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Iterative algorithm for solving mixed equilibrium problems and demigeneralized mappings

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Abstract. The purpose of this paper is to study a new iterative algorithm for finding a common element in the set of solutions of mixed equilibrium problem and common fixed point of finite family of Bregman demigeneralized mappings. We prove a strong convergence theorem of the sequences generated by the algorithm in a reflexive Banach space. Our result complement, extend and improve important recent results announced by many authors.

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1. Introduction

Let E be a real reflexive Banach space and C a nonempty closed and convex subset of E. The normalized duality map from E to 2^{E^*} (E^* is the dual space of E) denoted by J is defined by
$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2, \|x\| = \|f\| \right\}$ for all $x \in C$.
Numerous problems in optimization, economics and physics can be reduced to finding solutions of some equilibrium problems. Various methods have been studied for solutions of some equilibrium problems in Hilbert spaces, see for example Blum and Oetti [6] , Combettes and Hirstoaga [10] and the references contained therein.
Let $\Theta : C \times C \to R$ be a bifunction and $\varphi : C \to \mathbb{R}$ be a real valued function. The mixed equilibrium problem (MEP) is to find point
$z \in C$ such that $\Theta(z, y) + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C$.
If $\varphi \equiv 0$, the above MEP reduces to equilibrium problem (EP), which is to find a point
$z \in C$ such that $\Theta(z, y) \geq 0, \quad \forall y \in C$.

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Throughout this paper, we shall assume $f : E \to (-\infty, +\infty]$ is a proper, lower semi-continuous and convex function. We denote by $\mathrm{dom} f := \{x \in E : f(x) < +\infty\}$ the domain of f. Let $x \in \mathrm{int}(\mathrm{dom}(f))$; the subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \leq f(y),\ \forall y \in E\},$$

where the Fenchel conjugate of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}.$$

A function f on E is coercive [13] if the sublevel set of f is bounded; equivalently,

$$\lim_{\|x\| \to +\infty} f(x) = +\infty,$$

and f on E is said to be strongly coercive [26] if

$$\lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} = +\infty.$$

For any $x \in \mathrm{int}(\mathrm{dom}(f))$ and $y \in E$, the right-hand derivative of f at x in the direction y is defined by

$$f^\circ(x, y) := \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t}.$$

The function f is said to be Gâteaux differentiable at x if $\lim_{t \to 0} \frac{f(x + ty) - f(x)}{t}$ exists for any y. In this case, the gradient of f at x is the function $\nabla f(x) : E \to (-\infty, +\infty]$ defined by $\langle \nabla f(x), y\rangle = f^\circ(x, y)$ for any $y \in E$. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable for any $x \in \mathrm{int}(\mathrm{dom}(f))$. Furthermore f is said to be Fréchet differentiable at x if this limit is attained uniformly in y, $\|y\| = 1$. Also f is said to be uniformly Fréchet differentiable on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$. It is well known that if f is Gâteaux differentiable (resp. Fréchet differentiable) on $\mathrm{int}(\mathrm{dom}(f))$, then f is continuous and its Gâteaux derivative ∇f is norm-to-weak* continuous (resp. norm-to-norm continuous) on $\mathrm{int}(\mathrm{dom}(f))$ (see [11, 12]). We will need the following results in the sequel.

Lemma 1.1. *[19] If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Definition 1.2. [4] The function f is said to be:

(1) Essentially smooth, if ∂f is both locally bounded and single-valued on its domain;

(ii) Essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every subset of $\mathrm{dom} f$;

(iii) Legendre, if it is both essentially smooth and essentially strictly convex.

Remark 1.3. If E is reflexive Banach space, then we have the following results:

(i) f is essentially smooth if and only if f^* is essentially strictly convex (see [4] Theorem 5.4);

(ii) $(\partial f)^{-1} = \partial f^*$ (see [12])

(iii) f is Legendre if and only if f^* is Legendre (see [4],Corollary 5.5)

(iv) If f is Legendre, then ∇f is a bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, ran$\nabla f =$ dom$\nabla(f^*) =$ int(dom(f^*)) and ran $\nabla f^* =$ dom$(f) =$ int(dom(f)) (see [4], Theorem 5.10).

Examples of Legendre functions were given in [3, 4]. One important and interesting Legendre function is $\frac{1}{p}\|\cdot\|^p$ $(1 < p < \infty)$ when E is a smooth and strictly convex Banach space. In this case the gradient ∇f of f is coincident with the generalized duality mapping of E, i.e, $\nabla f = J_p$ $(1 < p < \infty)$. In particular, $\nabla f = I$ the identity mapping in Hilbert spaces.
In 1967, Bregman [7] introduced an effective technique using the Bregman distance function D_f for designing and analyzing feasibility and optimization algorithms. This opened a growing area of research in which Bregman's technique is applied in various ways in order to design and analyze iterative algorithm for solving not only feasibility and optimization problems, but also algorithms for solving fixed point problems for nonlinear maps (see, e.g [1, 8, 15, 24, 22] and the references therein).
In the rest of this paper, we always assume that $f : E \to (-\infty, +\infty]$ is Legendre.
Let $f : E \to (-\infty, +\infty]$ be a convex and Gateaux differentiable function. The function
D_f : dom$f\times$ intdom$f \to (-\infty, +\infty]$, defined as follows:

$$D_f(x, y) := f(x) - f(y) - \langle \nabla f(y), x - y\rangle, \tag{1.1}$$

is called the Bregman distance with respect to $f(x, y)$ (see [9]). It is obvious from the definition of D_f that

$$D_f(z, x) := D_f(z, y) + D_f(y, x) + \langle \triangledown f(y) - \triangledown f(x), z - y\rangle. \tag{1.2}$$

Let $T : C \to E$ be a map and $F(T) = \{x : Tx = x\}$ denote the set of fixed point of T. A point $p \in C$ is said to be asymptotic fixed point of a map T if there exist a sequence $\{x_n\}$ in C which converges weakly to p such that $\lim\limits_{n\to+\infty} \|x_n - Tx_n\| = 0$. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T. A point $p \in C$ is said to be strong asymptotic fixed point of a map T, if there exists a sequence $\{x_n\}$ in C which converges strongly to p such that $\lim\limits_{n\to+\infty} \|x_n - Tx_n\| = 0$. We denote by $\tilde{F}(T)$ the set of strong asymptotic fixed points of T. T is said to be quasi-Bregman relatively nonexpansive if $F(T) \neq \emptyset$,

$\hat{F}(T) = F(T)$ and $D_f(Tx, p) \leq D_f(x, p)$ for all $x \in C$ and $p \in F(T)$.
If E is a smooth Banach space, the Lyapunov functional $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = ||x||^2 - 2\langle x, Jy\rangle + ||y||^2, \quad \forall x, y \in E.$$

Let η and s be real numbers with $\eta \in (-\infty, 1)$ and $s \in [0, \infty)$, respectively. Then the map $T : C \to E$ with $F(T) \neq \emptyset$ is called $(\eta, s)-$demigeneralized if, for any $x \in C$ and $q \in F(T)$,

$$\langle x - q, Jx - JTx\rangle \geq (1 - \eta)\phi(x, Tx) + s\phi(Tx, x), \tag{1.3}$$

where $F(T)$ is the set of fixed points of T. In particular, if $s = 0$ in (1.3) then the map T becomes

$$\langle x - q, Jx - JTx\rangle \geq (1 - \eta)\phi(x, Tx),$$

for any $x \in C$ and $q \in F(T)$, which is called an $(\eta, 0)-$demigeneralized map.

