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# Mixed equilibrium and fixed point problems for a countable family of multi-valued Bregman quasi-nonexpansive mappings in reflexive Banach space

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**Abstract**

In this paper, we introduce a new iterative algorithm with Bregman distance approach for approximating a common solution of a finite family of Mixed Equilibrium Problem (MEP) with a relaxed monotone mapping and a countable family of Bregman multi-valued quasi-nonexpansive mappings in a reflexive Banach space. Under standard and mild assumption of relaxed monotonicity of the MEP associated mapping, we establish the strong convergence of the iterative sequence. A numerical example is presented to illustrate the performance of our method. The results obtained in this work extend and complement many related results in literature.

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## 1 Introduction

Let $E$ be a reflexive Banach space with $E^*$ its dual and $Q$ be a nonempty closed and convex subset of $E$. Let $f : E \to (-\infty, +\infty]$ be a proper, lower

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semicontinuous and convex function, then the Fenchel conjugate of $f$ denoted as $f^* : E^* \to (-\infty, +\infty]$ is defined as

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\},\ x^* \in E^*.$$

Let the domain of $f$ be denoted as $dom f = \{x \in E : f(x) < +\infty\}$, hence for any $x \in int dom f$ and $y \in E$, we define the right-hand derivative of $f$ at $x$ in the direction of $y$ by

$$f^0(x, y) = \lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}.$$

The function $f$ is said to be

(i) Gâteaux differentiable at $x$ if $\lim_{t\to 0^+} \frac{f(x+ty)-f(x)}{t}$ exists for any $y$. In this case, $f^0(x, y)$ coincides with $\bigtriangledown f(x)$ (the value of the gradient $\bigtriangledown f$ of $f$ at $x$);

(ii) Gâteaux differentiable, if it is Gâteaux differentiable for any $x \in int dom f$;

(iii) Fréchet differentiable at $x$, if its limit is attained uniformly in $||y|| = 1$;

(iv) Uniformly Fréchet differentiable on a subset $Q$ of $E$, if the above limit is attained uniformly for $x \in Q$ and $||y|| = 1$.

Let $f : E \to (-\infty, +\infty]$ be a function, then $f$ is said to be:

(i) essentially smooth, if the subdifferential of $f$ denoted as $\partial f$ is both locally bounded and single-valued on its domain, where $\partial f(x) = \{w \in E : f(x) - f(y) \geq \langle w, y - x\rangle,\ y \in E\}$;

(ii) essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and $f$ is strictly convex on every convex subset of $dom\ \partial f$;

(iii) Legendre, if it is both essentially smooth and essentially strictly convex. See [7, 8, 34] for more details on Legendre functions.

Alternatively, a function $f$ is said to be Legendre if it satisfies the following conditions:

(i) The $int dom f$ is nonempty, $f$ is Gâteaux differentiable on $int dom f$ and $dom \bigtriangledown f = int dom f$;

(ii) The $int dom f^*$ is nonempty, $f^*$ is Gâteaux differentiable on $int dom f^*$ and $dom \bigtriangledown f^* = int dom f$.

Let $E$ be a Banach space and $B_s := \{z \in E : ||z|| \leq s\}$ for all $s > 0$. Then, a function $f : E \to \mathbb{R}$ is said to be uniformly convex on bounded subsets of $E$, [ see pp. 203 and 221] [42] if $\rho_s t > 0$ for all $s, t > 0$, where $\rho_s : [0, +\infty) \to [0, \infty]$ is defined by

$$\rho_s(t) = \inf_{x,y\in B_s, ||x-y||=t, \alpha\in(0,1)} \frac{\alpha f(x) + (1 - \alpha)f(y) - f(\alpha(x) + (1 - \alpha)y)}{\alpha(1 - \alpha)},$$

for all $t \geq 0$, with $\rho_s$ denoting the gauge of uniform convexity of $f$. The function $f$ is also said to be uniformly smooth on bounded subsets of $E$, [ see pp. 221] [42], if $\lim_{t\downarrow 0} \frac{\sigma_s}{t}$ for all $s > 0$, where $\sigma_s : [0, +\infty) \to [0, \infty]$ is defined by

$$\sigma_s(t) = \sup_{x\in B, y\in S_E, \alpha\in(0,1)} \frac{\alpha f(x) + (1-\alpha)ty) + (1-\alpha)g(x - \alpha ty) - g(x)}{\alpha(1-\alpha)},$$

for all $t \geq 0$. The function $f$ is said to be uniformly convex if the function $\delta f : [0, +\infty) \to [0, +\infty)$ defined by

$$\delta f(t) := \sup \{ \frac{1}{2} f(x) + \frac{1}{2} f(y) - f(\frac{x+y}{2}) : ||y - x|| = t \},$$

satisfies $\lim_{t\downarrow 0} \frac{\delta f(t)}{t} = 0$.

