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Warped-twisted products and Einstein-like manifolds

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Abstract. We study warped-twisted product manifolds in the form ${}_{f_2}M_1 \times_{f_1} M_2$ with warping function f_2 on M_2 and twisting function f_1. We give a necessary and sufficient condition for a warped-twisted product to be a doubly warped product. We also give some conditions for such manifolds to be a twisted and a base conformal warped product. Moreover, we give some results for Einstein-like warped-twisted product manifolds of different classes.

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1. Introduction

Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds and f_1 and f_2 be positive smooth functions on $M_1 \times M_2$. Then *doubly twisted product manifold*[11] $M =_{f_2} M_1 \times_{f_1} M_2$ is the product manifold $M_1 \times M_2$ furnished with the metric tensor $g = {f_2}^2 \pi_1^* g_1 + {f_1}^2 \pi_2^* g_2$, where $\pi_i : M_1 \times M_2 \to M_i$ is canonical projections for $i = 1, 2$. Each function f_i is called a twisting function of $(M =_{f_2} M_1 \times_{f_1} M_2, g)$.

If the functions f_1 and f_2 in the above, depend only on the points of M_1 and M_2, respectively, then we get *doubly warped product manifold* $M =_{f_2} M_1 \times_{f_1} M_2$ [5] with the metric g given by $g = (f_2 \circ \pi_2)^2 \pi_1^* g_1 + (f_1 \circ \pi_1)^2 \pi_2^* g_2$. Then each function f_i is called a *warping function* of $({}_{f_2}M_1 \times_{f_1} M_2, g)$ for $i = 1, 2$. If $f_1 \equiv 1$ or $f_2 \equiv 1$ in the definition of doubly twisted product manifold, then $(M_1 \times_{f_1} M_2, g)$ or $({}_{f_2}M_1 \times M_2, g)$ is called a *twisted product manifold* [2]. In that case f_1 or f_2 is called a *twisting function.*

Moreover, $f_1 \equiv 1$ or $f_2 \equiv 1$, in the definition of doubly warped product manifold, then we get $(M_1 \times_{f_1} M_2, g)$ *warped product manifold* [1] with warping function f_1.

In [13], we defined a new subclass of doubly twisted product under the name of *nearly doubly twisted product of type 1.* In this article, we rename of such products as *warped-twisted products.*

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Let $({}_{f_2}M_1 \times_{f_1} M_2, g)$ be a doubly twisted product manifold. If f_2 depends only on the points of M_2, then $({}_{f_2}M_1 \times_{f_1} M_2, g)$ is a warped-twisted product manifold with the metric tensor $g = (f_2 \circ \pi_2)^2 \pi_1^* g_1 + {f_1}^2 \pi_2^* g_2$. In which case, f_2 is called a warping function and f_1 is called a twisting function of $({}_{f_2}M_1 \times_{f_1} M_2, g)$. In this case, if the function f_1 depends only on the the points of M_2, then the warped-twisted product ${}_{f_2}M_1 \times_{f_1} M_2$ becomes a *base conformal warped product* [4]. We say that a warped-twisted product is *non-trivial* if it is neither doubly warped product nor warped product or base conformal warped product.

Remark 1.1. Let (M_1, g_1) and (M_2, g_2) be pseudo-Riemannian manifolds with Levi-Civita connections ∇^1 and ∇^2, respectively. By usual convenience, we denote the set of lifts of vector fields on M_i by $\mathcal{L}(M_i)$ and use the same notation for a vector field and for its lift. On the other hand, each π_i is a positive homothety, so it preserves the Levi-Civita connection. Thus, there is no confusion using the same notation for a connection on M_i and for its pullback via π_i.

Now, we recall some facts for later use [10].

Let $(\bar{M}, \bar{g})$ be a Riemannian manifold of dimension m and $\bar{\nabla}$ be the Levi-Civita connnection on $\bar{M}$. Let $\{e_1, e_2, \ldots, e_m\}$ denote the local orthonormal frame field of $(\bar{M}, \bar{g})$.

The Ricci tensor S of $(\bar{M}, \bar{g})$ is defined by

$$S(\bar{U}, \bar{V}) = \sum_{i=1}^{m} \bar{g}(R(e_i, \bar{U})\bar{V}, e_i) \tag{1.1}$$

for any vector fields $\bar{U}, \bar{V}$ on $\bar{M}$, where R is the Riemannian curvature tensor of $\bar{M}$.

The scalar curvature τ of $(\bar{M}, \bar{g})$ is given by

$$\tau = \sum_{i=1}^{m} S(e_i, e_i). \tag{1.2}$$

For a vector field $\bar{U}$ on $\bar{M}$, divergence of $\bar{U}$ is defined by

$$div\bar{U} = \sum_{i=1}^{m} \bar{g}(\bar{\nabla}_{e_i}\bar{U}, e_i). \tag{1.3}$$

For a function $f \in C^\infty(\bar{M})$ and a vector field $\bar{U}$ on $\bar{M}$, the Hessian tensor of f is given by

$$H^f(\bar{U}) = \bar{\nabla}_{\bar{U}} \nabla f \tag{1.4}$$

and the Hessian form of f is given by

$$h^f(\bar{U}, \bar{V}) = \bar{g}(H^f(\bar{U}), \bar{V}). \tag{1.5}$$

For a function $f \in C^\infty(\bar{M})$, the Laplacian of f is defined by

$$\Delta f = div \nabla f. \tag{1.6}$$

2. Warped Twisted Product Manifolds

In this section, we give the covariant derivative formulas, Ricci tensor, Weyl conformal curvature tensor and scalar curvature of a warped-twisted product manifolds in the form ${}_{f_2}M_1 \times_{f_1} M_2$, where f_2 is a warping function on M_2 and f_1 is a twisting function. We also give a necessary and sufficient condition for such manifolds to be a doubly warped product. We also give some conditions for such manifolds to be a twisted and a base conformal warped product.

Remark 2.1. From now on, throughout this article, M denotes a warped-twisted product manifold in the form ${}_{f_2}M_1 \times_{f_1} M_2$ with warping function $f_2 \in C^\infty(M_2)$, twisting function f_1 and endowed with the Riemanninan metric g.

Lemma 2.2. *Let M be a warped-twisted product manifold. Then, we have*

$$\nabla_X Y = \nabla^1_X Y - g(X,Y)\nabla\tilde{l}, \tag{2.1}$$

$$\nabla_X U = \nabla_U X = U(\tilde{l})X + X(k)U, \tag{2.2}$$

$$\nabla_U V = \nabla^2_U V + U(k)V + V(k)U - g(U,V)\nabla k, \tag{2.3}$$

for $X, Y \in \mathcal{L}(M_1)$ and $U, V \in \mathcal{L}(M_2)$, where $k = \log f_1$, $l = \log f_2$ and $\tilde{l} = l \circ \pi_2$ which is the pullback of l via π_2.

Proof. The proof follows from Proposition 1 of [7] with $X(\tilde{l}) = 0$, for $X \in \mathcal{L}(M_1)$. □

For $X, Y \in \mathcal{L}(M_1)$, we define [7]

$$h^k_1(X,Y) = XY(k) - (\nabla^1_X Y)(k). \tag{2.4}$$

Then the Hessian form h^k of k on (M,g) satisfies

$$h^k(X,U) = XU(k) - X(k)U(l) - X(k)V(k), \tag{2.5}$$

$$h^k(X,Y) = h^k_1(X,Y) - X(l)Y(k) - X(k)Y(l) + g(X,Y)g(\nabla k, \nabla l), \tag{2.6}$$

where $U \in \mathcal{L}(M_2)$.