Remark 1.4. If E is smooth and strictly convex Banach space and $f(x) = ||x||^2$ for all $x \in E$, then we have $\nabla f(x) = 2Jx$, for all $x \in E$, and hence the function $D_f(x, y)$ reduces to $\phi(x, y)$

Definition 1.5. [2] Let E be a reflexive Banach space, C a nonempty closed and convex subset E, let η be a real number with $\eta \in (-\infty, 1)$. Then the map $T : C \to E$ with $F(T) \neq \emptyset$ is called $(\eta, 0)$-Bregman demigeneralized map, if for any $x \in C$ and $q \in F(T)$,

$$\langle x - q, \nabla f(x) - \nabla f(Tx)\rangle \geq (1 - \eta)D_f(x, Tx).$$

The following examples illustrate that the class of Bregman demigeneralized maps are is important.

(i) [24] Let C be a nonempty closed convex subset of a reflexive Banach space E. Let k be real number in $(0, 1)$, the map $T : C \to E$ is called quasi-Bregman strictly pseudocontractive map if $F(T) \neq \emptyset$, $\forall x \in C$ and $p \in F(T)$,

$$D_f(p, Tx) \leq D_f(p, x) + kD_f(x, Tx), \tag{1.4}$$

in fact T is a $(k, 0)$-Bregman demigeneralized map, from (1.2) and (1.4) we have,

$$D_f(p, Tx) \leq D_f(p, x) + D_f(x, Tx) - D_f(x, Tx) + kD_f(x, Tx),$$

which implies

$$\begin{aligned}(1 - k)D_f(x, Tx) &\leq D_f(p, x) + D_f(x, Tx) - D_f(p, Tx)\\ &= \langle x - p, \nabla f(x) - \nabla f(Tx)\rangle.\end{aligned}$$

This shows that T is $(k, 0)$-Bregman demigeneralized map.

(ii) Let E be a reflexive Banach space and C a nonempty closed convex subset of E and let $f : E \to \mathbb{R}$ be a Fréchet differentiable convex function. A map $T : C \to E$ is called quasi-Bregman nonexpansive map if $F(T) \neq \emptyset$ and for all $x \in C,\ p \in F(T)$,

$$D_f(p, Tx) \quad \leq \quad D_f(p, x),$$

then T is $(0, 0)$-Bregman demigeneralized map. For all $x \in C,\ p \in F(T)$ we have

$$D_f(p, Tx) \quad \leq \quad D_f(p, x),$$

which by (1.2) we get

$$D_f(p, x) + D_f(x, Tx) + \langle p - x, \nabla f(x) - \nabla f(Tx)\rangle \leq D_f(p, x)$$

and hence

$$D_f(x, Tx) \quad \leq \quad \langle x - p, \nabla f(x) - \nabla f(Tx)\rangle.$$

This implies that T is $(0, 0)$-Bregman demigeneralized map.

Example 1.6. [2] Let $E = \mathbb{R},\ C = [-1, 0]$ and define $T, f : [-1, 0] \to \mathbb{R}$ by $f(x) = x^3$ and $Tx = 2x$, for all $x \in [-1, 0]$. Then T is an $(\eta, 0)$-Bregman demigeneralized map but not an $(\eta, 0)$-demigeneralized map.

Kumam et.al [14] introduced the following iterative algorithm,

$$\begin{cases} x_1 = x \in C \\ z_n = Res_H^f(x_n); \quad j = 1, 2, \cdots, m \\ y_n = \nabla f^*\big(\beta_n \nabla f(x_n) + (1 - \beta_n)\nabla f(T_n z_n)\big); \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(x_n) + (1 - \alpha_n)\nabla f(T_n y_n)), \quad n \geq 1, \end{cases} \tag{1.5}$$

where H is an equilibrium bifunction and T_n is Bregman strongly nonexpansive map for any $n \in \mathbb{N}$. They proved that the sequence generated above sequence converges strongly.
Motivated by the result of Kumam et.al [14], Ugwunnadi and Bashir [23] introduced the following iterative scheme for finding the common fixed point of a finite family of quasi-Bregman nonexpansive maps which is the unique solution of some equilibrium problems,

$$\begin{cases} x_1 \in C \\ u_{j,n} = Res_{gj}^f(x_n); \quad j = 1, 2, \cdots, m \\ y_n = P_C\big(\nabla f^*\big((1 - \alpha_n)\nabla f(u_{j,n})\big)\big); \\ x_{n+1} = P_C\big(\nabla f^*(\beta_n \nabla f(y_n) + (1 - \beta_n)\nabla f(T_n y_n))\big), \quad n \geq 1, \end{cases} \tag{1.6}$$

where $T_n = T_{n(mod\ N)}$ are quasi-Bregman nonexpansive maps, under some appropriate conditions they proved that the sequence converges strongly to

$P^f_\Omega x$.
Also motivated by the result of Kumam et.al [14], Biranvand and Darvish [5], studied the following iterative algorithm,

$$\begin{cases} x_1 = x \in C \\ z_n = Res^f_{\Theta,\varphi}(x_n) \\ y_n = proj^f_C \nabla f^*\big(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(Tz_n)\big); \\ x_{n+1} = proj^f_C \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Ty_n)), \quad n \geq 1, \end{cases} \tag{1.7}$$

where $T = T_N \circ T_{N-1} \circ \cdots \circ T_1$ and T_i is a Bregman strongly nonexpansive map for each $i = 1, 2, \cdots, N$. They proved that the sequence $\{x_n\}$ converges strongly to $Proj^f_{(\cap_{i=1}^N F(T_i))\cap MEP(\Theta))}$.
Motivated and inspired by the above-mentioned results we propose a new iterative algorithm for finding a common element in the set of solutions of the mixed equilibrium problem and a common fixed point of a finite family of Bregman demigeneralized maps in reflexive Banach spaces. Our results complement and extend some results announced recently by some authors.

2. Preliminaries

Recall that the Bregman projection [7] of $x \in \text{intdom}f$ onto nonempty, closed and convex set $C \subset \text{dom}f$ is the unique vector $P_C(x) \in C$ satisfying

$$D_f(P_C(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

where P_C is the Bregman projection.
Concerning the Bregman projection, the following are well known.

Lemma 2.1. [8] *Let C be a nonempty, closed and convex subset of a reflexive Banach space E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. Then*

(a) $z = P_C(x)$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z\rangle \leq 0, \quad \forall y \in C$;

(b) $D_f(y, P_C(x)) + D_f(P_C(x), x) \leq D_f(y, x), \quad \forall x \in E,\ y \in C$.

Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The modulus of total convexity of f at $x \in \text{intdom}f$ is the function $v_f(x, \cdot) : [0, +\infty]$ defined by
$v_f(x, t) := \inf\{D_f(x, y) : y \in \text{dom}f, \|y - x\| = t\}$.
The function f is called totally convex at x if $v_f(x, t) > 0$ whenever $t > 0$. The function f is called convex if it is totally convex at any point $x \in \text{intdom}f$ and is said to be totally convex on a bounded set if $v_f(B, t) > 0$ for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $v_f : \text{intdom}f \times [0, +\infty) \to [0, +\infty)$ defined by
$v_f(B, t) := \inf\{v_f(x, t) : x \in B \cap\ \text{dom}f\}$.