**Definition 1.1.** [33] Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. Then, the function $D_f : E \times E \to [0, +\infty)$ defined by

$$D_f(x, y) := f(x) - f(y) - \langle \bigtriangledown f(y), x - y \rangle$$

is called the Bregman distance with respect to $f$.

It is well-known that Bregman distance $D_f$ does not satisfy the properties of a metric because $D_f$ fail to satisfy the symmetric and triangular inequality property. However, the Bregman distance satisfies the following so-called three point identity: for any $x \in dom f$ and $y, z \in int dom f$,

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \bigtriangledown f(z) - \bigtriangledown f(y), x - y \rangle. \tag{1.1}$$

Recall that $f$ is said to be totally convex at a point $x \in Dom f$, if the function $v_f : int dom f \times [0, +\infty) \to [0, +\infty)$ defined by

$$v_f(x, t) := \inf \{ D_f(y, x) : y \in \ int dom f, ||y - x|| = t \},$$

is positive whenever $t > 0$. Readers should check the following articles [10, 12, 16, 35] for more details on uniformly convex and totally convex functions.
Let $E$ be a real Banach space with $E^*$ its dual and $C$ be a nonempty subset of $E$. An element $p \in C$ is called a fixed point of a single-valued mapping $T : C \to C$, if $p = Tp$ and of a multi-valued mapping $T : C \to 2^C$ if $p \in Tp$. We denote by $F(T)$, the set of all fixed points of $T$.

**Definition 1.2.** Let $E$ be a Banach space and let $f : E \to (-\infty, +\infty)$ be a proper, lower semicontinuous function. Let $C$ be a nonempty subset of $int dom f$. A mapping $T : C \to int dom f$ is said to be:

(i) Bregman firmly nonexpansive (BFNE) if

$$\langle Tx - Ty, \bigtriangledown f(Tx) - \bigtriangledown f(Ty) \leq \langle Tx - Ty, \bigtriangledown f(x) - \bigtriangledown f(y) \rangle,$$

for any $x, y \in C$. Alternatively

$$\begin{aligned} D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \\ \leq D_f(Tx, y) + D_f(Ty, x). \end{aligned} \tag{1.2}$$

(ii) Bregman quasi firmly nonexpansive (BQFNE) if $F(T) \neq \emptyset$ and

$$\langle Tx - p, \nabla f(x) - \nabla f(Tx) \rangle \geq 0, \quad \forall x \in C, \ \ p \in F(T),$$

alternatively

$$D_f(p, Tx) + D_f(Tx, x) \leq D_f(p, x).$$

(iii) Bregman quasi nonexpansive (BQNE) if $F(T) \neq \emptyset$ and

$$D_f(p, Tx) \leq D_f(p, x), \quad \forall x \in E, \ \ p \in F(T).$$

Recall that a mapping $T : C \to C$ is said to be:

(i) nonexpansive, if $||Tx - Ty|| \leq ||x - y||, \ \forall \ x, y \in C$,

(ii) quasi-nonexpansive, if $F(T) \neq \emptyset$ and $||Tx - p|| \leq ||x - p||, \forall \ x \in C, \ p \in F(T)$.

Let $CB(E)$ denote the family of all nonempty closed bounded subsets of $E$ and $P(C)$ denote the family of all nonempty closed proximinal bounded subset of $C$. A subset $K$ of $E$ is said to be proximinal, if for each $x \in E$, there exists an element $k \in K$ such that $d(x, k) = d(x, K)$, where $d(x, K) = \inf\{||x - y|| : y \in K\}$ is the distance from the point $x$ to the set $K$.
For a multi-valued mapping, $T : C \to P(C)$, we define a multi-valued mapping $P_T : C \to P(C)$ by

$$P_T(x) = \{y \in T(x) : ||x - y|| = d(x, T(x))\}, \ \forall \ x \in C. \tag{1.3}$$

Let $T : C \to P(C)$ be a multi-valued mapping and $P_T : C \to P(C)$ be the mapping defined by (1.3), then, $F(T) = F(P_T)$ and $P_T(p) = \{p\}, \ \ $ for each $p \in F(T)$, see [14].
The Hausdorff metric on CB(E) is defined by

$$\mathcal{H}(A, B) = \max\big\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\big\}, \ \forall \ A, B \in CB(E). \tag{1.4}$$

A multi-valued mapping $T : C \to CB(C)$ is said to be
(i) nonexpansive, if for all $x, y \in C$,

$$\mathcal{H}(Tx, Ty) \leq ||x - y||, \ \forall \ x, y \in C;$$

(ii) quasi-nonexpansive, if $F(T) \neq \emptyset$ and

$$\mathcal{H}(Tx, Tp) \leq ||x - p||, \ \forall \ x \in C, \ p \in F(T).$$

In 1967, Bregman [10] discovered an effective technique (the Bregman distance function $D_f$) in the process of designing and analysing feasibility and optimization algorithms. In 2010, Reich and Sabach [37] introduced the class of Bregman strongly nonexpansive mappings and studied the convergence of two

iterative algorithms for finding common fixed points of finitely many Bregman strongly nonexpansive mappings in reflexive Banach spaces.
Also in 2012, Suantai et al. [38] considered the strong convergence results for fixed points of Bregman strongly nonexpansive mappings in reflexive Banach spaces. Very recently, Chang and Wang [14] proposed a shrinking projection method for a countable family of multi-valued Bregman quasi-nonexpansive mappings and obtained a strong convergence result under some mild conditions in the framework of a real reflexive Banach space. In fact, they proved the following theorem.

