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Recurrent and ϕ-recurrent curvature on mixed 3-Sasakian manifolds

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Abstract. In this paper we study the equation $\phi_i^2(\nabla R) = A_i R$ on a mixed 3-structure manifold $(M, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$, where ∇R is the covariant derivative of the curvature tensor and A_i's are some 1-forms. We give some examples of such manifolds and show that this equation leads to the important equation $\nabla R = A_i R$. That means that the manifold is curvature recurrent. Moreover, we prove that on a mixed 3-Sasakian manifold that equation implies $\nabla R = 0$.

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1. Introduction

Para-hyper Hermitian and para-quaternionic structures have many applications in theoretical physics and mechanics [8, 11]. Recently, as an odd dimensional analogue of para-quaternionic manifolds, Ianus et al. [9] have introduced the notion of mixed 3-structures. Some lightlike hypersurfaces of almost para-quaternionic Hermitian manifolds and normal semi invariant submanifolds of para-quaternionic Kaehler manifolds admit mixed 3-structure [10]. An important and interesting type of these manifolds is the mixed 3-Sasakian manifold which has been shown to be an Einstein manifold [4].

On the other hand, locally symmetric manifolds, recurrent manifolds and their generalizations were very interesting for many authors [1, 7, 14]. It is well-known that locally symmetric Sasakian manifolds have constant curvature and this condition is too strong. Hence, Takahashi introduced locally ϕ-symmetric Sasakian manifolds [15]. Later, Boeckx and Vanhecke extended this notion to the contact metric structures [3]. As a generalization of this concept, De and collaborators [5] defined ϕ-recurrent Sasakian manifolds. It has been shown that locally ϕ-symmetric and ϕ-recurrent Sasakian manifolds are Einstein and η-Einstein manifolds, respectively [6, 12]. In the present paper, we introduce and investigate ϕ-recurrent mixed 3-structures. By constructing some non-trivial examples, we show their existence. We also study ϕ-recurrent mixed 3-Sasakian manifolds.

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This paper is organized as follows. In Section 2, we review some basic information about mixed 3-structures and 3-Sasakian manifolds. In Section 3, we introduce ϕ-recurrent mixed 3-structures, construct some examples and show that ϕ-recurrent manifolds are recurrent manifolds. Moreover, we prove that ϕ-recurrent mixed 3-Sasakian manifolds are ϕ-symmetric and locally symmetric.

2. Preliminaries

Let (N, g) be a smooth semi-Riemannian manifold. A subbundle σ of TN is said to be almost para-quaternionic Hermitian, if there exists a local basis $\{J_1, J_2, J_3\}$ on sections of σ such that for any X, $Y \in TN$

(a) $J_i^2 = \epsilon_i$,

(b) $J_iJ_j = -J_jJ_i = \epsilon_k J_k$,

(c) $g(J_iX, J_iY) = \epsilon_i g(X, Y)$,

where (i, j, k) is a permutation of $(1, 2, 3)$ and $\epsilon_1 = \epsilon_2 = -\epsilon_3 = -1$ [11]. If the bundle σ is preserved by the Levi-Civita connection of g, then N is said to be a para-quaternionic Kaehler manifold. It is easy to show that any almost para-quaternionic Hermitian manifolds are of dimension $4m, m \geq 1$. The counterparts in odd dimension of these manifolds are studied in this paper.

Let M be a $2n + 1$-dimensional semi-Riemannian manifold which admits a vector field ξ, a 1-form η and a (1,1)-tensor field ϕ such that

$$\phi^2 X = \epsilon(-X + \eta(X)\xi)\ ,\ \eta(\xi) = 1 \quad \forall X \in TM, \tag{2.1}$$

then (M, ξ, η, ϕ) is called an almost contact manifold for $\epsilon = 1$ and an almost para-contact manifold for $\epsilon = -1$ [2, 13].