Lemma 2.2. *[17] Let E be a Banach space and $f : E \to \mathbb{R}$ be a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in E. Then*

$$\lim_{n\to\infty} D_f(x_n, y_n) = 0 \quad if \text{ and only } if \quad \lim_{n\to\infty} ||x_n - y_n|| = 0.$$

Lemma 2.3. *[20] Let $f : E \to \mathbb{R}$ be Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, the sequence $\{x_n\}$ is bounded too.*

Recall that the function f is called sequentially consistent if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first one is bounded

$$\lim_{n\to+\infty} D_f(y_n, u_n) = 0 \qquad implies \qquad \lim_{n\to+\infty} \|y_n - u_n\| = 0.$$

The following result was first proved in [8].

Lemma 2.4. *Let E be a reflexive Banach space, let $f : E \to \mathbb{R}$ be a strongly coercive Bregman function and let V be the function defined by*

$$V(x, x^*) = f(x) - \langle x, x^*\rangle + f^*(x^*), \quad x \in E, \quad x^* \in E^*.$$

Then the following hold:

(1) $D_f(x, \nabla f(x^*)) = V(x, x^*)$ for all $x \in E$ and $x^* \in E^*$.

(2) $V(x, x^*)+\langle \nabla f^*(x^*)-x, y^*\rangle \le V(x, x^*+y^*)$ for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.5. *[25] Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the condition*

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n\beta_n,\ n \ge 0,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers such that

i. $\{\alpha_n\} \subset [0, 1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

ii. $\limsup_{n\to\infty} \beta_n \le 0$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6. *[18] Let E be a real reflexive Banach space, $f : E \to -(\infty, +\infty]$ be a proper lower semi-continuous function, then $f : E \to -(\infty, +\infty]$ is a proper weak* lower semi-continuous and convex function. Thus, for all $z \in E$, we have*

$$D_f\left(z, \triangledown f^*\left(\sum_{i=1}^{N} t_i \triangledown f(x_i)\right)\right) \le \sum_{i=1}^{N} t_i D_f(z, x_i), \quad where \sum_{i=1}^{N} t_i = 1.$$

Lemma 2.7. *[17] Let E be a Banach space, let $r > 0$ be a constant, and let $f : E \to \mathbb{R}$ be a continuous and convex function which is uniformly convex on bounded subsets of E. Then*

$$f\left(\sum_{k=0}^{\infty}\alpha_k x_k\right) = \sum_{k=0}^{\infty}\alpha_k f(x_k) - \alpha_i\alpha_j\rho_r(||x_i - x_j||),$$

for all $i, j \in \mathbb{N}\cup\{0\}$ $x_k \in B_r$, $\alpha_k \in (0,1)$ and $k \in \mathbb{N}\cup\{0\}$ with $\sum_{k=0}^{\infty}\alpha_k = 1$. where ρ_r is the gauge of uniform convexity of f.

Theorem 2.8. *[26] Let E be a reflexive Banach space and let $f : E \to \mathbb{R}$ be a continuous convex function which is strongly coercive. Then the following are equivalent:*

(1) f is bounded on bounded subsets and uniformly smooth on bounded subsets of E.

(2) f^* is Fréchet differentiable and f^* is uniformly norm-to-norm continuous on bounded subsets of E^*.

(3) $\mathrm{dom} f^* = E^*, f^*$ is strongly coercive and uniformly convex on bounded subsets of E^*.

Lemma 2.9. *[2] Let E be a reflexive Banach space and C a nonempty closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a strongly coercive, Legendre function, which is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of E. Let η be a real number with $\eta \in (-\infty, 1)$ and T an $(\eta, 0)$-Bregman demigeneralized map of C onto E. Then $F(T)$ is closed and convex.*

Lemma 2.10. *[16] Let Γ_n be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_j}\}_{j\geq 0}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_j} \leq \Gamma_{n_j+1}$ for all $j \geq 0$. Also consider a sequence of integers $\{\tau(n)\}_{n\geq n_0}$ defined by*

$$\tau(n) = \max\{k \leq n \mid \Gamma_{n_k} \leq \Gamma_{n_k+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ is a nondecreasing sequence verifying $\lim_{n\to\infty}\tau(n) = \infty$. If it holds that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ then we have

$$\Gamma_n \leq \Gamma_{\tau(n)+1}, \quad \forall n \geq n_0.$$

In order to solve equilibrium problems, we shall assume that the bifunction $\Theta : C \times C \to \mathbb{R}$ satisfies the following conditions [6] :
(A1) $\Theta(x, x) = 0$, $\forall x \in C$;
(A2) Θ is monotone, *i.e.*, $G(x, y) + G(y, x) \leq 0 \quad \forall x, y \in C$;
(A3) for each $y \in C$, $x \mapsto \Theta(x, y)$ is weakly upper semicontinuous

(A4) for each $x \in C$, $y \mapsto \Theta(x, y)$ is convex;
(A5) for each $x \in C$, $y \mapsto \Theta(x, y)$ is lower semi-continuous.

Definition 2.11. Let C be a nonempty closed and convex subset of a reflexive Banach space E and let φ be a lower semi-continuous and convex functional from C to $\mathbb{R}$. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunctional satisfying $(A1)-(A5)$. The mixed resolvent of Θ is the operator $Res^f_{\Theta,\varphi} : E \to 2^C$ defined by

$$Res^f_{\Theta,\varphi}(x) = \big\{z \in \Theta(z, y) + \varphi(y)\langle \nabla f(z) - \nabla f(x), y - z\rangle \geq \varphi(z), \quad \forall y \in C\big\}. \tag{2.1}$$

If $f : (-\infty, +\infty] \to \mathbb{R}$ is strongly coercive and Gâteaux differentiable function, and G satisfies condition $(A1)-(A4)$, then dom($Res^f_{\Theta,\varphi}$)$= E$ (see [21], Lemma 1).
The following lemma gives some characterization of the resolvent $Res^f_{\Theta,\varphi}$.

Lemma 2.12. *[21] Let E be a real reflexive Banach space and C be a nonempty closed convex subset of E. Let $f : E \to (-\infty, +\infty]$ be a Legendre function. If the bifunction $\Theta : C \times C \to \mathbb{R}$ satisfies conditions (A1)-(A4), then the following hold:*

(i) $Res^f_{\Theta,\varphi}$ is singled-valued;

(ii) $Res^f_{\Theta,\varphi}$ is a Bregman firmly nonexpansive operator

(iii) $F(Res^f_{\Theta,\varphi})$=$MEP(\Theta)$;

(iv) $MEP(\Theta)$ is a closed and convex subset of C;

(v) For all $x \in E$ and for all $q \in F(Res^f_{\Theta,\varphi})$ we have $D_f(q, Res^f_{\Theta,\varphi}(x)) + D_f(Res^f_{\Theta,\varphi}(x), x) \leq D_f(q, x) \quad \forall q \in F(Res^f_{\Theta,\varphi}), x \in E.$

3. Main Result

Lemma 3.1. *Let E be a reflexive Banach space and C a nonempty closed convex subset of E and let $f : E \to \mathbb{R}$ be a Fréchet differentiable convex function. Let η be a real number with $\eta \in (-\infty, 0]$ and let $T : C \to E$ be an $(\eta, 0)$-Bregman demigeneralized map with $F(T) \neq \emptyset$. Let α be real number in $[0, 1)$ and let $S = \nabla f^*(\alpha\nabla f + (1 - \alpha)\nabla f(T))$, then $S : C \to E$ is a quasi-Bregman nonexpansive map.*