Remark 2.3. From now on, throughout this paper, we denote by l the pullback of l, i.e., $l = l \circ \pi_2$.

Now let S and S^i be Ricci tensor of (M,g) and (M_i, g_i), respectively. Then we have the following relations:

Lemma 2.4. *Let M be a warped-twisted product manifold. Then, we have*

$$\begin{aligned} S(X,Y) &= S^1(X,Y) + h^l(X,Y) - m_2\Big\{h^k_1(X,Y) + X(k)Y(k)\Big\} \\ &\quad -g(X,Y)\Big\{\Delta l + g(\nabla l, \nabla l)\Big\}, \end{aligned} \tag{2.7}$$

$$S(X,U) = (1-m_2)XU(k) + (m_1+m_2-2)X(k)U(l), \tag{2.8}$$

$$\begin{aligned} S(U,V) &= S^2(U,V) + h^k(U,V) + (1-m_2)h_2^k(U,V) + m_2U(k)V(k) \\ &\quad -g(U,V)\Big\{\Delta k + g(\nabla k, \nabla k)\Big\} \\ &\quad -m_1\Big\{h_2^l(U,V) + U(l)V(l) - U(l)V(k) - U(k)V(l)\Big\} \end{aligned} \tag{2.9}$$

for $X, Y \in \mathcal{L}(M_1)$ *and* $U, V \in \mathcal{L}(M_2)$.

Proof. The above equations can be easily obtained from Proposition 3 of [7] with $X(l) = 0$, for $X \in \mathcal{L}(M_1)$. □

Now let τ^1 and τ^2 be scalar curvature tensors of (M_1, g_1) and (M_2, g_2), respectively. Let $\{e_1, \ldots, e_{m_1}, e_{m_1+1}, \ldots, e_{m_1+m_2}\}$ be an orthonormal basis of M, where $\{f_2e_1, \ldots, f_2e_{m_1}\}$ is an orthonormal basis of M_1 and $\{f_1e_{m_1+1}, \ldots, f_1e_{m_1+m_2}\}$ is an orthonormal basis of M_2. Then we have the following relation from Lemma 2.4:

Lemma 2.5. *Let M be a warped-twisted product manifold and let τ be the scalar curvature of M. Then, we have*

$$\begin{aligned} \tau &= \frac{\tau^1}{{f_2}^2} + \frac{\tau^2}{{f_1}^2} + \tilde{\Delta}_1(l) + \tilde{\Delta}_2(k) - \frac{m_2}{{f_2}^2}\Delta_1(k) - \frac{m_1}{{f_1}^2}\Delta_2(l) \\ &\quad + \frac{(1-m_2)}{{f_1}^2}\Delta_2(k) - m_2 g(P_1\nabla k, P_1\nabla k) - m_1\Delta l - 2m_1 g(\nabla l, \nabla l) \\ &\quad -m_2\Big\{\Delta k + g(\nabla k, \nabla k)\Big\} + m_2 g(P_2\nabla k, P_2\nabla k) + 2m_1 g(P_2\nabla k, \nabla l) \end{aligned} \tag{2.10}$$

for $X, Y \in \mathcal{L}(M_1)$ *and* $U, V \in \mathcal{L}(M_2)$, *where* $\tilde{\Delta}_1(k) = \sum_{i=1}^{m_1} h^k(e_i, e_i)$,

$\tilde{\Delta}_2(k) = \sum_{i=m_1+1}^{m_1+m_2} h^k(e_i, e_i)$, $\Delta k = \tilde{\Delta}_1(k) + \tilde{\Delta}_2(k)$ *and* $\nabla k = P_1\nabla k + P_2\nabla k$.

Definition 2.6. Let $(\bar{M}, \bar{g})$ be a Riemannian manifold of dimension m. Then the *Weyl conformal curvature tensor* field of $\bar{M}$ is the tensor field $\mathcal{W}$ of type $(1,3)$ defined as

$$\begin{aligned} &\mathcal{W}(\bar{X}, \bar{Y})\bar{Z} \\ &= R_{\bar{X}\bar{Y}}\bar{Z} \\ &\quad + \frac{1}{m-2}\Big\{S(\bar{X}, \bar{Z})\bar{Y} - S(\bar{Y}, \bar{Z})\bar{X} + \bar{g}(\bar{X}, \bar{Z})Q\bar{Y} - \bar{g}(\bar{Y}, \bar{Z})Q\bar{X}\Big\} \\ &\quad - \frac{\tau}{(m-1)(m-2)}\Big\{\bar{g}(\bar{X}, \bar{Z})\bar{Y} - \bar{g}(\bar{Y}, \bar{Z})\bar{X}\Big\} \end{aligned} \tag{2.11}$$

for any vector fields $\bar{X}, \bar{Y}$ and $\bar{Z}$ on $\bar{M}$, where Q is the Ricci operator and τ is the scalar curvature of $\bar{M}$ [17].

Lemma 2.7. *Let M be a warped-twisted product manifold. Then, for $X \in \mathcal{L}(M_1)$ and $U, V \in \mathcal{L}(M_2)$, the Weyl conformal curvature tensor $\mathcal{W}$ satisfies*

$$\mathcal{W}(U,V)X = \frac{(m_1-1)}{(m_1+m_2-2)}\Big\{XU(k)V - XV(k)U\Big\}, \tag{2.12}$$

where $m_i = dimM_i$, i=1,2.

Proof. Let $X \in \mathcal{L}(M_1)$ and $U, V \in \mathcal{L}(M_2)$. Then we have

$$\mathcal{W}(U,V)X = R(U,V)X + \frac{1}{(m_1+m_2-2)}\Big\{S(U,X)V - S(V,X)U\Big\}, \tag{2.13}$$

from (2.11), since $g(U,X) = g(V,X) = 0$. If we use (2.8) in (2.13), we get (2.12). □

Definition 2.8. Let M be a warped-twisted product manifold. Then we say that M_2 is *Weyl conformal flat* along M_1 if $\mathcal{W}(U,V) = 0$ and M_1 is *Weyl conformal flat* along M_2 if $\mathcal{W}(X,Y) = 0$, where $X, Y \in \mathcal{L}(M_1)$ and $U, V \in \mathcal{L}(M_2)$.

Now, we are ready to give an another main result.

Theorem 2.9. *Let M be a warped-twisted product manifold and $dimM_1 > 1$. Then (M,g) can be expressed as a doubly warped product manifold ${}_{f_2}M_1 \times_f M_2$ of (M_1, g_1) and $(M_2, \tilde{g_2})$ with warping functions f_1 and f, if and only if M_2 is Weyl conformal flat on M_1, where $\tilde{g_2} = f^2 g_2$ for some positive smooth function f on M_2.*

Proof. If (M,g) is a doubly warped product manifold of (M_1, g_1) and $(M_2, \tilde{g_2})$ with warping functions f_1 and f, then we have $V(k) = 0$ for $V \in \mathcal{L}(M_2)$. Thus, the assertion comes immediately from (2.12).