Definition 2.1. Let M be a semi-Riemannian manifold which admits two almost para-contact structures (ξ_i, η_i, ϕ_i), $i = 1, 2$, and an almost contact structure (ξ_3, η_3, ϕ_3) satisfying

$$\eta_i(\xi_j) = 0, \quad \phi_i(\xi_j) = \epsilon_j\xi_k, \ \phi_j(\xi_i) = -\epsilon_i\xi_k, \quad \eta_i(\phi_j) = -\eta_j(\phi_i) = \epsilon_k\eta_k, \tag{2.2}$$

$$\phi_i o \phi_j - \epsilon_i\eta_j \otimes \xi_i = -\phi_j o \phi_i + \epsilon_j\eta_i \otimes \xi_j = \epsilon_k\phi_k, \tag{2.3}$$

where (i,j,k) is an even permutation of $(1, 2, 3)$. Then $(M, \xi_i, \eta_i, \phi_i)_{i\in\{1,2,3\}}$ is called a mixed 3-structure manifold [9].

In addition, if there is a semi-Riemannian metric g on M such that

$$g(\phi_iX, \phi_iY) = \epsilon_i[g(X, Y) - \tau_i\eta_i(X)\eta_i(Y)], \quad \forall X, Y \in TM, \tag{2.4}$$

in which $\tau_i = g(\xi_i, \xi_i) = \pm 1$, then $(M, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ is called a metric mixed 3-structure manifold. In such a manifold, (2.4) implies,

$$g(\phi_iX, Y) = -g(X, \phi_iY). \tag{2.5}$$

We can choose a local basis $\{e_i, \phi_1 e_i, \phi_2 e_i, \phi_3 e_i, \xi_1, \xi_2, \xi_3\}$ for T_pM, and the dimension of manifold is $4n+3$.
Moreover, a metric mixed 3-structure manifold is said to be a mixed 3-Sasakian manifold if

$$(\nabla_X \phi_i)Y = \epsilon_i[g(X,Y)\xi_i - \tau_i \eta_i(Y)X], \ \forall X, Y \in TM, \ i \in \{1,2,3\}. \tag{2.6}$$

If the Ricci tensor of a manifold satisfies $S(X,Y) = fg(X,Y)$, for a scalar function f, then it is called an Einstein manifold.

Theorem 2.2. *[4] Let $(M, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ be a mixed 3-Sasakian manifold. Then M is an Einstein manifold*

3. ϕ-recurrent mixed 3-structures

A Riemannian manifold M is called a recurrent manifold, if there exists a 1-form A such that

$$(\nabla_W R)(X,Y,Z) = A(W)R(X,Y)Z, \tag{3.1}$$

for any vector fields X,Y,Z,W $\in TM$, in which R is the curvature tensor of M [1].

Definition 3.1. An almost contact manifold (M, ξ, η, ϕ) is called a ϕ-recurrent manifold, if there exists a 1-form A such that

$$\phi^2(\nabla_W R)(X,Y,Z) = A(W)R(X,Y)Z, \tag{3.2}$$

for any vector fields $X, Y, Z, W \in TM$ [5].

Moreover, if the 1-form A is equal to zero in (3.1) and (3.2), then M is said to be locally symmetric and ϕ-symmetric, respectively. Now, we introduce the notion of ϕ-recurrent mixed 3-structure manifolds and give some examples.

Definition 3.2. Let $(M, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ be a metric mixed 3-structure manifold. We say that M is a 3-ϕ-recurrent manifold, if

$$\phi_i^2(\nabla_W R)(X,Y,Z) = A_i(W)R(X,Y)Z, \quad i = 1,2,3, \tag{3.3}$$

where $X, Y, Z, W \in TM$ and A_is are 1-forms.