Proof. It is obvious that $F(T) = F(S)$. Since S is an $(\eta, 0)$-Bregman demigeneralized map, for any $x \in C$, we obtain

$$\begin{aligned} D_f(x, Sx) &= D_f(x, \nabla f^*(\alpha\nabla f(x) + (1 - \alpha)\nabla f(T(x)))) \\ &\leq \alpha D_f(x, x) + (1 - \alpha)D_f(x, Tx) \\ &= (1 - \alpha)D_f(x, Tx), \end{aligned} \tag{3.1}$$

and letting $p \in F(S)$ we get

$$\begin{aligned} \langle x-p, \nabla f(x) - \nabla f(Sx)\rangle &= \langle x-p, \nabla f(x) - \alpha\nabla f(x) - (1-\alpha)\nabla f(T(x))\rangle \\ &= (1-\alpha)\langle x-p, \nabla f(x) - \nabla f(Tx)\rangle \\ &\geq (1-\alpha)(1-\eta)D_f(x,Tx), \end{aligned} \tag{3.2}$$

from (3.1), (3.2) and the fact that $\eta \in (-\infty, 0]$,we have

$$\begin{aligned} \langle x-p, \nabla f(x) - \nabla f(Sx)\rangle &\geq (1-\alpha)(1-\eta)D_f(x,Tx) \\ &\geq (1-\alpha)D_f(x,Tx), \end{aligned} \tag{3.3}$$

from (1.2) and (3.3) we have

$$D_f(p,x) - D_f(p,Sx) + D_f(x,Sx) \geq (1-\alpha)D_f(x,Tx)$$

this and (3.1) implies

$$\begin{aligned} D_f(p,x) - D_f(p,Sx) &\geq (1-\alpha)D_f(x,Tx) - D_f(x,Sx) \\ &\geq (1-\alpha)D_f(x,Tx) - (1-\alpha)D_f(x,Tx), \end{aligned}$$

this implies

$$D_f(p,Sx) \leq D_f(p,x).$$

This completes the proof. □

Theorem 3.2. *Let C be a nonempty closed and convex subset of a reflexive Banach space E and $f : E \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $T_i : C \to C$ $i = 1, 2, \cdots, N$ be finite family of $(k_i, 0)$-Bregman demigeneralized and demiclosed maps, where $k_i \in (-\infty, 0]$ for each $i = 1, 2, \cdots, N$. Let $\Theta : C \times C \to \mathbb{R}$ satisfy conditions (A1)-(A5) such that $\Omega := (\cap_{i=1}^{N} F(T_i)) \cap MEP(\Theta) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm; $x_1 = x \in C$*

$$\begin{cases} y_n = \nabla f^*\big(\lambda_n \nabla f(x_n) + (1-\lambda_n)\nabla f(T_n x_n)\big); \\ z_n = Res^f_{\Theta,\varphi}(y_n); \\ w_n = \nabla f^*(\alpha_n \nabla f(x_n) + \beta_n \nabla f(z_n) + \gamma_n \nabla f(y_n)), \\ x_{n+1} = P^f_C \nabla f^*(\tau_n \nabla f(u) + (1-\tau_n)\nabla f(w_n)) \quad n \geq 1, \end{cases} \tag{3.4}$$

where $T_n = n(mod\ N)$, $0 < c \leq \lambda_n \leq \min\{1-k_1, \ldots, 1-k_N\}$, let $k = \max\limits_{1\leq i\leq N}\{k_i\}$ $\lim\limits_{n\to\infty} \tau_n = 0$ and $\sum_{n=1}^{\infty} \tau_n = \infty$, $0 < \liminf \gamma_n \leq \limsup \gamma_n < 1$, $\beta_n \in [a, b]$ $0 < a,\ b < 1$ and $\alpha_n + \beta_n + \gamma_n = 1$ $\forall n \geq 1$. Then the sequence $\{x_n\}$ generated by (3.4) converges strongly to $p = P^f_\Omega x$.

Proof. From Lemma 2.9, $F(T_i)$ for each $i = 1, 2, \cdots, N$ is closed and convex, hence $\cap_{i=1}^{N} F(T_i)$ is close and convex. Also from Lemma 2.12, $MEP(G)$ is closed and convex. Hence Ω is closed and convex and so $p = P^f_\Omega x$ is well

defined.
Let $p \in \Omega$. Then from (3.4), Lemma 3.1 and Lemma 2.6, we have

$$\begin{aligned} D_f(p, y_n) &= D_f\big(p, \nabla f^*\big(\lambda_n \nabla f(x_n) + (1-\lambda_n)\nabla f(T_n x_n)\big)\big) \\ &= D_f(p, S_n x_n) \\ &\leq D_f(p, x_n). \end{aligned}$$

Also from Lemma 2.12 we have

$$D_f(p, z_n) = D_f(p, Res^f_{\theta,\varphi}(y_n)) \leq D_f(p, y_n) \leq D_f(p, x_n). \tag{3.5}$$

$$\begin{aligned} D_f(p, w_n) &= D_f(p, \nabla f^*(\alpha_n \nabla f(x_n) + \beta_n \nabla f(z_n) + \gamma_n \nabla f(y_n))) \\ &\leq \alpha_n D_f(p, x_n) + \beta_n D_f(p, z_n) + \gamma_n D_f(p, y_n) \\ &\leq \alpha_n D_f(p, x_n) + \beta_n D_f(p, x_n) + \gamma_n D_f(p, x_n) \\ &= D_f(p, x_n). \end{aligned}$$

$$\begin{aligned} D_f(p, x_{n+1}) &\leq D_f(p, \nabla f^*(\tau_n \nabla f(u) + (1-\tau_n)\nabla f(w_n))) \\ &\leq \tau_n D_f(p, u) + (1-\tau_n) D_f(p, w_n) \\ &\leq \tau_n D_f(p, u) + (1-\tau_n) D_f(p, x_n) \\ &\leq \max\Big\{D_f(p, u), D_f(p, x_n)\Big\} \quad \forall n \geq 1. \end{aligned} \tag{3.6}$$

By induction we have

$$D_f(p, x_n) \leq \max\Big\{D_f(p, u), D_f(p, x_1)\Big\}, \ \forall n \geq 1$$

Hence, $\{D_f(p, x_n)\}$ is bounded, which by Lemma 2.3 implies $\{x_n\}$ is bounded. Furthermore, $\{y_n\}$, $\{z_n\}$ and $\{T_i x_n\}$ are bounded for $i = 1, 2, \ldots, N$. Since f is bounded on a bounded subset of E, then ∇f is bounded on bounded subset of E^*, which implies $\{\nabla f(x_n)\}$, $\{\nabla f(y_n)\}$, $\{\nabla f(z_n)\}$ and $\{\nabla f(T_i x_n)\}$ are bounded.
Let $\rho^*_r : E \to \mathbb{R}$ be the gauge function of uniform convexity of the conjugate