Conversely, if $\mathcal{W}(U,V)X = 0$ for all $X \in \mathcal{L}(M_1)$ and $U, V \in \mathcal{L}(M_2)$, then we have

$$XV(k)U - XU(k)V = 0, \tag{2.14}$$

from (2.12). For linearly independent vector field U and V, we deduce that $XV(k) = XU(k) = 0$ from (2.14). Hence, it follows that $f_1 = e^k = a(x)b(y)$, where a and b are positive smooth functions on M_1 and M_2, respectively. Then, we can write the metric g as $g = f_2^2(y)g_1 + a^2(x)\tilde{g_2}$, where $\tilde{g_2} = b^2(y)g_2$. Therefore, $({}_{f_2}M_1 \times_{f_1} M_2, g)$ is a doubly warped product with $f = a$. □

Definition 2.10. A vector field $\bar{V}$ on a Riemannian manifold $(\bar{M}, \bar{g})$ is called *torse-forming*, if it satisfies [15]

$$\bar{\nabla}_{\bar{X}}\bar{V} = \lambda\bar{X} + \mu(\bar{X})\bar{V} \tag{2.15}$$

for any vector field $\bar{X}$ on $\bar{M}$, where λ is a function, μ is a 1-form. If the 1-form μ in (2.15) vanishes identically, then the vector field $\bar{V}$ is called *concircular* [3, 12, 16, 14].

Now, we give another characterization for a warped-twisted product admitting a concircular vector field.

Proposition 2.11. *Let M be a warped-twisted product manifold. Then*

a) *If M admits a concircular vector field in $\mathcal{L}(M_1)$, then ${}_{f_2}M_1 \times_{f_1} M_2$ is a twisted product.*

b) *If M admits a concircular vector field in $\mathcal{L}(M_2)$, then ${}_{f_2}M_1 \times_{f_1} M_2$ is a base-conformal warped product.*

Proof. Let M be a warped-twisted product manifold. Then

a) Let X be a concircular vector field in $\mathcal{L}(M_1)$, then for any $V \in \mathcal{L}(M_2)$, we have

$$\nabla_V X = \lambda V \tag{2.16}$$

from (2.15). By using (2.2), we obtain

$$V(l)X + X(k)V = \lambda V$$

from (2.16). Hence, we get $V(l) = 0$ for all $V \in \mathcal{L}(M_2)$. Which means that the function l is constant, so f_2 is also constant. Thus, ${}_{f_2}M_1 \times_{f_1} M_2$ is a twisted product.

b) Let U be a concircular vector field in $\mathcal{L}(M_2)$, then for any $Y \in \mathcal{L}(M_1)$, we have

$$\nabla_Y U = \lambda Y \tag{2.17}$$

from (2.15). By using (2.2), we obtain

$$U(l)Y + Y(k)U = \lambda Y$$

from (2.17). Hence, we get $Y(k) = 0$ for all $Y \in \mathcal{L}(M_1)$. Which means that the function $k = \ln f_1$ depends only points of M_2. Thus, ${}_{f_2}M_1 \times_{f_1} M_2$ is a base-conformal warped product. □

3. Einstein-Like Warped-Twisted Products

In [8], Gray defined Einstein-like Riemannian manifolds of different classes. Mantica and Shenawy studied Einstein-like warped product manifolds in [9]. Besides this, El-Sayied and etc. [6] consider the Einstein-like doubly warped products and its applications. In this section, we introduce the different classes of Einstein-like warped-twisted product manifolds.

Class $\mathcal{A}$. A Riemannian manifold $(\bar{M}, \bar{g})$ admitting a *cyclic parallel Ricci tensor*, that is,

$$(\bar{\nabla}_{\bar{X}} S)(\bar{Y}, \bar{Z}) + (\bar{\nabla}_{\bar{Y}} S)(\bar{Z}, \bar{X}) + (\bar{\nabla}_{\bar{Z}} S)(\bar{X}, \bar{Y}) = 0,$$

for any vector fields $\bar{X}, \bar{Y}, \bar{Z}$ on $\bar{M}$ is called an *Einstein-like manifold of class $\mathcal{A}$*. It is noted that the above condition is equivalent to

$$(\bar{\nabla}_{\bar{X}} S)(\bar{X}, \bar{X}) = 0, \tag{3.1}$$

for any vector field $\bar{X}$ on $\bar{M}$. The inheritance property of Class $\mathcal{A}$ is given by the following result.

Theorem 3.1. *Let M be an Einstein-like warped-twisted product of class $\mathcal{A}$. Then*

a) *(M_1, g_1) is an Einstein-like manifold of class $\mathcal{A}$ if and only if*

$$\begin{aligned}(\nabla^1_X h^l)(X,X) &= m_2\{(\nabla^1_X h^k_1)(X,X) + 2X(k)h^k_1(X,X)\} \\ &\quad -2g(X,X)S(\nabla l, X),\end{aligned} \tag{3.2}$$

b) *(M_2, g_2) is an Einstein-like manifold of class $\mathcal{A}$ if and only if*

$$\begin{aligned}&(\nabla^2_U h^k)(U,U) + (1-m_2)(\nabla^2_U h^k_2)(U,U) \\ &= m_1(\nabla^2_U h^l_2)(U,U) - 2m_2 h^k_2(U,U)U(k) \\ &\quad +2U(f_1)f_1 g_2(U,U)k^\diamond + g(U,U)U(k^\diamond) \\ &\quad +2m_1\{h^l_2(U,U)U(l) - h^l_2(U,U)U(k) \\ &\quad -h^k_2(U,U)U(l)\} \\ &\quad +4U(k)S(U,U) - 2g(U,U)S(\nabla k, U)\end{aligned} \tag{3.3}$$

for $X \in \mathcal{L}(M_1)$ and $U \in \mathcal{L}(M_2)$, where $k^\diamond = \triangle k + \|\nabla k\|^2$.

Proof. Let M be an Einstein-like warped-twisted product of class $\mathcal{A}$. Then, for any vector field X on M, we have $(\nabla_X S)(X,X) = 0$, from (3.1).

a) (M_1, g_1) is an Einstein-like manifold of class $\mathcal{A}$ if and only if

$$(\nabla^1_X S^1)(X,X) = 0. \tag{3.4}$$

for any vector field X on M_1. On the other hand, from (2.1) and (2.7), we have

$$\begin{aligned}&(\nabla_X S)(X,X) \\ &= X(S(X,X)) - 2S(\nabla_X X, X) \\ &= X(S^1(X,X) + h^l(X,X) - m_2\{h^k_1(X,X) + X(k)X(k)\} - g(X,X)l^\diamond) \\ &\quad -2S(\nabla^1_X X - g(X,X)\nabla l, X). \\ &= X(S^1(X,X)) - 2S^1(\nabla^1_X X, X) + X(h^l(X,X)) - 2h^l(\nabla^1_X X, X) \\ &\quad -m_2\{X(h^k_1(X,X)) - 2h^k_1(X,X)\} - m_2 X(X(k)X(k)) - X(g(X,X)l^\diamond) \\ &\quad +2m_2\nabla^1_X X(k)X(k) + 2g(\nabla^1_X X, X)l^\diamond + 2g(X,X)S(\nabla l, X).\end{aligned}$$

for $X \in \mathcal{L}(M_1)$. Then, we find

$$\begin{aligned}0 &= (\nabla_X S)(X,X) \\ &= (\nabla^1_X S^1)(X,X) + (\nabla^1_X h^l)(X,X) \\ &\quad -m_2(\nabla^1_X h^k_1)(X,X) - 2m_2 X(X(k))X(k) - (\nabla^1_X g)(X,X)l^\diamond \\ &\quad +2m_2\nabla^1_X X(k)X(k) + 2g(X,X)S(\nabla l, X).\end{aligned} \tag{3.5}$$