**Theorem 1.3.** *Let $C$ be a nonempty, closed and convex subset of a real reflexive Banach space $E$. Let $f : E \to (-\infty, +\infty]$ be a Legendre function which is bounded on bounded subsets of $E$. For $i = 1, 2\dots,$ let $T_i : C \to P(C)$ be Bregman multi-valued nonexpansive mappings with $\Gamma := \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ such that all $P_{T_i} : C \to P(C)$ defined by (1.3) are Bregman quasi-nonexpansive. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in C, \quad \textit{chosen arbitrarily}, \quad C_1 = C, \\ y_{n,m} = \bigtriangledown f^*[\alpha_n \bigtriangledown f(x_1) + (1-\alpha_n) \bigtriangledown f(u_{n,m})], \quad u_{n,m} \in P_{T_m} x_m, \; m \geq 1, \\ C_{n+1} = \left\{ z \in C_n : \sup_{m \geq 1} D_f(z, y_{n,m}) \leq \alpha_n D_f(z, x_1) + (1-\alpha_n) D_f(z, x_n) \right\}, \\ x_{n+1} = Proj_{C_{n+1}}^f (x_1), \quad \forall n \geq 1, \end{cases}$$

*where $Proj_{C_{n+1}}^f$ is the Bregman projections of $intdomf$ onto $C_{n+1}$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\alpha_n \to 0$ as $n \to \infty$, then $\{x_n\}$ converges strongly to $Proj_{C_{n+1}}^{\Gamma}(x_1)$.*

Let $E$ be a real reflexive Banach space and $f : E \to (-\infty, +\infty]$ be a Legendre function. Let $C$ be a subset of $intdom(f)$ and $T : C \to P(C)$ be a multivalued mapping. $T$ is said to be multivalued Bregman quasi-nonexpansive, if $F(T) \neq \emptyset$ and the mapping defined by (1.3) satisfies the following condition

$$D_f(p, w) \leq D_f(p, x), \quad \forall x \in C, \quad w \in P_T(x), \quad p \in F(T).$$

In particular, if $T : C \to C$ is a single valued mapping (It is easy to show that $P_T = T$). Then, $T$ is said to be single valued Bregman quasi-nonexpansive, if $F(T) \neq \emptyset$ and the following condition is satisfied:

$$D_f(p, Tx) \leq D_f(p, x), \quad \forall x \in C, \quad p \in F(T).$$

An example of a multi-valued Bregman quasi-nonexpansive mapping can be found in [15].
Equilibrium Problems (EP) involving monotone bifunctions, their generalizations and related optimization problems have been studied extensively by many authors, (see [1, 2, 9, 19, 18, 23, 22, 26, 27, 30, 31, 32, 39]). Let $C$ be a nonempty, closed and convex subset of a reflexive Banach space $E$, the EP for a bifunction $g : C \times C \to \mathbb{R}$ is defined as follows: Find $x^* \in C$ such that

$$g(x^*, y) \geq 0, \; \forall \; y \in C. \tag{1.5}$$

We denote the set of solutions of (1.5) by $\triangle$. To solve EP (1.5), the bifunction $g$ is assumed to satisfy the following conditions, see [4, 21, 20, 28]:

(L1) $g(x,x) = 0$, for all $x, y \in C$,

(L2) $g$ is monotone, that is $g(x,y) + g(y,x) \leq 0$, for all $x, y \in C$,

(L3) for all $x, y, z \in C$, $\limsup_{t \downarrow 0} g(tz + (1-t)x, y) \leq g(x,y)$,

(L4) for all $x \in C$, $g(x, \cdot)$ is convex and lower semicontinuous.

Let $\phi : C \to \mathbb{R} \cup \{+\infty\}$ be a function. The Generalized Equilibrium Problem (GEP) is finding $x^* \in C$ such that

$$g(x^*, y) + \phi(y) - \phi(x^*) \geq 0, \quad \forall y \in C. \tag{1.6}$$

The set of solution of GEP (1.6) is denoted by GEP(g,$\phi$). If $\phi = 0$, (1.1) reduces to (1.6) and if $g = 0$, then (1.6) reduces to the following Convex Minimization Problem (CMP):

$$\text{Find} \quad x^* \in C \quad \text{such that} \quad \phi(y) \geq \phi(x^*), \forall y \in C. \tag{1.7}$$

The set of solutions of (1.7) is denoted by CMP($\phi$), (see [5, 29]).
Fang and Huang [17] introduced the concept of relaxed $\eta$-$\alpha$ monotone mappings for solving mixed equilibrium problems.