function f^*. By Lemma 2.4, Lemma 2.7 and Theorem 2.8 we obtain

$$\begin{aligned}
D_f(p,w_n) &= D_f(p,\nabla f^*(\alpha_n\nabla f(x_n)+\beta_n\nabla f(z_n)+\gamma_n\nabla f(y_n)))\\
&= V_f(p,\alpha_n\nabla f(x_n)+\beta_n\nabla f(z_n)+\gamma_n\nabla f(y_n))\\
&= f(p)-\langle p,\alpha_n\nabla f(x_n)+\beta_n\nabla f(z_n)+\gamma_n\nabla f(y_n)\rangle\\
&\quad +f^*(\alpha_n\nabla f(x_n)+\beta_n\nabla f(z_n)+\gamma_n\nabla f(y_n))\\
&\leq \alpha_n f(p)+\beta_n f(p)+\gamma_n f(p)-\alpha_n\langle p,\nabla f(x_n)\rangle-\beta_n\langle p,\nabla f(z_n)\rangle\\
&\quad -\gamma_n\langle p,\nabla f(y_n)\rangle+\alpha_n f^*(\nabla f(x_n))+\beta_n f^*(\nabla f(z_n))\\
&\quad +\gamma_n f^*(\nabla f(y_n))-\beta_n\gamma_n\rho_r^*(||\nabla f(x_n)-\nabla f(z_n)||)\\
&\leq \alpha_n(f(p)-\langle p,\nabla f(x_n)\rangle+f^*(\nabla f(x_n)))+\beta_n(f(p)-\langle p,\nabla f(z_n)\rangle\\
&\quad +f^*(\nabla f(z_n)))+\gamma_n(f(p)-\langle p,\nabla f(y_n)\rangle+f^*(\nabla f(y_n)))\\
&\quad -\beta_n\gamma_n\rho_r^*(||\nabla f(x_n)-\nabla f(z_n)||)\\
&= \alpha_n V_f(p,\nabla f(x_n))+\beta_n V_f(p,\nabla f(z_n))+\gamma_n V_f(p,\nabla f(y_n))\\
&\quad -\beta_n\gamma_n\rho_r^*(||\nabla f(x_n)-\nabla f(z_n)||)\\
&= \alpha_n D_f(p,x_n)+\beta_n D_f(p,z_n)+\gamma_n D_f(p,y_n)\\
&\quad -\beta_n\gamma_n\rho_r^*(||\nabla f(x_n)-\nabla f(z_n)||)\\
&\leq \alpha_n D_f(p,x_n)+\beta_n D_f(p,z_n)+\gamma_n D_f(p,y_n)\\
&\quad -\beta_n\gamma_n\rho_r^*(||\nabla f(x_n)-\nabla f(z_n)||)
\end{aligned}$$

(3.7)

$$\begin{aligned}
&\leq \alpha_n D_f(p,x_n)+\beta_n D_f(p,x_n)+\gamma_n D_f(p,x_n)\\
&\quad -\alpha_n\beta_n\rho_r^*(||\nabla f(x_n)-\nabla f(z_n)||)\\
&= D_f(p,x_n)-\alpha_n\beta_n\rho_r^*(||\nabla f(x_n)-\nabla f(z_n)||),
\end{aligned}$$

therefore

$$D_f(p,w_n) \leq D_f(p,x_n)-\alpha_n\beta_n\rho_r^*(||\nabla f(x_n)-\nabla f(z_n)||). \tag{3.8}$$

Similarly,

$$D_f(p,w_n) \leq D_f(p,x_n)-\alpha_n\gamma_n\rho_r^*(||\nabla f(x_n)-\nabla f(y_n)||). \tag{3.9}$$

From (3.6),(3.8) and (3.9) we have

$$\begin{aligned}
D_f(p,x_{n+1}) &\leq \tau_n D_f(p,u)+(1-\tau_n)D_f(p,w_n)\\
&\leq \tau_n D_f(p,u)+(1-\tau_n)\big[D_f(p,x_n)\\
&\quad -\alpha_n\beta_n\rho_r^*(||\nabla f(x_n)-\nabla f(z_n)||)\big]\\
&= D_f(p,x_n)-\alpha_n\beta_n\rho_r^*(||\nabla f(x_n)-\nabla f(z_n)||)\\
&\quad -\tau_n\big[D_f(p,x_n)-D_f(p,u)-\alpha_n\beta_n\rho_r^*(||\nabla f(x_n)-\nabla f(z_n)||)\big].
\end{aligned}$$

Thus,

$$\begin{aligned}
D_f(p,x_{n+1}) &\leq D_f(p,x_n)-\alpha_n\beta_n\rho_r^*(||\nabla f(x_n)-\nabla f(z_n)||)\\
&\quad -\tau_n\big[D_f(p,x_n)-D_f(p,u)\\
&\quad -\alpha_n\beta_n\rho_r^*(||\nabla f(x_n)-\nabla f(z_n)||)\big]
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned} D_f(p, x_{n+1}) &\le D_f(p, x_n) - \alpha_n\gamma_n\rho_r^*(||\nabla f(x_n) - \nabla f(y_n)||) \\ &\quad -\tau_n\big[D_f(p, x_n) - D_f(p, u) \\ &\quad -\alpha_n\gamma_n\rho_r^*(||\nabla f(x_n) - \nabla f(y_n)||)\big]. \qquad (3.11) \end{aligned}$$

Let $v_n = \nabla f^*(\tau_n \nabla f(u) + (1-\tau_n)\nabla f(w_n))$ and using Lemma 2.4 we have

$$\begin{aligned} D_f(p, x_{n+1}) &= D_f(p, P_C^f \nabla f^*(\tau_n \nabla f(u) + (1-\tau_n)\nabla f(w_n))) \\ &\le D_f(p, \nabla f^*(\tau_n \nabla f(u) + (1-\tau_n)\nabla f(w_n))) \\ &= V_f(p, \tau_n \nabla f(u) + (1-\tau_n)\nabla f(w_n)) \\ &\le V_f(p, \tau_n \nabla f(u) + (1-\tau_n)\nabla f(w_n) - \tau_n(\nabla f(u) - \nabla f(p))) \\ &\quad +\tau_n\langle v_n - p, \nabla f(u) - \nabla f(p)\rangle \\ &= V_f(p, \tau_n \nabla f(p) + (1-\tau_n)\nabla f(w_n)) \\ &\quad +\tau_n\langle v_n - p, \nabla f(u) - \nabla f(p)\rangle \\ &= \tau_n V_f(p, \nabla f(p)) + (1-\tau_n)V_f(p, \nabla f(w_n)) \\ &\quad +\tau_n\langle v_n - p, \nabla f(u) - \nabla f(p)\rangle \\ &\le \tau_n D_f(p, p) + (1-\tau_n)D_f(p, w_n) \\ &\quad +\tau_n\langle v_n - p, \nabla f(u) - \nabla f(p)\rangle \\ &\le (1-\tau_n)D_f(p, x_n) + \tau_n\langle v_n - p, \nabla f(u) - \nabla f(p)\rangle. \qquad (3.12) \end{aligned}$$