If we use (2.4) and (3.4), we have

$$\begin{aligned} 0 &= (\nabla_X S)(X,X) \quad (3.6)\\ &= (\nabla^1_X h^l)(X,X) - m_2(\nabla^1_X h^k_1)(X,X)\\ &\quad -2m_2 X(k)h^k_1(X,X) + 2g(X,X)S(\nabla l, X). \quad (3.7)\end{aligned}$$

So, (3.2) follows from (3.6).

b) (M_2, g_2) is an Einstein-like manifold of class $\mathcal{A}$ if and only if

$$(\nabla^2_U S^2)(U,U) = 0 \quad (3.8)$$

for any vector field U on M_2. On the other hand, from (2.3) and (2.9), we have

$$\begin{aligned} &(\nabla_U S)(U,U)\\ &= U(S(U,U)) - 2S(\nabla_U U, U)\\ &= U(S^2(U,U) + h^k(U,U) + (1-m_2)h^k_2(U,U) + m_2 U(k)U(k) - g(U,U)k^\diamond)\\ &\quad -m_1 U(h^l_2(U,U) + U(l)U(l) - U(l)U(k) - U(k)U(l))\\ &\quad -2S(\nabla^2_U U + 2U(k)U - g(U,U)\nabla k, U)\\ &= U(S^2(U,U)) - 2S^2(\nabla^2_U U, U) + U(h^k(U,U)) - 2h^k(\nabla^2_U U, U)\\ &\quad +(1-m_2)\{U(h^k_2(U,U)) - 2h^k_2(\nabla^2_U U, U)\}\\ &\quad -m_1\{U(h^l_2(U,U)) - 2h^l_2(\nabla^2_U U, U)\} + m_2\{U(U(k)U(k)) - 2\nabla^2_U U(k)U(k)\}\\ &\quad -U(g(U,U))k^\diamond + 2g(\nabla^2_U U, U)k^\diamond - g(U,U)U(k^\diamond)\\ &\quad -m_1\{U(U(l)U(l)) - 2U(U(l)U(k))\}\\ &\quad -m_1\{-2\nabla^2_U U(l)U(l) + 2\nabla^2_U U(l)U(k) + 2\nabla^2_U U(k)U(l)\}\\ &\quad -4U(k)S(U,U) + 2g(U,U)S(\nabla k, U)\end{aligned}$$

for $U \in \mathcal{L}(M_2)$. Then, we find

$$\begin{aligned} 0 &= (\nabla_U S)(U,U)\\ &= (\nabla^2_U S^2)(U,U) + (\nabla^2_U h^k)(U,U)\\ &\quad +(1-m_2)(\nabla^2_U h^k_2)(U,U) - m_1(\nabla^2_U h^l_2)(U,U)\\ &\quad +2m_2 h^k_2(U,U)U(k) - (\nabla^2_U g)(U,U)k^\diamond - g(U,U)U(k^\diamond)\\ &\quad -m_1\{2U(U(l))U(l) - 2U(U(l))U(k) - 2U(U(k))U(l)\} \quad (3.9)\\ &\quad -m_1\{-2\nabla^2_U U(l)U(l) + 2\nabla^2_U U(l)U(k) + 2\nabla^2_U U(k)U(l)\}\\ &\quad -4U(k)S(U,U) + 2g(U,U)S(\nabla k, U).\end{aligned}$$

If we use (2.4) and (3.8), we have

$$\begin{aligned} 0 &= (\nabla^2_U h^k)(U,U) + (1-m_2)(\nabla^2_U h^k_2)(U,U)\\ &\quad -m_1(\nabla^2_U h^l_2)(U,U) + 2m_2 h^k_2(U,U)U(k)\\ &\quad -2U(f_1)f_1 g_2(U,U)k^\diamond - g(U,U)U(k^\diamond) \quad (3.10)\\ &\quad -2m_1\{h^l_2(U,U)U(l) - h^l_2(U,U)U(k) - h^k_2(U,U)U(l)\}\\ &\quad -4U(k)S(U,U) + 2g(U,U)S(\nabla k, U).\end{aligned}$$

So, (3.3) follows from (3.10).

□

For a singly warped product, one can obtain the following result.

Corollary 3.2. *Let M be an Einstein-like singly warped product of class $\mathcal{A}$. Then*

a) (M_1, g_1) *is an Einstein-like manifold of class $\mathcal{A}$ if and only if*

$$(\nabla^1_X h^l)(X, X) = 0, \tag{3.11}$$

b) (M_2, g_2) *is an Einstein-like manifold of class $\mathcal{A}$ if and only if*

$$(\nabla^2_U h^l_2)(U, U) + 2h^l_2(U, U)U(l) = 0 \tag{3.12}$$

for $X \in \mathcal{L}(M_1)$ and $U \in \mathcal{L}(M_2)$.

Class $\mathcal{B}$. If the Ricci tensor of a Riemannian manifold $(\bar{M}, \bar{g})$ is a Codazzi tensor, i.e.,

$$(\bar{\nabla}_{\bar{X}} S)(\bar{Y}, \bar{Z}) = (\bar{\nabla}_{\bar{Y}} S)(\bar{X}, \bar{Z}), \tag{3.13}$$

for any vector fields $\bar{X}, \bar{Y}, \bar{Z}$ on $\bar{M}$, then $(\bar{M}, \bar{g})$ is called an *Einstein-like manifold of Class $\mathcal{B}$.* This condition is equivalent to one of the following conditions:

i) The Riemann tensor of $(\bar{M}, \bar{g})$ is harmonic, or

ii) The Weyl conformal tensor of $(\bar{M}, \bar{g})$ is harmonic and the scalar curvature of $(\bar{M}, \bar{g})$ is constant.

The factor manifolds are Einstein-like of Class $\mathcal{B}$ according to the following result.

Theorem 3.3. *Let M be an Einstein-like warped-twisted product of class $\mathcal{B}$. Then*

a) (M_1, g_1) *is an Einstein-like manifold of class $\mathcal{B}$ if and only if*

$$\begin{aligned}(\nabla^1_X h^l)(Y, Z) - (\nabla^1_Y h^l)(X, Z) \\ = \; & m_2\{(\nabla^1_X h^k_1)(Y, Z) - (\nabla^1_Y h^k_1)(X, Z)\} \\ & + m_2\{h^k_1(X, Z)Y(k) - h^k_1(Y, Z)X(k)\} \\ & - g(X, Z)S(Y, \nabla l) + g(Y, Z)S(X, \nabla l),\end{aligned} \tag{3.14}$$

b) *(M_2, g_2) is an Einstein-like manifold of class $\mathcal{B}$ if and only if*

$$\begin{aligned}
&(\nabla^2_U h^k)(V,W) - (\nabla^2_V h^k)(U,W) \\
&+(1-m_2)\{(\nabla^2_U h^k_2)(V,W) - (\nabla^2_V h^k_2)(U,W)\} \\
&\quad = \quad m_1\{(\nabla^2_U h^l_2)(V,W) - (\nabla^2_V h^l_2)(U,W)\} \\
&\qquad +\{2U(f_1)f_1 g_2(V,W) - 2V(f_1)f_1 g_2(U,W)\}k^\diamond \\
&\qquad +g(V,W)U(k^\diamond) - g(U,W)V(k^\diamond) \\
&\qquad -m_2\{h^k_2(U,W)V(k) - h^k_2(V,W)U(k)\} \\
&\qquad +m_1\{h^l_2(U,W)V(l) - h^k_2(U,W)V(l)\} \\
&\qquad +m_1\{-h^l_2(U,W)V(k) - h^l_2(V,W)U(l)\} \\
&\qquad +m_1\{h^k_2(V,W)U(l) + h^l_2(V,W)U(k)\} \\
&\qquad -U(k)S(V,W) + V(k)S(U,W) \\
&\qquad +g(U,W)S(V,\nabla k) - g(V,W)S(U,\nabla k)
\end{aligned} \tag{3.15}$$

for $X, Y, Z \in \mathcal{L}(M_1)$ and $U, V, W \in \mathcal{L}(M_2)$, where $k^\diamond = \triangle k + \|\nabla k\|^2$.