**Definition 1.4.** A mapping $A : C \to E^*$ is said to be relaxed $\eta$-$\alpha$ monotone, if there exists a mapping $\eta : C \times C \to E$ and a function $\alpha : E \to \mathbb{R}$ with $\alpha(tz) = t^p \alpha(z)$ for all $t > 0$ and $z \in E$, where $p > 1$ such that

$$\langle Ax - Ay, \eta(x,y) \rangle \geq \alpha(x-y), \forall x, y \in C. \tag{1.8}$$

In particular, if $\eta(x,y) = x - y$ for all $x, y \in C$ and $\alpha(z) = k||z||^p$, where $p > 1$ and $k > 1$ are two constants, then $A$ is called $p$ monotone (see [17]).

The Mixed Equilibrium Problem (MEP) with relaxed $\eta$-$\alpha$ monotone mapping consists of finding a point $\bar{x} \in C$ such that

$$g(\bar{x}, y) + \langle Ay, \eta(y, \bar{x}) \rangle + \phi(y) - \phi(\bar{x}) \geq 0. \tag{1.9}$$

We shall denote the set of solutions of (1.9) by $EP(g, A)$.
The MEP with relaxed $\eta$-$\alpha$ monotone mapping reduces to a Variational-Like Inequality Problem (VLIP) if in (1.9), we set $g = 0$. That is, the VLIP is to find a point $\bar{x} \in C$ such that

$$\langle Ay, \eta(y, \bar{x}) \rangle + \phi(y) - \phi(\bar{x}) \geq 0. \tag{1.10}$$

We shall denote by $VLIP(C, A)$ the set of solutions of (1.10).
In 2016, Bashir and Harbau [6] introduced and proved the existence of solutions of the mixed equilibrium problem with relaxed $\eta$-$\alpha$ monotone mapping in reflexive Banach spaces. Using the Bregman distance, they introduced the concept of

$K$-mapping for a finite family of Bregman quasi-asymptotically nonexpansive mappings. They proposed an iterative algorithm for finding a common element in the set of fixed points of a finite family of Bregman quasi-asymptotically nonexpansive mappings and the set of solutions of mixed equilibrium problem with relaxed $\eta$-$\alpha$ monotone mapping.

**Definition 1.5.** Let $C$ be a nonempty, closed and convex subset of a real Banach space $E$. Let $\{T_i\}_i^N$ be a finite family of Bregman quasi-asymptotically nonexpansive mappings. For any $n \in \mathbb{N}$, define a mapping $K_n : C \to C$ as follows:

$$\begin{aligned}
S_{n,0}x &= x \\
S_{n,1}x &= P_C^f(\bigtriangledown f^*(\alpha_{n,1} \bigtriangledown f(T_1^n x) + (1-\alpha_{n,1}) \bigtriangledown f(x))) \\
S_{n,2}x &= P_C^f(\bigtriangledown f^*(\alpha_{n,2} \bigtriangledown f(T_2^n S_{n,1}x) + (1-\alpha_{n,2}) \bigtriangledown f(S_{n,1}x))) \\
S_{n,3}x &= P_C^f(\bigtriangledown f^*(\alpha_{n,3} \bigtriangledown f(T_1^n S_{n,2}x) + (1-\alpha_{n,3}) \bigtriangledown f(S_{n,2}x))) \\
&\vdots \\
S_{n,N-1}x &= P_C^f(\bigtriangledown f^*(\alpha_{n,N-1} \bigtriangledown f(T_1^n S_{n,N-2}x) \\
&\quad +(1-\alpha_{n,N-1}) \bigtriangledown f(S_{n,N-2}x))) \\
K_n x &= S_{n,N}x = P_C^f(\bigtriangledown f^*(\alpha_{n,N} \bigtriangledown f(T_1^n S_{n,N-1}x) \\
&\quad +(1-\alpha_{n,N}) \bigtriangledown f(S_{n,N-1}x))).
\end{aligned} \tag{1.11}$$

Such a mapping $K_n$ is called the Bregman $K$-mapping generated by $T_i$ and $\alpha_{n,i} \in (0,1)$, with $i = 1, 2, 3 \cdots, N$.

