] Mordeson, J. N., Bhutani, K. R., and Rosenfeld, A. *Fuzzy Group Theory*, vol. 182 of *Studies in Fuzziness and Soft Computing.* Springer, 2005.

[7] Petrich, M. *Inverse Semigroups.* John Wiley & Sons, 1984.

[8] Rosenfeld, A. Fuzzy groups. *J. Math. Anal. Appl. 35* (1971), 512–517.

[9] Sivaramakrishna Das, P. Fuzzy groups and level subgroups. *J. Math. Anal. Appl. 84* (1981), 264–269.

[10] Szendrei, M. B. A note on birget-rhodes expansions of groups. *J. Pure Appl. Algebra 58*, 1 (1989), 93–99.

[11] Zadeh, L. A. Fuzzy sets. *Inform. and Comput. 8*, 3 (1965), 338–353.

Received by the editors January 18, 2021
First published online January 13, 2022