Proof. Let M be an Einstein-like warped-twisted product of class $\mathcal{B}$.

a)(M_1, g_1) is an Einstein-like manifold of class $\mathcal{B}$ if and only if

$$(\nabla^1_X S^1)(Y,Z) = (\nabla^1_Y S^1)(X,Z), \tag{3.16}$$

for any vector fields X, Y, Z on M_1. On the other hand, from (2.1) and (2.7), we have

$$\begin{aligned}
&(\nabla_X S)(Y,Z) \\
&\quad = \quad X(S(Y,Z)) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z) \\
&\qquad = X(S(Y,Z)) - S(\nabla^1_X Y - g(X,Y)\nabla l, Z) - S(Y, \nabla^1_X Z - g(X,Z)\nabla l) \\
&\qquad = X(S^1(Y,Z)) + X(h^l(Y,Z)) - m_2\{X(h^k_1(Y,Z)) + X(Y(k)Z(k))\} \\
&\qquad -X(g(Y,Z))l^\diamond \\
&\qquad -\{S^1(\nabla^1_X Y, Z) + h^l(\nabla^1_X Y, Z) - m_2\{h^k_1(\nabla^1_X Y, Z) + \nabla^1_X Y(k)Z(k)\} \\
&\qquad -g(\nabla^1_X Y, Z)l^\diamond\} \\
&\qquad -\{S^1(Y, \nabla^1_X Z) + h^l(Y, \nabla^1_X Z) - m_2\{h^k_1(Y, \nabla^1_X Z) + \nabla^1_X Z(k)Y(k)\} \\
&\qquad -g(Y, \nabla^1_X Z)l^\diamond\} \\
&\qquad +g(X,Y)S(\nabla l, Z) + g(X,Z)S(Y, \nabla l)
\end{aligned}$$

for $X, Y, Z \in \mathcal{L}(M_1)$. Then, we find

$$\begin{aligned}
&(\nabla_X S)(Y,Z)\\
&= X(S^1(Y,Z)) - S^1(\nabla^1_X Y, Z) - S^1(Y, \nabla^1_X Z)\\
&\quad + X(h^l(Y,Z)) - h^l(\nabla^1_X Y, Z) - h^l(Y, \nabla^1_X Z)\\
&\quad - m_2\{X(h^k_1(Y,Z)) - h^k_1(\nabla^1_X Y, Z) - h^k_1(Y, \nabla^1_X Z)\}\\
&\quad - \{X(g(Y,Z))l^\diamond - g(\nabla^1_X Y, Z)l^\diamond - g(Y, \nabla^1_X Z)l^\diamond\}\}\\
&\quad - m_2\{X(Y(k)Z(k)) - \nabla^1_X Y(k)Z(k) - \nabla^1_X Z(k)Y(k)\}\\
&\quad + g(X,Y)S(\nabla l, Z) + g(X,Z)S(Y, \nabla l)
\end{aligned}$$

If we use (2.4), we have

$$\begin{aligned}
&(\nabla_X S)(Y,Z)\\
&= (\nabla^1_X S^1)(Y,Z) + (\nabla^1_X h^l)(Y,Z) - m_2(\nabla^1_X h^k_1)(Y,Z)\\
&\quad - m_2\{h^k_1(X,Y)Z(k) + h^k_1(X,Z)Y(k)\}\\
&\quad + g(X,Y)S(\nabla l, Z) + g(X,Z)S(Y, \nabla l).
\end{aligned} \tag{3.17}$$

If we interchange X and Y in (3.17), we obtain

$$\begin{aligned}
&(\nabla_Y S)(X,Z)\\
&= (\nabla^1_Y S^1)(X,Z) + (\nabla^1_Y h^l)(X,Z) - m_2(\nabla^1_Y h^k_1)(X,Z)\\
&\quad - m_2\{h^k_1(Y,X)Z(k) + h^k_1(Y,Z)X(k)\}\\
&\quad + g(Y,X)S(\nabla l, Z) + g(Y,Z)S(X, \nabla l).
\end{aligned} \tag{3.18}$$

Hence, from (3.13), (3.17), (3.18) and (3.16), we have

$$\begin{aligned}
0 &= (\nabla^1_X h^l)(Y,Z) - (\nabla^1_Y h^l)(X,Z)\\
&\quad - m_2\{(\nabla^1_X h^k_1)(Y,Z) - (\nabla^1_Y h^k_1)(X,Z)\}\\
&\quad - m_2\{h^k_1(X,Z)Y(k) - h^k_1(Y,Z)X(k)\}\\
&\quad + g(X,Z)S(Y,\nabla l) - g(Y,Z)S(X,\nabla l).
\end{aligned} \tag{3.19}$$

So, (3.14) follows from (3.19).

b) (M_2, g_2) is an Einstein-like manifold of class $\mathcal{B}$ if and only if

$$(\nabla^2_U S^2)(V,W) = (\nabla^2_V S^2)(U,W), \tag{3.20}$$

for any vector fields U, V, W on M_2. On the other hand, from (2.3) and (2.9), we have

$$\begin{aligned}
&(\nabla_U S)(V,W)\\
&= U(S(V,W)) - S(\nabla_U V, W) - S(V, \nabla_U W)\\
&= U(S^2(V,W) + h^k(V,W) + (1-m_2)h^k_2(V,W) + m_2 V(k)W(k)\\
&\quad - g(V,W)k^\diamond)\\
&\quad - m_1 U(h^l_2(V,W) + V(l)W(l) - V(l)W(k) - V(k)W(l))\\
&\quad - S(\nabla^2_U V + U(k)V + V(k)U - g(U,V)\nabla k, W)\\
&\quad - S(V, \nabla^2_U W + U(k)W + W(k)U - g(U,W)\nabla k)
\end{aligned}$$

for $U, V, W \in \mathcal{L}(M_2)$. Then, we find

$$
\begin{aligned}
&(\nabla_U S)(V,W) \\
&= U(S^2(V,W)) + U(h^k(V,W)) + (1-m_2)U(h_2^k(V,W)) \\
&\quad +m_2 U(V(k)W(k)) - U(g(V,W)k^\diamond) \\
&\quad -m_1\{U(h_2^l(V,W)) + U(V(l)W(l)) \\
&\quad -U(V(l)W(k)) - U(V(k)W(l))\} \\
&\quad -\{S^2(\nabla_U^2 V,W)) + h^k(\nabla_U^2 V,W) + (1-m_2)h_2^k(\nabla_U^2 V,W) \\
&\quad +m_2\nabla_U^2 V(k)W(k) - g(\nabla_U^2 V,W)k^\diamond\} \\
&\quad -\{-m_1\{h_2^l(\nabla_U^2 V,W) + \nabla_U^2 V(l)W(l) \\
&\quad -\nabla_U^2 V(l)W(k) - \nabla_U^2 V(k)W(l)\}\} \\
&\quad -\{S^2(V,\nabla_U^2 W)) + h^k(V,\nabla_U^2 W) + (1-m_2)h_2^k(V,\nabla_U^2 W) \\
&\quad +m_2\nabla_U^2 W(k)V(k) - g(V,\nabla_U^2 W)k^\diamond\} \\
&\quad -\{-m_1\{h_2^l(V,\nabla_U^2 W) + \nabla_U^2 W(l)V(l) \\
&\quad -\nabla_U^2 W(k)V(l) - \nabla_U^2 W(l)V(k)\}\} \\
&\quad -2U(k)S(V,W) - V(k)S(U,W) - W(k)S(V,U)
\end{aligned}
$$