They proved a strong convergence theorem using the following iterative method: Let $\{x_n\}$ be iteratively defined as follows:

$$\begin{cases}
x_0 = x \in C, \text{chosen arbitrarily}, \\
C_{1,j} = C = C_0 \\
y_n = \bigtriangledown f^*[\beta_n \bigtriangledown f(x_n) + (1-\beta_n) \bigtriangledown f(K_n x_n)], \\
u_{n,j} \in C, \text{such that} \\
g_j(u_{n,j}, y) + \langle A_j u_{n,j}, \eta(y, u_{n,j})\rangle + \psi_j(y) - \psi_j(u_{n,j}) \\
+\dfrac{1}{r_n}\langle \bigtriangledown f(u_{n,j}) - \bigtriangledown f(y_n), y - u_{n,j}\rangle \geq 0, \forall y \in C, \\
C_{n+1,j} = \{z \in C_n : D_f(z, u_{n,j}) \leq D_f(z, x_n) + \theta_n\}, \\
C_{n+1} = \cap_{j=1}^M C_{n+1,j}, \\
x_{n+1} = P_{C_{n+1}}^f x_0, \quad n \geq 0.
\end{cases}$$

where $\{\beta_n\}$ is a sequence in $(0,1)$ satisfying $\liminf\limits_{n\to\infty} \beta_n(1-\beta_n) > 0$, $\{r_n\} \subset (a, \infty)$ for some $a > 0$ and $\theta_n = (1-\beta_n)t_n \sup_{p\in\Gamma} D_f(p, x_n)$. Then, $\{x_n\}$ converges to $u = P_\Gamma^f x_0$.

Motivated by the above works, we introduce an iterative algorithm and employ the Bregman distance approach for approximating a common solution of a finite family of mixed equilibrium problem with a relaxed $\eta$-$\alpha$ monotone mappings and a countable family of Bregman multivalued quasi-nonexpansive mappings in a real reflexive Banach space. Using our iterative algorithm, we state and prove a strong convergence result for the aformentioned problems. We give some consequences of our main result and we display a numerical example to show the applicability of the main result.

# 2 Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively.

**Definition 2.1.** A function $f : E \to \mathbb{R}$ is said to be super coercive if

$$\lim_{x\to\infty} \frac{f(x)}{||x||} = +\infty,$$

and strongly coercive if

$$\lim_{||x_n||\to\infty} \frac{f(x_n)}{||x_n||} = \infty.$$

**Lemma 2.2.** *[16] Let $g : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions $(L1)$-$(L4)$, then $\triangle$ is closed and convex.*

**Definition 2.3.** [10] Let $C$ be a nonempty, closed and convex subset of a reflexive real Banach space $E$. A Bregman projection of $x \in intdomf$ onto $C \subset intdomf$ is the unique vector $Proj_C^f \in C$ which satisfies

$$D_f(Proj_C^f x, x) = \inf\{D_f(y, x) : y \in C\}.$$

**Lemma 2.4.** *[13] Let $C$ be a nonempty, closed and convex subset of $E$ and $x \in E$. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. Then*

*(i) $q = Proj_C^f(x)$ if and only if $\langle \bigtriangledown f(x) - \bigtriangledown f(q), y - q\rangle$, for all $y \in C$;*

*(ii) $D_f(y, Proj_C^f(x)) + D_f(Proj_C^f(x), x) \leq D_f(y, x)$, for all $y \in C$.*

**Lemma 2.5.** *[25] Let $E$ be a Banach space, $r > 0$ be a constant, $\rho_r$ be the gauge of uniform convexity of $f$ and $f : E \to \mathbb{R}$ be a continuous uniformly convex function on bounded subset of $E$. Then, for any $x, y \in B_r$, we have*

$$f\left(\sum_{k=0}^{\infty} \alpha_k x_k\right) \leq \sum_{k=0}^{\infty} \alpha_k f(x_k) - \alpha_i \alpha_j \rho_r(||x_i - x_j||)$$

*for all $i, j \in \mathbb{N} \cup \{0\}$, $x_k \in B_r, \alpha_k \in (0, 1)$ and $k \in \mathbb{N} \cup \{0\}$ with $\sum_{k=0}^{\infty} \alpha_k = 1$. Here, $B_r := \{z \in E : ||z|| \leq r\}$.*

**Lemma 2.6.** *[13] Let $E$ be a reflexive Banach space, $f : E \to \mathbb{R}$ be a strongly coercive Bregman function and $V$ be a function defined by*

$$V(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*),\ x \in E,\ x^* \in E^*.$$

*The following assertions also hold:*

$$D_f(x, \bigtriangledown f^*(x^*)) = V(x, x^*),\ \textit{for all } x \in E \quad \textit{and } x^* \in E^*.$$

$$V(x, x^*) + \langle \bigtriangledown f^*(x^*) - x, y^* \rangle \leq V(x, x^* + y^*) \quad \textit{for all} \quad x \in E \textit{and} \quad x^*, y^* \in E^*.$$

**Lemma 2.7.** *[6] Let $E$ be a reflexive Banach space with the dual $E^*$ and let $C$ be a nonempty closed convex and bounded subset of $E$. Let $f : E \to (-\infty, +\infty]$ be a Legendre and Gâteaux differentiable function. Let $A : C \to E^*$ be $\eta$-hemicontinuous and relaxed $\eta$-$\alpha$ monotone mapping and $g : C \times C \to \mathbb{R}$ be a bifunction satisfying (L1), (L2) and (L4). Let $\psi : C \to \mathbb{R}$ be proper, convex and lower semicontinuous. For $r > 0$ and $x \in E$, define a map $T_r : E \to 2^C$ by*

$$T_r(x) = \{z \in C : g(z, y) + \langle Ay, \eta(y, z) \rangle + \psi(y) - \psi(z) \tag{2.1}$$

$$+\frac{1}{r}\langle \bigtriangledown f(z) - \bigtriangledown f(x), y - z \rangle \geq 0,\ \forall\ y \in C\}. \tag{2.2}$$