Example 3.3. Let $M = \{(x_i)_{i=\overline{1,7}} \in \mathbb{R}^7 | x_1 + x_2 + x_3 + x_4 \neq 0\}$, and ϕ_i's are defined as follows

$$\phi_1((x_i)_{i=\overline{1,7}}) = (-x_2, -x_1, -x_4, -x_3, 0, x_7, x_6),$$

$$\phi_2((x_i)_{i=\overline{1,7}}) = (x_4, -x_3, -x_2, x_1, -x_6, -x_5, 0),$$

$$\phi_3((x_i)_{i=\overline{1,7}}) = (x_3, -x_4, -x_1, x_2, x_7, 0, -x_5).$$

Suppose that $q = x_1 + x_2 + x_3 + x_4$ and

$$g = \sum_{i=1}^{4} (-1)^i q^2 dx_i dx_i + dx_5 dx_5 - dx_6 dx_6 + dx_7 dx_7,$$

Let $\xi_1 = \partial x_5$, $\xi_2 = \partial x_7$, $\xi_3 = -\partial x_6$ and η_i's be the duals of ξ_i's. Then $(M, \xi_i, \eta_i, \phi_i, g)_{i \in \{1,2,3\}}$ is a metric mixed 3-structure manifold. By some computation, we obtain the non-zero Christoffel symbols and components of curvature tensor R as follows

$$\Gamma^i_{ii} = \Gamma^i_{ij} = \frac{1}{q}, \ i \neq j, \ i, j = 1, \dots, 4.$$

$$-\Gamma^r_{ii} = -\Gamma^i_{rr} = \Gamma^j_{ii} = \Gamma^s_{rr} = -\frac{1}{q}, \ i \neq j, r \neq s, \ i, j \in \{1,3\}, r, s \in \{2,4\}.$$

$$R_{rssr} = -R_{ijji} = 4, \quad i \neq j, r \neq s, \ i, j \in \{1,3\}, r, s \in \{2,4\},$$
$$-R_{riis} = -R_{jiir} = R_{irrj} = R_{srri} = 2.$$

By computing covariant derivative of R, we get

$$R_{rssr,k} = -R_{ijji,k} = \frac{-24}{q},$$

$$R_{riis;k} = R_{jiir;k} = -R_{irrj;k} = -R_{srri;k} = \frac{12}{q},$$

for $i \neq j, r \neq s, \ i, j \in \{1,3\}, r, s \in \{2,4\}$ and $k = 1, \dots, 4$.

Thus by taking

$$A(\partial x_k) = \begin{cases} \frac{-6}{q}, & k = 1, \dots, 4; \\ 0, & k = 5, 6, 7, \end{cases}$$

for any $X, Y, Z, W \in TM$, we have

$$\phi_i^2 (\nabla_W R)(X, Y, Z) = A_i(W) R(X, Y, Z, U), \ i = 1, 2, 3,$$

where $A_1(W) = A_2(W) = -A_3(W) = A(W)$. So, M is a 3-ϕ-recurrent.

Example 3.4. Let $(M, \xi_i, \eta_i, \phi_i)_{i \in \{1,2,3\}}$ be the manifold in Example 3.3 and

$$g = \sum_{i=1}^{4} (-1)^i e^{2q} dx_i dx_i + dx_5 dx_5 - dx_6 dx_6 + dx_7 dx_7.$$

Then it is easy to check that $(M, \xi_i, \eta_i, \phi_i, g)_{i \in \{1,2,3\}}$ is a metric mixed 3-structure manifold and the non-zero Christoffel symbols are

$$\Gamma^i_{ii} = \Gamma^i_{ij} = 1, \ i \neq j, \ i, j = 1, \dots, 4$$

$$-\Gamma^r_{ii} = -\Gamma^i_{rr} = \Gamma^j_{ii} = \Gamma^s_{rr} = -1, \ i \neq j, r \neq s, \ i, j \in \{1,3\}, r, s \in \{2,4\}.$$

So, the non-zero components of curvature tensor R and its covariant derivatives are obtained as follows

$$R_{rssr} = -R_{ijji} = 2e^{2q},$$

$$-R_{riis} = -R_{jiir} = R_{irrj} = R_{srri} = e^{2q}.$$

$$R_{rssr,k} = -R_{ijji,k} = -8e^{2q},$$

$$-R_{riis;k} = -R_{jiir;k} = R_{irrj;k} = R_{srri;k} = -4e^{2q},$$

for $i \neq j, r \neq s,\ i,j \in \{1,3\}, r,s \in \{2,4\}$ and $k = 1,\dots,4$.
By putting

$$A(\partial x_k) = \begin{cases} -4, & k = 1,\dots,4; \\ 0, & k = 5,6,7, \end{cases}$$

we can see that $(M,\xi_i,\eta_i,\phi_i,g)_{i\in\{1,2,3\}}$ is a 3-ϕ-recurrent manifold.