Case 1. Suppose there exists $n_0 \in \mathbb{N}$ such that $\{D_f(p, x_n)\}$ is nonincreasing $\forall n \ge n_0$. Then $\{D_f(p, x_n)\}$ is convergent. Thus from equations (3.10) and (3.11) we have

$$\begin{aligned} \alpha_n\beta_n\rho_r^*(||\nabla f(x_n) - \nabla f(z_n)||) &\le D_f(p, x_n) - D_f(p, x_{n+1}) \\ &\quad -\tau_n\big[D_f(p, x_n) - D_f(p, u) \\ &\quad -\alpha_n\beta_n\rho_r^*(||\nabla f(x_n) - \nabla f(z_n)||)\big] \qquad (3.13) \end{aligned}$$

and

$$\begin{aligned} \alpha_n\gamma_n\rho_r^*(||\nabla f(x_n) - \nabla f(y_n)||) &\le D_f(p, x_n) - D_f(p, x_{n+1}) \\ &\quad -\tau_n\big[D_f(p, x_n) - D_f(p, u) \\ &\quad -\alpha_n\gamma_n\rho_r^*(||\nabla f(x_n) - \nabla f(y_n)||)\big], \qquad (3.14) \end{aligned}$$

which implies

$$\lim_{n\to\infty} \big(\alpha_n\beta_n\rho_r^*(||\nabla f(x_n) - \nabla f(z_n)||)\big) = 0$$

and

$$\lim_{n\to\infty} \big(\alpha_n\gamma_n\rho_r^*(||\nabla f(x_n) - \nabla f(y_n)||)\big) = 0.$$

Using the property of ρ_s^* and conditions on α_n, γ_n and β_n we obtain

$$\lim_{n\to\infty} ||\nabla f(x_n) - \nabla f(z_n)|| = 0 \qquad (3.15)$$

and

$$\lim_{n\to\infty} ||\nabla f(x_n) - \nabla f(y_n)|| = 0. \tag{3.16}$$

Since ∇f is norm to norm uniformly continuous on bounded subset of E^* we have

$$\lim_{n\to\infty} ||x_n - z_n|| = 0 \tag{3.17}$$

and

$$\lim_{n\to\infty} ||x_n - y_n|| = 0. \tag{3.18}$$

But

$$||z_n - y_n|| \leq ||z_n - x_n|| + ||x_n - y_n||,$$

which from (3.17) and (3.18) implies

$$\lim_{n\to\infty} ||z_n - y_n|| = 0.$$

From (3.4) we have

$$\begin{aligned} ||\nabla f(w_n) - \nabla f(x_n)|| &\leq \alpha_n ||\nabla f(x_n) - \nabla f(x_n)|| + \beta_n ||\nabla f(z_n) - \nabla f(x_n)|| \\ &\quad + \gamma_n ||\nabla f(y_n) - \nabla f(x_n)||, \end{aligned}$$

which by (3.15) and (3.16) implies

$$\lim_{n\to\infty} ||\nabla f(w_n) - \nabla f(x_n)|| = 0.$$

Since ∇f is norm to norm uniformly continuous on bounded subset of E^* we have

$$\lim_{n\to\infty} ||w_n - x_n|| = 0. \tag{3.19}$$

From (3.17) and (3.19) we have

$$\lim_{n\to\infty} ||w_n - z_n|| = 0.$$

Also from (3.18) and (3.19) we have

$$\lim_{n\to\infty} ||w_n - y_n|| = 0.$$

Now, by definition 1.5, we obtain

$$\begin{aligned} \langle x_n - p, \nabla f(x_n) - \nabla f(y_n) \rangle &= \langle x_n - p, \nabla f(x_n) - \lambda_n \nabla f(x_n) \\ &\quad - (1-\lambda_n) \nabla f(T_n x_n) \rangle \\ &= (1-\lambda_n) \langle x_n - p, \nabla f(x_n) - \nabla f(T_n x_n) \rangle \\ &\geq (1-\lambda_n)(1-K) D_f(x_n, T_n x_n). \end{aligned}$$

Hence

$$(3.20)\quad (1-\lambda_n)(1-K)D_f(x_n, T_nx_n) \quad \leq \quad ||x_n - p||||\nabla f(x_n) - \nabla f(y_n)||,$$

which from (3.15) gives

$$\lim_{n\to\infty} D_f(x_n, T_nx_n) = 0.$$

Since f is totally convex on bounded subset of E, f is sequentially consistent. It follows that

$$\lim_{n\to\infty} ||x_n - T_nx_n|| = 0. \tag{3.21}$$

Since $v_n = \nabla f^*(\tau_n \nabla f(u) + (1-\tau_n)\nabla f(w_n))$ we have

$$\begin{aligned} D_f(w_n, v_n) &= D_f(w_n, \nabla f^*(\tau_n \nabla f(u) + (1-\tau_n)\nabla f(w_n))) \\ &\leq \tau_n D_f(w_n, u) + (1-\tau_n)D_f(w_n, w_n) \\ &= \tau_n D_f(w_n, u), \end{aligned}$$

which implies that

$$\lim_{n\to\infty} D_f(w_n, v_n) = 0.$$

Since f is totally convex on bounded subset of E, f is sequentially consistent, it follows that

$$\lim_{n\to\infty} ||w_n - v_n|| = 0. \tag{3.22}$$

Also,

$$||x_n - v_n|| \leq ||x_n - w_n|| + ||w_n - v_n||,$$

which from (3.19) and (3.22) implies that

$$\lim_{n\to\infty} ||x_n - v_n|| = 0.$$

Since $\{x_n\}$ is bounded and E is reflexive, then there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup q$, which implies that $y_{n_k} \rightharpoonup q$ and $z_{n_k} \rightharpoonup q$ as $k \to \infty$. From the fact that T_i is demiclosed and (3.21) it follows that $q \in F(T_i)$ for each $i = 1, 2, \cdots, \mathbb{N}$.
Next, we show that $q \in MEP(\Theta)$. The equation $z_n = Res^f_{\Theta,\varphi}(x_n)$ implies that for each $n \geq 1$.

$$\Theta(z_n, y) + \varphi(y) - \varphi(z_n) + \langle y - z_n, \nabla f(z_n) - \nabla f(x_n)\rangle \geq 0, \quad \forall y \in C.$$

Hence

$$\Theta(z_{n_k}, y) + \varphi(y) - \varphi(z_{n_k}) + \langle y - z_{n_k}, \nabla f(z_{n_k}) - \nabla f(x_{n_k})\rangle \geq 0, \quad \forall y \in C.$$

By applying $(A2)$, we have

$$\varphi(y) - \varphi(z_{n_k}) + \langle y - z_{n_k}, \nabla f(z_{n_k}) - \nabla f(x_{n_k})\rangle \geq -\Theta(z_{n_k}, y) \geq \Theta(y, z_{n_k}) \quad \forall y \in C.$$

From the fact that φ is lower semicontinous and also Θ is lower semicontinuous in the second variable, (3.17) and $z_{n_k} \rightharpoonup q$, we have

$$\Theta(y,q) + \varphi(q) - \varphi(y) \leq 0 \quad \forall y \in C.$$

Let $y_t = ty + (1-t)q$ for $t \in [0,1]$ and $y \in C$. This implies that $y_t \in C$. This yields $\Theta(y_t,q) + \varphi(q) - \varphi(y_t) \leq 0$. It follows from $(A1)$ and $(A5)$ that

$$\begin{aligned} 0 &= \Theta(y_t,y_t) + \varphi(y_t) - \varphi(y_t) \\ &\leq t\Theta(y_t,y) + (1-t)\Theta(y_t,q) + t\varphi(y) + (1-t)\varphi(q) - \varphi(y_t) \\ &\leq t[\Theta(y_t,y) + \varphi(y) - \varphi(y_t)]. \end{aligned}$$