*Assume that*

*(i)* $\eta(x, x) = 0$, *for all* $x \in C$;

*(ii)* $\eta(z, y) + \eta(y, z) = 0,\ \forall\ y, z \in C$;

*(iii)* $\langle Au, \eta(., v) \rangle$ *is convex and lower semicontinuous for fixed* $u, v \in C$;

*(iv)* $\alpha : E \to \mathbb{R}$ *is weakly lower semicontinuous;*

*(v)* $\alpha(x - y) + \alpha(y - z) \geq 0,\ \forall\ x, y \in C$.

*Then,*

*(1)* $T_r$ *is single-valued,*

*(2)* $T_r$ *is a Bregman firmly nonexpansive type mapping, that is*

$$\langle \bigtriangledown f(T_r x) - \bigtriangledown f(T_r y), T_r x - T_r y \rangle \leq \langle \bigtriangledown f(x) - \bigtriangledown f(y), T_r x - T_r y \rangle,$$

$\forall\ x, y \in C$;

*(3)* $F(T_r) = EP(g, A)$;

*(4)* $T_r$ *is Bregman quasi nonexpansive satisfying*

$$D_f(u, T_r x) + D_f(T_r x, x) \leq D_f(u, x);$$

*(5) EP(g, A) is closed and convex.*

**Lemma 2.8.** *[13] Let* $E$ *be a Banach space and* $f : E \to \mathbb{R}$ *a Gâteaux differentiable function which is uniformly convex on bounded subsets of* $E$*. Let* $\{x_n\}_{n\in\mathbb{N}}$ *and* $\{y_n\}_{n\in\mathbb{N}}$ *be bounded sequences in* $E$*. Then,*

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\to\infty} ||y_n - x_n|| = 0.$$

**Lemma 2.9.** *[42] Let* $E$ *be a reflexive Banach space and* $f : E \to \mathbb{R}$ *a convex function which is bounded on bounded subsets of* $E$*. Then, the following assertions are equivalent:*
*(i)* $f$ *is strongly coercive and uniformly convex on bounded subsets of* $E$*;*
*(ii) dom* $f^* = E^*, f^*$ *is bounded on bounded subsets and uniformly smooth on bounded subsets of* $E^*$;
*(iii) dom* $f^* = E^*, f^*$ *is Fréchet differentiable and* $\bigtriangledown f^*$ *is uniformly norm-to-norm continuous on bounded subset of* $E^*$*.*

**Lemma 2.10.** *[11] If* $domf$ *contains at least two points, then the function* $f$ *is totally convex on bounded sets if and only if the function* $f$ *is sequentially consistent.*

**Lemma 2.11.** *[37] Let* $f : E \to \mathbb{R}$ *be a Gâteaux differentiable and totally convex function. If* $x_0 \in E$ *and the sequence* $\{D_f(x_n, x_0)\}$ *is bounded, then the sequence* $\{x_n\}$ *is also bounded.*

**Lemma 2.12.** *[42] Let* $f : E \to \mathbb{R}$ *be a continuous convex function which is strongly coercive. Then, the following assertions are equivalent:*

*(i)* $f$ *is bounded on bounded subsets and uniformly smooth on bounded subsets of E,*

*(ii)* $f$ *is Fréchet differentiable and* $\bigtriangledown f^*$ *is uniformly norm-to-norm continuous on bounded subset of* $E^*$*,*

*(iii)* $domf^* = E^*$*,* $f^*$ *is strongly coercive and uniformly convex on bounded subsets of* $E^*$*.*

**Definition 2.13.** Let $E$ be a reflexive Banach space and $C$ be a nonempty closed and convex subset of $E$. A Bregman projection of $x \in intdomf$ onto $C \subset intdomf$ is the unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x), x) = int\{D_f(y, x) : y \in C\}.$$

**Lemma 2.14.** *[36] Let* $C$ *be a nonempty closed and convex subset of a reflexive Banach space* $E$ *and* $x \in E$*. Let* $f : E \to \mathbb{R}$ *be a Gâteaux differentiable and totally convex function. Then,*
*(i)* $z = P_C^f(x)$ *if and only if* $\langle \bigtriangledown f(x) - \bigtriangledown f(z), y - z\rangle \leq 0, \ \forall\ y \in C$*.*
*(ii)* $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x)\ \forall\ y \in C$*.*