Lemma 3.5. *Let $(M,\xi_i,\eta_i,\phi_i)_{i\in\{1,2,3\}}$ be a metric mixed 3-structure manifold. Then*

$$\phi_i^2 o \phi_j^2 = -\epsilon_k[\epsilon_i\phi_i^2 + \eta_j \otimes \xi_j] = -\epsilon_k[\epsilon_j\phi_j^2 + \eta_i \otimes \xi_i], \tag{3.4}$$

where $-\epsilon_1 = -\epsilon_2 = \epsilon_3 = 1$ and (i,j,k) is an even permutation of $(1,2,3)$.

Proof. Since the composition of tensors is associative, by using (2.3) and straightforward computation, we have

$$\phi_i o((\phi_i o \phi_j) o \phi_j) = -\epsilon_k[\epsilon_i\phi_i^2 + \eta_j \otimes \xi_j]. \tag{3.5}$$

On the other hand,

$$(\phi_i o(\phi_i o \phi_j)) o \phi_j = -\epsilon_k[\epsilon_j\phi_j^2 + \eta_i \otimes \xi_i]. \tag{3.6}$$

So, comparing (3.5) and (3.6) completes the proof. □

Theorem 3.6. *Let $(M,\xi_i,\eta_i,\phi_i)_{i\in\{1,2,3\}}$ be a non-flat 3-ϕ-recurrent manifold. Then $A_j(W) = \epsilon_k A_i(W)$, where $\{i,j,k\}$ is an even permutation of $\{1,2,3\}$.*

Proof. Suppose

$$\phi_j^2(\nabla_W R)(X,Y,Z) = A_j(W)R(X,Y)Z. \tag{3.7}$$

By applying ϕ_i^2 on (3.7) and using Lemma 3.5, we have

$$-\epsilon_k[\epsilon_i\phi_i^2(\nabla_W R)(X,Y,Z) + \eta_j((\nabla_W R)(X,Y,Z))\xi_j] =$$

$$\epsilon_i A_j(W)[-R(X,Y)Z + \eta_i(R(X,Y)Z)\xi_i]. \tag{3.8}$$

Since

$$\phi_i^2(\nabla_W R)(X,Y,Z) = A_i(W)R(X,Y)Z,$$

from (3.8) we have

$$\begin{aligned}(\epsilon_i A_j(W) - \epsilon_k\epsilon_i A_i(W))R(X,Y)Z &= \epsilon_k\eta_j((\nabla_W R)(X,Y,Z))\xi_j \\ &+ \epsilon_i\eta_i(A_j(W)R(X,Y)Z)\xi_i.\end{aligned} \tag{3.9}$$

Now, by applying ϕ_j and then ϕ_k on (3.9), we get

$$(\epsilon_i A_j(W) - \epsilon_k\epsilon_i A_i(W))\phi_k o\phi_j(R(X,Y)Z) = 0. \tag{3.10}$$

Assume $\phi_k o\phi_j(R(X,Y)Z) = 0$. Then (2.3) implies

$$\epsilon_i\phi_i(R(X,Y)Z) = -\epsilon_k\eta_j(R(X,Y)Z)\xi_k, \tag{3.11}$$

we apply η_j on (3.11) and obtain $\phi_k(R(X,Y)Z) = 0$. This means

$$R(X,Y)Z = \eta_k(R(X,Y)Z)\xi_k. \tag{3.12}$$

On the other hand, applying η_k on (3.9) implies, $\eta_k(R(X,Y)Z) = 0$ and therefore, $R(X,Y)Z = 0$. This contradicts the assumption that M is a non-flat manifold. Therefore, from (3.10) we get $A_j(W) = \epsilon_k A_i(W)$. □