If we arrange this equation, we have

$$
\begin{aligned}
&(\nabla_U S)(V,W) \\
&= U(S^2(V,W)) - S^2(\nabla_U^2 V,W)) - S^2(V,\nabla_U^2 W)) \\
&\quad +U(h^k(V,W)) - h^k(\nabla_U^2 V,W) - h^k(V,\nabla_U^2 W) \\
&\quad +(1-m_2)\{U(h_2^k(V,W)) - h_2^k(\nabla_U^2 V,W) - h_2^k(V,\nabla_U^2 W)\} \\
&\quad -m_1\{U(h_2^l(V,W)) - h_2^l(\nabla_U^2 V,W) - h_2^l(V,\nabla_U^2 W)\} \\
&\quad -\{U(g(V,W))k^\diamond - g(\nabla_U^2 V,W)k^\diamond - g(V,\nabla_U^2 W)k^\diamond\} \\
&\quad -g(V,W)U(k^\diamond) \\
&\quad +m_2\{U(V(k))W(k) + U(W(k))V(k) - \nabla_U^2 V(k)W(k) \\
&\quad -\nabla_U^2 W(k)V(k)\} \\
&\quad -m_1\{U(V(l)W(l)) - U(V(l)W(k)) - U(V(k)W(l))\} \\
&\quad -m_1\{-\nabla_U^2 V(l)W(l) + \nabla_U^2 V(l)W(k) + \nabla_U^2 V(k)W(l)\} \\
&\quad -m_1\{-\nabla_U^2 W(l)V(l) + \nabla_U^2 W(k)V(l) + \nabla_U^2 W(l)V(k)\} \\
&\quad -2U(k)S(V,W) - V(k)S(U,W) - W(k)S(V,U) \\
&\quad +g(U,V)S(\nabla k,W) + g(U,W)S(V,\nabla k).
\end{aligned}
$$

If we use (2.4), we have

$$
\begin{aligned}
&(\nabla_U S)(V,W)\\
&= (\nabla^2_U S^2)(V,W) + (\nabla^2_U h^k)(V,W) + (1-m_2)(\nabla^2_U h^k_2)(V,W)\\
&\quad -m_1(\nabla^2_U h^l_2)(V,W)\\
&\quad -(\nabla^2_U g)(V,W)k^\diamond - g(V,W)U(k^\diamond)\\
&\quad +m_2\{h^k_2(U,V)W(k) + h^k_2(U,W)V(k)\}\\
&\quad -m_1\{h^l_2(U,V)W(l) + h^l_2(U,W)V(l) - h^l_2(U,V)W(k)\}\\
&\quad -m_1\{-h^k_2(U,W)V(l) - h^k_2(U,V)W(l) - h^l_2(U,W)V(k)\}\\
&\quad -2U(k)S(V,W) - V(k)S(U,W) - W(k)S(V,U)\\
&\quad +g(U,V)S(\nabla k,W) + g(U,W)S(V,\nabla k).
\end{aligned}
\tag{3.21}
$$

If we interchange U and V in (3.21), we obtain

$$
\begin{aligned}
&(\nabla_V S)(U,W)\\
&= (\nabla^2_V S^2)(U,W) + (\nabla^2_V h^k)(U,W) + (1-m_2)(\nabla^2_V h^k_2)(U,W)\\
&\quad -m_1(\nabla^2_V h^l_2)(U,W)\\
&\quad -(\nabla^2_V g)(U,W)k^\diamond - g(U,W)V(k^\diamond)\\
&\quad +m_2\{h^k_2(V,U)W(k) + h^k_2(V,W)U(k)\}\\
&\quad -m_1\{h^l_2(V,U)W(l) + h^l_2(V,W)U(l) - h^l_2(V,U)W(k)\}\\
&\quad -m_1\{-h^k_2(V,W)U(l) - h^k_2(V,U)W(l) - h^l_2(V,W)U(k)\}\\
&\quad -2V(k)S(U,W) - U(k)S(V,W) - W(k)S(U,V)\\
&\quad +g(V,U)S(\nabla k,W) + g(V,W)S(U,\nabla k).
\end{aligned}
\tag{3.22}
$$

Hence, from (3.13), (3.21),(3.22) and (3.20), we obtain

$$
\begin{aligned}
0 &= (\nabla^2_U h^k)(V,W) - (\nabla^2_V h^k)(U,W)\\
&\quad +(1-m_2)\{(\nabla^2_U h^k_2)(V,W) - (\nabla^2_V h^k_2)(U,W)\}\\
&\quad -m_1\{(\nabla^2_U h^l_2)(V,W) - (\nabla^2_V h^l_2)(U,W)\}\\
&\quad -2U(f_1)f_1g_2(V,W)k^\diamond + 2V(f_1)f_1g_2(U,W)k^\diamond\\
&\quad -g(V,W)U(k^\diamond) + g(U,W)V(k^\diamond)\\
&\quad +m_2\{h^k_2(U,W)V(k) - h^k_2(V,W)U(k)\}\\
&\quad -m_1\{h^l_2(U,V)W(l) + h^l_2(U,W)V(l) - h^l_2(U,V)W(k)\}\\
&\quad -m_1\{-h^k_2(U,W)V(l) - h^k_2(U,V)W(l) - h^l_2(U,W)V(k)\}\\
&\quad +m_1\{-h^k_2(V,W)U(l) - h^k_2(V,U)W(l) - h^l_2(V,W)U(k)\}\\
&\quad -2U(k)S(V,W) - V(k)S(U,W) - W(k)S(V,U)\\
&\quad +2V(k)S(U,W) + U(k)S(V,W) + W(k)S(U,V)\\
&\quad -g(V,U)S(\nabla k,W) - g(V,W)S(U,\nabla k).
\end{aligned}
$$

So, (3.15) follows from the above equation. □

For a singly warped product, one can obtain the following result.

Corollary 3.4. *Let M be an Einstein-like singly warped product of class $\mathcal{B}$. Then*

a) (M_1, g_1) *is an Einstein-like manifold of class $\mathcal{B}$ if and only if*

$$(\nabla^1_X h^l)(Y,Z) - (\nabla^1_Y h^l)(X,Z) = 0 \tag{3.23}$$

b) (M_2, g_2) *is an Einstein-like manifold of class $\mathcal{B}$ if and only if*

$$(\nabla^2_U h^l_2)(V,W) - (\nabla^2_V h^l_2)(U,W) = -h^l_2(U,W)V(l) + h^l_2(V,W)U(l) \tag{3.24}$$

for $X, Y, Z \in \mathcal{L}(M_1)$ and $U, V, W \in \mathcal{L}(M_2)$.

Class $\mathcal{P}$. If a Riemannian manifold $(\bar{M}, \bar{g})$ has a parallel Ricci tensor, i.e.,

$$(\bar{\nabla}_{\bar{X}} S)(\bar{Y}, \bar{Z}) = 0 \tag{3.25}$$

for any vector fields $\bar{X}, \bar{Y}, \bar{Z}$ on $\bar{M}$, then $(\bar{M}, \bar{g})$ are called *Einstein-like manifolds of Class $\mathcal{P}$*. In this case, $(\bar{M}, \bar{g})$ are also called *Ricci symmetric* manifolds.