**Lemma 2.15.** *[13] Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a convex function whose domain contains at least two points. Then, the following statement hold:*
*(i) $f$ is sequentially consistent if and only if it is totally convex on bounded subsets.*
*(ii) If $f$ is lower semicontinuous, then $f$ is sequentially consistent if and only if it is uniformly convex on bounded subsets.*
*(iii) If $f$ is uniformly strictly convex on bounded subsets, then it is sequentially consistent, and the converse implication holds when $f$ is lower semicontinuous, Fréchet differentiable on its domain and the Fréchet differentiable $\bigtriangledown f$ is uniformly continuous on bounded subsets.*

**Lemma 2.16.** *[24, 40] Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}_{i\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$ such that $a_{n_i} < a_{n_{i+1}}$ for all $i \in \mathbb{N}$. Then there exists a subsequence $\{m_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_{k+1}} \text{ and } a_k \leq a_{k+1}.$$

*In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.*

**Lemma 2.17.** *[3, 41] Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\sigma_n)a_n + \sigma_n\delta_n, \;\; n > 0,$$

*where $\{\sigma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a real sequence such that*

*(i)* $\sum\limits_{n=1}^{\infty} \sigma_n = \infty$;

*(ii)* $\limsup\limits_{n\to\infty} \delta_n \leq 0$ *or* $\sum\limits_{n=1}^{\infty} |\sigma_n\delta_n| < \infty$.

*Then,* $\lim\limits_{n\to\infty} a_n = 0$.

# 3 Main Results

**Theorem 3.1.** *Let $E$ be a real reflexive Banach space with $E^*$ its dual and $C$ be a nonempty closed convex subset of $\text{int}\,\text{dom} f$. Let $f : E \to (-\infty, +\infty)$ be a strongly coercive, Legendre, uniformly Fréchet differentiable and totally convex function which is bounded on bounded subsets of $E$. Let $T_i : C \to P(C), i = 1, 2, ...$ be multivalued nonexpansive mappings such that $P_{T_i} : C \to P(C)$ are Bregman quasi-nonexpansive. For each $j = 1, 2, 3, ..M$ let $A_j : C \to E^*$ be $\eta$-hemicontinuous and relaxed $\eta$-$\alpha$ monotone mappings satisfying the assumptions of Lemma 2.7, $g_j : C \times C \to \mathbb{R}$ be bifunctions satisfying $(L1)-(L4)$, $\psi_j : C \to \mathbb{R}$ be proper, convex and lower semi-continuous functions. Suppose $\Gamma := \big[\cap_{i=1}^{\infty} F(T_i) \bigcap \cap_{j=1}^{M} EP(g_j, A_j)\big] \neq \emptyset$. For arbitrary $u, x_1 \in C$, let $\{x_n\}$ be the sequence* *generated by*

$$
\left\{\begin{array}{l}
y_n = \nabla f^*[\beta_{n,0} \nabla f(x_n) + \sum\limits_{i=1}^{\infty} \beta_{n,i} \nabla f(z_n^i)],\;\; z_n^i \in P_{T_i}x_n;\\
u_{n_j} \in C, \quad \textit{such that},\\
g(u_{n,j}, y) + \langle A_j u_{n,j}, \eta(y, u_{n_j})\rangle + \psi_j(y) - \psi_j(u_{n,j})\\
+\frac{1}{r_n}\langle \nabla f(u_{n,j}) - \nabla f(y_n), y - u_{n_j}\rangle \geq 0,\;\; \forall\; y \in C;\\
j_n \in Argmax\{D_f(u_{n_j}, y_n),\;\; j = 1, 2, ..., M\}, \overline{u_n} = u_{n_j};\\
x_{n+1} = \nabla f^*\left[\alpha_n \nabla f(u) + (1-\alpha_n) \nabla f(\overline{u_n})\right], n \in \mathbb{N};
\end{array}\right. \tag{3.1}
$$

*where* $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$, $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0, 1)$ *satisfying the following conditions:*

(i) $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, *and* $\liminf\limits_{n\to\infty} \alpha_n = 0$;

(ii) $\liminf\limits_{n\to\infty} \beta_{n,i} > 0$ *and* $\sum\limits_{i=0}^{\infty} \beta_{n,i} = 1$.

*Then* $\{x_n\}$ *converges strongly to* $z = Proj_\Gamma^f u$, *where* $Proj_\Gamma^f$ *is the Bregman projection of* $C$ *onto* $\Gamma$.