This implies

$$0 \leq \Theta(y_t,y) + \varphi(y) - \varphi(y_t).$$

Therefore, we obtain

$$\Theta(q,y) + \varphi(y) - \varphi(q) \geq 0, \quad \forall y \in C.$$

This implies $q \in MEP(\Theta)$. Hence $q \in \Omega := F(T_i) \cap MEP(\Theta)$.
Next, we show that $\{x_n\}$ converges strongly to $p = P^f_\Omega x_0$. Since $\{x_n\}$ is bounded, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that $x_{n_k} \rightharpoonup q$ and

$$\begin{aligned} \lim_{k\to\infty} \langle x_{n_k} - p, \nabla f(u) - \nabla f(p)\rangle &= \limsup_{n\to\infty} \langle x_n - p, \nabla f(u) - \nabla f(p)\rangle \\ &= \limsup_{n\to\infty} \langle v_n - p, \nabla f(u) - \nabla f(p)\rangle. \end{aligned}$$

Using Lemma 2.1 we have

$$\lim_{k\to\infty} \langle x_{n_k} - p, \nabla f(u) - \nabla f(p)\rangle = \langle q - p, \nabla f(u) - \nabla f(p)\rangle \leq 0.$$

Hence,

$$\limsup_{n\to\infty} \langle v_n - p, \nabla f(u) - \nabla f(p)\rangle \leq 0. \tag{3.23}$$

Hence from (3.12), (3.23) and Lemma 2.5, we have $x_n \to p$.
Case 2. If the assumption in Case 1 does not hold, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $D_f(p,x_{n_k}) < D_f(p,x_{n_k+1}), \forall k \in \mathbb{N}$. From Lemma 2.10, there exists a nondecreasing sequence $\{m_j\} \subset \mathbb{N}$, such that $\lim_{n\to\infty} m_j = \infty$ and the following inequalities hold
$D_f(p,x_{m_j}) \leq D_f(p,x_{m_j+1})$ and $D_f(p,x_j) \leq D_f(p,x_{m_j+1})$ for all $j \in \mathbb{N}$.
Combining this together with (3.13) and (3.14), we have

$$\begin{aligned} \alpha_{m_j}\beta_{m_j}\rho_r^*(||\nabla f(x_{m_j}) - \nabla f(z_{m_j})||) &\leq D_f(p,x_{m_j}) - D_f(p,x_{m_j+1}) \\ &\quad -\tau_{m_j}\big[D_f(p,x_{m_j}) - D_f(p,m_j) \\ &\quad -\alpha_{m_j}\beta_{m_j}\rho_r^*(||\nabla f(x_{m_j}) - \nabla f(z_{m_j})||)\big] \end{aligned}$$

and

$$\begin{aligned}\alpha_{m_j}\gamma_{m_j}\rho_r^*(||\nabla f(x_{m_j})-\nabla f(y_{m_j})||) &\leq D_f(p,x_{m_j})-D_f(p,x_{m_j+1})\\ &\quad -\tau_{m_j}\big[D_f(p,x_{m_j})-D_f(p,m_j)\\ &\quad -\alpha_{m_j}\gamma_{m_j}\rho_r^*(||\nabla f(x_{m_j})-\nabla f(y_{m_j})||)\big].\end{aligned}$$

The following can be obtained using the same argument as in Case 1 above,

1. $$\lim_{j\to\infty}||y_{m_j}-x_{m_j}||=0, \quad \lim_{j\to\infty}||z_{m_j}-x_{m_j}||=0.$$

2. $$\lim_{j\to\infty}||w_{m_j}-x_{m_j}||=0, \qquad \lim_{j\to\infty}||x_{m_j}-T_ix_{m_j}||=0.$$

3. $$\limsup_{j\to\infty}\langle v_{m_j}-p,\nabla f(u)-\nabla f(p)\rangle\leq 0. \tag{3.24}$$

It follows from (3.12) that

$$D_f(p,x_{m_j+1})\leq(1-\alpha_{m_j})D_f(p,x_{m_j})+\alpha_{m_j}\langle v_{m_j}-p,\nabla f(u)-\nabla f(p)\rangle. \tag{3.25}$$

Since $D_f(p,x_{m_j})\leq D_f(p,x_{m_j+1})$, we have

$$\begin{aligned}\alpha_{m_j}D_f(p,x_{m_j}) &\leq D_f(p,x_{m_j})-D_f(p,x_{m_j+1})\\ &\quad +\alpha_{m_j}\langle v_{m_j}-p,\nabla f(u)-\nabla f(p)\rangle\\ &\leq \alpha_{m_j}\langle v_{m_j}-p,\nabla f(u)-\nabla f(p)\rangle. \end{aligned}\tag{3.26}$$

From (3.24) we have

$$\limsup_{j\to\infty}D_f(p,x_{m_j})=0.$$

Putting this together with (3.25), we have

$$\limsup_{j\to\infty}D_f(p,x_{m_j+1})=0.$$

On the other hand, we have $D_f(p,x_j)\leq D_f(p,x_{m_j+1})$ for all $j\in\mathbb{N}$. This implies that $x_j\to p$ as $j\to\infty$. Hence $x_n\to p$ as $n\to\infty$. □

Corollary 3.3. *Let C be a nonempty closed and convex subset of a reflexive Banach space E and $f:E\to\mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $T_i:C\to C$ $i=1,2,\cdots,N$ be a finite family of Bregman quasi nonexpansive maps. Let $\Theta:C\times C\to\mathbb{R}$ satisfy conditions (A1)-(A5) such that*

$\Omega := (\cap_{i=1}^{N} F(T_i)) \cap MEP(\Theta) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm; $x_1 = x \in C$

$$
\begin{cases} y_n = \nabla f^*\big(\lambda_n \nabla f(x_n) + (1-\lambda_n)\nabla f(T_n x_n)\big); \\ z_n = Res^f_{\Theta,\varphi}(y_n); \\ w_n = \nabla f^*(\alpha_n \nabla f(x_n) + \beta_n \nabla f(z_n) + \gamma_n \nabla f(y_n)), \\ x_{n+1} = P^f_C \nabla f^*(\tau_n \nabla f(u) + (1-\tau_n)\nabla f(w_n)) \quad n \geq 1, \end{cases} \tag{3.27}
$$

where $T_n = n(mod\ N)$, $0 < c \leq \lambda_n \leq \min\{1-k_1, \ldots, 1-k_N\}$, $\lim_{n\to\infty} \tau_n = 0$ and $\sum_{n=1}^{\infty} \tau_n = \infty$, $0 < \liminf \gamma_n \leq \limsup \gamma_n < 1$, $\beta_n \in [a, b]$ $\ 0 < a,\ b < 1$ and $\alpha_n + \beta_n + \gamma_n = 1\ \forall n \geq 1$. Then the sequence $\{x_n\}$ generated by (1.5) converges strongly to $p = P^f_{\Omega} x$.