Theorem 3.5. *Let M be an Einstein-like warped-twisted product of class $\mathcal{P}$. Then*

a) (M_1, g_1) *is an Einstein-like manifold of class $\mathcal{P}$ if and only if*

$$\begin{aligned}(\nabla^1_X h^l)(Y,Z) &= m_2(\nabla^1_X h^k_1)(Y,Z) \\ &\quad + m_2\{h^k_1(X,Y)Z(k) + h^k_1(X,Z)\}Y(k) \\ &\quad - g(X,Y)S(\nabla l, Z) - g(X,Z)S(Y, \nabla l),\end{aligned} \tag{3.26}$$

b) (M_2, g_2) *is an Einstein-like manifold of class $\mathcal{P}$ if and only if*

$$\begin{aligned}&(\nabla^2_U h^k)(V,W) + (1-m_2)(\nabla^2_U h^k_2)(V,W) \\ &= m_1(\nabla^2_U h^l_2)(V,W) + 2U(f_1)f_1 g_2(V,W)k^\diamond \\ &\quad + g(V,W)U(k^\diamond) - m_2\{h^k_2(U,V)W(k) + h^k_2(U,W)V(k)\} \\ &\quad + m_1\{h^l_2(U,V)W(l) + h^l_2(U,W)V(l) - h^l_2(U,V)W(k)\} \\ &\quad + m_1\{-h^k_2(U,W)V(l) - h^k_2(U,V)W(l) - h^l_2(U,W)V(k)\} \\ &\quad + 2U(k)S(V,W) + V(k)S(U,W) + W(k)S(V,U) \\ &\quad - g(U,V)S(\nabla k, W) - g(U,W)S(V, \nabla k)\end{aligned} \tag{3.27}$$

for $X, Y, Z \in \mathcal{L}(M_1)$ and $U, V, W \in \mathcal{L}(M_2)$, where $k^\diamond = \triangle k + \|\nabla k\|^2$.

Proof. Let M be an Einstein-like warped-twisted product of class $\mathcal{P}$.

a)(M_1, g_1) is an Einstein-like manifold of class $\mathcal{P}$ if and only if

$$(\nabla^1_X S^1)(Y,Z) = 0 \tag{3.28}$$

for any vector fields X, Y, Z on M_1. On the other hand, from (3.17), we have

$$\begin{aligned}(\nabla_X S)(Y,Z) &= (\nabla^1_X S^1)(Y,Z) + (\nabla^1_X h^l)(Y,Z) - m_2(\nabla^1_X h^k_1)(Y,Z)\\ &\quad -m_2\{h^k_1(X,Y)Z(k) + h^k_1(X,Z)Y(k)\}\\ &\quad +g(X,Y)S(\nabla l, Z) + g(X,Z)S(Y,\nabla l)\end{aligned} \tag{3.29}$$

for $X, Y, Z \in \mathcal{L}(M_1)$. Thus, using (3.25) and (3.28) in (3.29), we obtain (3.26).

b) (M_2, g_2) is an Einstein-like manifold of class $\mathcal{P}$ if and only if

$$(\nabla^2_U S^2)(V,W) = 0 \tag{3.30}$$

for any vector fields U, V, W on M_2. On the other hand, from (3.21), we have

$$\begin{aligned}&(\nabla_U S)(V,W)\\ &\quad = (\nabla^2_U S^2)(V,W) + (\nabla^2_U h^k)(V,W)\\ &\qquad +(1-m_2)(\nabla^2_U h^k_2)(V,W) - m_1(\nabla^2_U h^l_2)(V,W)\\ &\qquad -2U(f_1)f_1 g_2(V,W)k^\diamond - g(V,W)U(k^\diamond)\\ &\qquad +m_2\{h^k_2(U,V)W(k) + h^k_2(U,W)V(k)\}\\ &\qquad -m_1\{h^l_2(U,V)W(l) + h^l_2(U,W)V(l) - h^l_2(U,V)W(k)\}\\ &\qquad -m_1\{-h^k_2(U,W)V(l) - h^k_2(U,V)W(l) - h^l_2(U,W)V(k)\}\\ &\qquad -2U(k)S(V,W) - V(k)S(U,W) - W(k)S(V,U)\\ &\qquad +g(U,V)S(\nabla k, W) + g(U,W)S(V,\nabla k)\end{aligned} \tag{3.31}$$

for $U, V, W \in \mathcal{L}(M_2)$. Thus, using (3.25) and (3.30) in (3.31), we obtain (3.27). □

For a singly warped product, one can obtain the following result.

Corollary 3.6. *Let M be an Einstein-like singly warped product of class $\mathcal{P}$. Then*

a) *(M_1, g_1) is an Einstein-like manifold of class $\mathcal{P}$ if and only if*

$$(\nabla^1_X h^l)(Y,Z) = 0, \tag{3.32}$$

b) *(M_2, g_2) is an Einstein-like manifold of class $\mathcal{P}$ if and only if*

$$(\nabla^2_U h^l_2)(V,W) = -h^l_2(U,V)W(l) - h^l_2(U,W)V(l) \tag{3.33}$$

for $X, Y, Z \in \mathcal{L}(M_1)$ and $U, V, W \in \mathcal{L}(M_2)$.

Class $\mathcal{I} \oplus \mathcal{A}$. Let $(\bar{M}, \bar{g})$ be a Riemannian manifold of dimension m and a tensor $\mathcal{T}$ is defined by

$$\mathcal{T} = S - \frac{2\tau}{m+2}\bar{g}.$$

If the tensor $\mathcal{T}$ is Killing, then $(\bar{M}, \bar{g})$ are called *Einstein-like manifolds of class* $\mathcal{I} \oplus \mathcal{A}$. This condition is equivalent to

$$0 = (\bar{\nabla}_{\bar{X}} \mathcal{T})(\bar{X}, \bar{X}) \tag{3.34}$$

for any vector field $\bar{X}$ on $\bar{M}$. The inheritance property of Class $\mathcal{I} \oplus \mathcal{A}$ is given by the following result.

Theorem 3.7. *Let M be an Einstein-like warped-twisted product of class $\mathcal{I} \oplus \mathcal{A}$. Then*

a) *(M_1, g_1) is an Einstein-like manifold of class $\mathcal{I} \oplus \mathcal{A}$ if and only if*

$$\begin{aligned} &(\nabla^1_X h^l)(X,X) \\ &\quad = m_2(\nabla^1_X h^k_1)(X,X) \\ &\qquad +2m_2 X(k) h^k_1(X,X) - 2g(X,X)S(\nabla l, X) \\ &\qquad +\frac{2}{m+2}\{\nabla_X \tau g(X,X) - \frac{m+2}{m_1+2}\nabla^1_X \tau^1 g_1(X,X)\}, \end{aligned} \tag{3.35}$$

b) *(M_2, g_2) is an Einstein-like manifold of class $\mathcal{I} \oplus \mathcal{A}$ if and only if*

$$\begin{aligned} &(\nabla^2_U h^k)(U,U) + (1-m_2)(\nabla^2_U h^k_2)(U,U) \\ &\quad = m_1(\nabla^2_U h^l_2)(U,U) - 2m_2 h^k_2(U,U)U(k) \\ &\qquad +2U(f_1)f_1 g_2(U,U)k^\diamond + g(U,U)U(k^\diamond) \\ &\qquad +m_1\{2h^l_2(U,U)U(l) - 2h^l_2(U,U)U(k) - 2h^k_2(U,U)U(l)\} \\ &\qquad +4U(k)S(U,U) - 2g(U,U)S(\nabla k, U) \\ &\qquad +\frac{2}{m+2}\{(\nabla_U \tau)g(U,U) - \frac{m+2}{m_2+2}\nabla^2_U \tau^2 g_2(U,U)\} \end{aligned} \tag{3.36}$$

for $X \in \mathcal{L}(M_1)$ and $U \in \mathcal{L}(M_2)$, where $k^\diamond = \triangle k + \|\nabla k\|^2$.