Theorem 3.7. *Let $(M, \xi_i, \eta_i, \phi_i)_{i\in\{1,2,3\}}$ be a 3-ϕ-recurrent manifold. Then M is a recurrent manifold.*

Proof. Since M is a 3-ϕ-recurrent manifold, by using Theorem 3.6, we put $A(W) = A_1(W) = A_2(W) = -A_3(W)$ and get

$$[-(\nabla_W R)(X,Y,Z) + \eta_i((\nabla_W R)(X,Y,Z))\xi_i] = A(W)R(X,Y)Z, \tag{3.13}$$

for $i = 1,2,3$. We obtain $\eta_i(A(W)R(X,Y)Z) = 0$ and $\eta_j((\nabla_W R)(X,Y,Z)) = -\eta_j(A(W)R(X,Y)Z)$, by applying η_i and η_j on (3.13), respectively. Therefore,

$$\eta_i((\nabla_W R)(X,Y,Z)) = -\eta_i(A(W)R(X,Y)Z) = 0, \tag{3.14}$$

so, (3.13) and (3.14) imply $(\nabla_W R)(X,Y,Z) = -A(W)R(X,Y)Z$. □

The following example shows that the inverse of the previous theorem is not necessarily correct.

Example 3.8. Let $M = \{(x_i)_{i=\overline{1,7}} \in \mathbb{R}^7 | x_1 + x_2 \neq 0\}$, (ξ_i, η_i, ϕ_i) be the same in Example 3.3 and

$$g = e^{2(x_1+x_2)}[\sum_{i=1}^{4}(-1)^{(i)}dx_i dx_i + dx_5 dx_5 - dx_6 dx_6 + dx_7 dx_7].$$

$(M, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ is a metric mixed 3-structure manifold.

For $i \neq k,\ i \in \{1,2\},\ k \in \{1,..,7\},\ j \in \{2,4,5,7\}, r \in \{1,3,6\}$, we have

$$\Gamma^i_{ii} = \Gamma^i_{ik} = \Gamma^k_{ik} = 1,$$

$$\Gamma^1_{jj} = -\Gamma^1_{rr} = -\Gamma^2_{jj} = \Gamma^2_{rr} = 1,$$

$$-R_{riir} = R_{jiij} = -R_{1rr2} = R_{1jj2} = e^{2(x_1+x_2)}.$$

By taking

$$A(\partial x_k) = \begin{cases} -4, & k = 1, 2; \\ 0, & k = 3, .., 7, \end{cases}$$

M is a recurrent manifold with associated 1-form A. Since ∇R has only two non-zero components, it is not ϕ_i-recurrent for any $i = 1, 2, 3$.

Theorem 3.9. *Let $(M, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ be a mixed 3-Sasakian manifold. If M is a 3-ϕ-recurrent manifold, then M is a ϕ-symmetric and locally symmetric manifold.*

Proof. Since M is a mixed 3-Sasakian manifold, Theorem 2.2 implies that is an Einstein manifold. Thus its Ricci tensor S satisfies $S(X, Y) = fg(X, Y)$. By contracting this formula, we have $r = f(dimM)$, where r is the scalar curvature of M and $dimM = 4n + 3$. On the other hand, from contracted Bianchi Identity, we obtain $\frac{1}{2}dr = \frac{1}{4n+3}dr$, hence for $n \geq 0$, r is constant and therefore f is constant. So,

$$(\nabla_W S)(Y, Z) = f(\nabla_W g)(Y, Z) = 0, \ \forall W, Y, Z \in TM. \tag{3.15}$$

On the other hand, M is a 3-ϕ-recurrent manifold and from Theorem 3.7, we conclude

$$(\nabla_W R)(X, Y, Z) = -A(W)R(X, Y, Z). \tag{3.16}$$

By contracting (3.16), we obtain

$$(\nabla_W S)(Y, Z) = -A(W)S(Y, Z). \tag{3.17}$$

Since $S(Y, Z) \neq 0$, comparing (3.15) and (3.17) implies $A(W) = 0$. Therefore, M is locally symmetric and ϕ-symmetric. □

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