Proof. Since T_i is a finite family of Bregman quasi nonexpansive mappings, then T_i is $(0,0)$-Bregman demigeneralized mappings. Therefore the result follows from Theorem 3.2. □

Corollary 3.4. *(see [5], theorem 3.1) Let C be a nonempty closed and convex subset of a reflexive Banach space E and $f : E \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $T_i : C \to C$ $i = 1, 2, \cdots, N$ be a finite family of Bregman strongly nonexpansive maps with respect to f such that $F(T_i) = \hat{F}(T_i)$ and each T_i is uniformly continous. Let $\Theta : C \times C \to \mathbb{R}$ satisfy conditions (A1)-(A5) such that $\Omega := (\cap_{i=1}^{N} F(T_i)) \cap MEP(\Theta) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$
\begin{cases} x_1 = x \in C \\ z_n = Res^f_{\Theta,\varphi}(x_n) \\ y_n = proj^f_C \nabla f^*\big(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(Tz_n)\big); \\ x_{n+1} = proj^f_C \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Ty_n)), \quad n \geq 1, \end{cases} \tag{3.28}
$$

where $T = T_N \circ T_{N-1} \circ \cdots \circ T_1$ and T_i is a Bregman strongly nonexpansive map for each $i = 1, 2, \cdots, N$. Then the sequence $\{x_n\}$ converges strongly to $Proj^f_{(\cap_{i=1}^{N} F(T_i)) \cap MEP(\Theta))}$

Proof. Since T_i are Bregman strongly nonexpansive mappings, then T_i are $(0,0)$-Bregman demigeneralized mappings. Therefore the result follows from Theorem 3.2. □

Corollary 3.5. *(see [23], theorem 18) Let C be a nonempty closed and convex subset of a real reflexive Banach space E and $f : E \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. For each $j = 1, 2, \cdots, m$ and let Θ_j be a bifunction from $C \times C$ to $\mathbb{R}$ satisfy conditions (A1)-(A5)and let $\{T^N_{i=1}\}$ be a finite family of quasi-Bregman nonexpansive mappings of C such that $F := \cap_{i=1}^{N} F(T_i) \neq \emptyset$ where $F = F(T_N T_{N-1} \ldots T_2 T_1) = F(T_1 T_N T_{N-1} \ldots T_2) = \cdots =$*

$F(T_{N-1}T_{N-2}\dots T_2T_1T_N) \neq \emptyset$ and $\Omega := (\cap_{j=1}^{m} F(\Theta_j)) \cap F \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm; $x_1 = x \in C$

$$\begin{cases} C_1 \in C \\ u_{j,n} = Res^f_{\Theta_j}(x_n); \quad j = 1, 2, \cdots, m \\ y_n = P_C\big(\nabla f^*\big((1-\tau_n)\nabla f(u_{j,n})\big)\big); \\ x_{n+1} = P_C\big(\nabla f^*(\beta_n \nabla f(y_n) + (1-\beta_n)\nabla f(T_n y_n))\big), \quad n \geq 1, \end{cases} \tag{3.29}$$

where $T_{[n]} = T_{n(mod\ N)}$, and $\{\tau_n\}_{n=1}^{\infty} \subset (0,1)$ and $\{\beta_n\}_{n=1}^{\infty} \subset [c,d] \subset (0,1)$, satisfying $\lim\limits_{n\to\infty} \tau_n = 0$ and $\sum_{n=1}^{\infty} \tau_n = \infty$. Then the sequence $\{x_n\}$ generated by (3.29) converges strongly to $P_\Omega x$.

Proof. Since T_i are quasi-Bregman nonexpansive mappings, then T_i are $(0,0)$-Bregman demigeneralized mappings. Therefore the result follows from Theorem 3.2. □

Corollary 3.6. *Let C be a nonempty closed and convex subset of a uniformly smooth and uniformly convex Banach space E and $T_i : C \to C$, $i = 1, 2, \cdots, N$ be a finite family of quasi nonexpansive mappings. Let $\Theta : C \times C \to \mathbb{R}$ satisfy conditions (A1)-(A5) such that $\Omega := (\cap_{i=1}^{N} F(T_i)) \cap MEP(\Theta) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm; $x_1 = x \in C$*

$$\begin{cases} y_n = J^{-1}\big(\sum_{i=1}^{N} \eta_i\big((1-\lambda_n)Jx_n + \lambda_n JT_i x_n\big)\big); \\ z_n = Res^f_{\Theta,\varphi}(y_n); \\ w_n = J^{-1}(\alpha_n Jx_n + \beta_n Jz_n + \gamma_n Jy_n), \\ x_{n+1} = P_C J^{-1}(\tau_n Ju + (1-\tau_n)Jw_n) \quad n \geq 1, \end{cases} \tag{3.30}$$

where $T_n = n(mod\ N)$, $0 < c \leq \lambda_n \leq \min\{1-k_1, \dots, 1-k_N\}$, $\lim\limits_{n\to\infty} \tau_n = 0$ and $\sum_{n=1}^{\infty} \tau_n = \infty$, $0 < \liminf \gamma_n \leq \limsup \gamma_n < 1$, $\beta_n \in [a,b]$ $\quad 0 < a,\ b < 1$ and $\alpha_n + \beta_n + \gamma_n = 1\ \forall n \geq 1$ Then, the sequence $\{x_n\}$ generated by (3.30) converges strongly to $p = P_\Omega x$.

Proof. Since T_i are quasi nonexpansive mappings, then T_i are $(0,0)$-demigeneralized mappings. From the fact that every demigeneralized mapping is Bregman demigeneralized mapping, the result follows from Theorem 3.2. □

4. Application

In this section we present the application of Theorem 3.2

Theorem 4.1. *Let C be a nonempty closed and convex subset of a reflexive Banach space E and $f : E \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $T_i : C \to C$, $i = 1, 2, \cdots, N$ be a finite family of k_i quasi-Bregman strictly pseudocontractive and demiclosed maps, where $k_i \in (0,1)$ for each $i = 1, 2, \cdots, N$. Let $\Theta : C \times C \to \mathbb{R}$ satisfy conditions (A1)-(A5) such that $\Omega :=$*

$(\cap_{i=1}^N F(T_i)) \cap MEP(\Theta) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following algorithm; $x_1 = x \in C$

$$
\begin{cases} y_n = \nabla f^*\big(\lambda_n \nabla f(x_n) + (1-\lambda_n)\nabla f(T_n x_n)\big); \\ z_n = Res^f_{\Theta,\varphi}(y_n); \\ w_n = \nabla f^*(\alpha_n \nabla f(x_n) + \beta_n \nabla f(z_n) + \gamma_n \nabla f(y_n)), \\ x_{n+1} = P^f_C \nabla f^*(\tau_n \nabla f(u) + (1-\tau_n)\nabla f(w_n)) \quad n \geq 1, \end{cases} \tag{4.1}
$$

where $T_n = n(mod\ N)$, $0 < c \leq \lambda_n \leq \min\{1-k_1, \ldots, 1-k_N\}$, let $k = \max\limits_{1\leq i\leq N}\{k_i\}$ $\lim\limits_{n\to\infty} \tau_n = 0$ and $\sum_{n=1}^\infty \tau_n = \infty$, $0 < \liminf \gamma_n \leq \limsup \gamma_n < 1$, $\beta_n \in [a,b]$ $0 < a$, $b < 1$ and $\alpha_n + \beta_n + \gamma_n = 1$ $\forall n \geq 1$. Then, the sequence $\{x_n\}$ generated by (4.1) converges strongly to $p = P^f_\Omega x$.

Proof. Since T is a quasi-Bregman strictly pseudocontractive mapping with $F(T) \neq \emptyset$, then T is a $(k,0)$-Bregman demigeneralized mapping. Therefore the result follows from Theorem 3.2. □

Remark 4.2. Our result extends and generalizes the result of Biranvand and Darvish [5], from a Bregman strongly nonexpansive map to a finite family of Bregman demigeneralized maps. Also our work extends and generalizes the results of Kumam et.al [14] and Ugwunnadi and Bashir [23] from finite family of quasi-Bregman nonexpansive maps to finite family of quasi-Bregman demigeneralized maps.

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