Proof. Let M be an Einstein-like warped-twisted product of class $\mathcal{I} \oplus \mathcal{A}$.

a)(M_1, g_1) is an Einstein-like manifold of class $\mathcal{I} \oplus \mathcal{A}$ if and only a tensor $\mathcal{T}^1 = S^1 - \frac{2\tau^1}{m_1+2} g_1$ is Killing, i.e.,

$$0 = (\nabla^1_X \mathcal{T}^1)(X,X) \tag{3.37}$$

for any vector field X on M_1. On the other hand, from (3.34), we have

$$\begin{aligned} 0 &= (\nabla_X \mathcal{T})(X,X) \\ &= (\nabla_X S)(X,X) - \frac{2}{m+2}(\nabla_X(\tau g))(X,X) \\ &= (\nabla_X S)(X,X) - \frac{2}{m+2}(\nabla_X \tau)g(X,X). \end{aligned}$$

for all $X \in \mathcal{L}(M_1)$. Then, using (3.5), we find

$$\begin{aligned} 0 &= (\nabla^1_X S^1)(X,X) + (\nabla^1_X h^l)(X,X) \\ &\quad -m_2(\nabla^1_X h^k_1)(X,X) - 2m_2 X(k) h^k_1(X,X) \\ &\quad -(\nabla^1_X g)(X,X) l^\diamond + 2g(X,X) S(\nabla l, X) \\ &\quad -\frac{2}{m+2}(\nabla_X \tau) g(X,X). \end{aligned} \tag{3.38}$$

If we add and subtract $\nabla^1_X(\frac{2\tau^1}{m_1+2} g_1)(X,X)$ right side of the equation (3.38), we have

$$\begin{aligned} 0 &= (\nabla^1_X S^1)(X,X) - \nabla^1_X(\frac{2\tau^1}{m_1+2} g_1)(X,X) \\ &\quad +(\nabla^1_X h^l)(X,X) - m_2(\nabla^1_X h^k_1)(X,X) - 2m_2 X(k) h^k_1(X,X) \\ &\quad +2g(X,X) S(\nabla l, X) - \frac{2}{m+2}(\nabla_X \tau) g(X,X) \\ &\quad +\nabla^1_X(\frac{2\tau^1}{m_1+2} g_1)(X,X). \end{aligned} \tag{3.39}$$

So, using (3.39) and (3.37), we obtain (3.35).

b)(M_2, g_2) is an Einstein-like manifold of class $\mathcal{I} \oplus \mathcal{A}$ if and only a tensor $\mathcal{T}^2 = S^2 - \frac{2\tau^2}{m_2+2} g_2$ is Killing, i.e.,

$$0 = (\nabla^2_U \mathcal{T}^2)(U,U) \tag{3.40}$$

for any vector field U on M_2. On the other hand, we have

$$\begin{aligned} 0 &= (\nabla_U \mathcal{T})(U,U) \\ &= (\nabla_U S)(U,U) - \frac{2}{m+2}(\nabla_U(\tau g))(U,U) \\ &= (\nabla_U S)(U,U) - \frac{2}{m+2}(\nabla_U \tau) g(U,U). \end{aligned}$$

for all $U \in \mathcal{L}(M_2)$, from (3.34). Then, using (3.9), we find

$$\begin{aligned} 0 &= (\nabla^2_U S^2)(U,U) + (\nabla^2_U h^k)(U,U) + (1-m_2)(\nabla^2_U h^k_2)(U,U) \\ &\quad -m_1(\nabla^2_U h^l_2)(U,U) \\ &\quad +2m_2 h^k_2(U,U) U(k) - (\nabla^2_U g)(U,U) k^\diamond - g(U,U) U(k^\diamond) \\ &\quad -m_1\{2h^l_2(U,U)U(l) - 2h^l_2(U,U)U(k) - 2h^k_2(U,U)U(l)\} \\ &\quad -4U(k) S(U,U) + 2g(U,U) S(\nabla k, U) \\ &\quad -\frac{2}{m+2}(\nabla_U \tau) g(U,U). \end{aligned} \tag{3.41}$$

If we add and subtract $\nabla^2_U(\frac{2\tau^2}{m_2+2} g_2)(U,U)$ right side of the equation (3.41), we have

$$
\begin{aligned}
0 \;=\; & (\nabla_U^2 S^2)(U,U) - \nabla_U^2(\frac{2\tau^2}{m_2+2}g_2)(U,U) \\
& +2m_2 h_2^k(U,U)U(k) - 2U(f_1)f_1 g_2(U,U)k^\diamond - g(U,U)U(k^\diamond) \\
& -m_1\{2h_2^l(U,U)U(l) - 2h_2^l(U,U)U(k) - 2h_2^k(U,U)U(l)\} \\
& -4U(k)S(U,U) + 2g(U,U)S(\nabla k, U) \\
& -\frac{2}{m+2}(\nabla_U \tau)g(U,U) \qquad (3.42) \\
& +\nabla_U^2(\frac{2\tau^2}{m_2+2}g_2)(U,U).
\end{aligned}
$$

So, using (3.42) and (3.40), we obtain (3.36).

□

Class $\mathcal{A}\oplus\mathcal{B}$. Let $(\bar{M},\bar{g})$ be a Riemannian manifold of dimension m. If $(\bar{M},\bar{g})$ has a constant scalar curvature, then $(\bar{M},\bar{g})$ are called *Einstein-like manifolds of class $\mathcal{A}\oplus\mathcal{B}$.* Hence, from (2.10), one can prove the following theorem.

Theorem 3.8. *Let M be an Einstein-like warped-twisted product of class $\mathcal{A}\oplus\mathcal{B}$. Then, (M_1,g_1) and (M_2,g_2) are Einstein-like manifolds of class $\mathcal{A}\oplus\mathcal{B}$ if and only if*

$$
\begin{aligned}
c \;=\; & \frac{c_1}{{f_2}^2} + \frac{c_2}{{f_1}^2} + \tilde{\Delta}_1(l) + \tilde{\Delta}_2(k) - \frac{m_2}{{f_2}^2}\Delta_1(k) - \frac{m_1}{{f_1}^2}\Delta_2(l) \\
& +\frac{(1-m_2)}{{f_1}^2}\Delta_2(k) - m_2 g(P_1\nabla k, P_1\nabla k) - m_1\Delta l - 2m_1 g(\nabla l,\nabla l) \\
& -m_2\left\{\Delta k + g(\nabla k,\nabla k)\right\} + m_2 g(P_2\nabla k, P_2\nabla k) + 2m_1 g(P_2\nabla k,\nabla l)
\end{aligned}
$$

where c, c_1 and c_2 are constant scalar curvature of (M,g), (M_1,g_1) and (M_2,g_2), respectively.

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