*Proof.* Let $\bar{x} \in \Gamma$, then from (3.1) and (2.1), we have that $u_{n,j} = T_{r_n}^j y_n$, $j = 1, 2\cdots, M$. Using Lemma 2.7 (3), we obtain that

$$
D_f(\bar{x}, u_{n,j}) = D_f(\bar{x}, T_{r_n}^j y_n) \leq D_f(\bar{x}, y_n). \tag{3.2}
$$

From (3.1) and using the assumption that $P_{T_i}, i = 1, 2, ...$ are Bregman quasi-nonexpansive mappings, we obtain

$$
\begin{aligned}
D_f(\bar{x}, y_n) &= D_f(\bar{x}, \nabla f^*(\beta_{n,0} \nabla f(x_n) + \sum_{i=1}^{\infty} \beta_{n,i} \nabla f(z_n^i)))\\
&\leq \beta_{n,0} D_f(\bar{x}, x_n) + \sum_{i=1}^{\infty} \beta_{n,i} D_f(\bar{x}, z_n^i)\\
&\leq \beta_{n,0} D_f(\bar{x}, x_n) + \sum_{i=1}^{\infty} \beta_{n,i} D_f(\bar{x}, x_n)\\
&= D_f(\bar{x}, x_n). \qquad (3.3)
\end{aligned}
$$

We conclude from (3.2), (3.3) and the definition of $u_{n,j}$ in (3.1) that

$$
D_f(\bar{x}, \overline{u_n}) \leq D_f(\bar{x}, x_n). \tag{3.4}
$$

Now, using (3.1) and (3.4), we have that

$$
\begin{aligned}
D_f(\bar{x}, x_{n+1}) &= D_f(\bar{x}, \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n) \nabla f(\overline{u_n})))\\
&\leq \alpha_n D_f(\bar{x}, u) + (1-\alpha_n) D_f(\bar{x}, \overline{u_n})
\end{aligned}
$$

$$\leq \alpha_n D_f(\bar{x}, u) + (1-\alpha_n) D_f(\bar{x}, x_n)$$
$$\leq \max\{D_f(\bar{x}, u), D_f(\bar{x}, x_n)\}. \tag{3.5}$$

By induction, we obtain

$$D_f(\bar{x}, x_{n+1}) \leq \max\{D_f(\bar{x}, u), D_f(\bar{x}, x_1)\}.$$

From Lemma 2.12, we have that $f^*$ is bounded on bounded subsets of $E^*$. Hence, $\bigtriangledown f^*$ is also bounded on bounded subsets of $E^*$. Therefore, $\{D_f(\bar{x}, x_n)\}$ is bounded and in view of Lemma 2.11, we obtain that $\{x_n\}$ is bounded. Let $s \geq \sup\{|| \bigtriangledown f(x_n)||, \ || \bigtriangledown f(z_n^i)|| : n \in \mathbb{N}\}$ and let $\rho_r^* : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function $f^*$. We have from Lemma 2.5, Lemma 2.6, (3.3) and the assumption that $P_{T_i}, i = 1, 2, ...$ are Bregman quasi-nonexpansive mappings that

$$\begin{aligned}
D_f(\bar{x}, y_n) &= D_f(\bar{x}, \bigtriangledown f^*[\beta_{n,0} \bigtriangledown f(x_n) + \sum_{i=1}^{\infty} \beta_{n,i} \bigtriangledown f(z_n^i)]) \\
&= V_f(\bar{x}, \beta_{n,0} \bigtriangledown f(x_n) + \sum_{i=1}^{\infty} \beta_{n,i} \bigtriangledown f(z_n^i)) \\
&= f(\bar{x}) - \langle \bar{x}, \beta_{n,0} \bigtriangledown f(x_n) + \sum_{i=1}^{\infty} \beta_{n,i} \bigtriangledown f(z_n^i)\rangle \\
&+ f^*(\beta_{n,0} \bigtriangledown f(x_n) + \sum_{i=1}^{\infty} \beta_{n,i} \bigtriangledown f(z_n^i)) \\
&\leq \beta_{n,0} f(\bar{x}) + \sum_{i=1}^{\infty} \beta_{n,i} f(\bar{x}) - \beta_{n,0} \langle \bar{x}, \bigtriangledown f(x_n)\rangle \\
&- \sum_{i=1}^{\infty} \beta_{n,i} \langle V, \bigtriangledown f(z_n^i)\rangle + \beta_{n,0} f^*(\bigtriangledown f(x_n)) \\
&+ \sum_{i=1}^{\infty} \beta_{n,i} f^*(\bigtriangledown f(z_n^i)) - \beta_{n,0} \sum_{i=1}^{\infty} \beta_{n,i} \rho_r^*(|| \bigtriangledown f(x_n) - \bigtriangledown f(z_n^i)||) \\
&= \beta_{n,0}[f(\bar{x}) - \langle \bar{x}, \bigtriangledown f(x_n)\rangle + f^*(\bigtriangledown f(x_n))] \\
&+ \sum_{i=1}^{\infty} \beta_{n,i}[f(\bar{x}) - \langle \bar{x}, \bigtriangledown f(z_n^i)\rangle + f^*(\bigtriangledown f(z_n^i))] \\
&- \beta_{n,0} \sum_{i=1}^{\infty} \beta_{n,i} \rho_r^*(|| \bigtriangledown f(x_n) - \bigtriangledown f(z_n^i)||) \\
&= \beta_{n,0} D_f(\bar{x}, x_n) + \sum_{i=1}^{\infty} \beta_{n,i} D_f(\bar{x}, z_n^i) \\
&- \beta_{n,0} \sum_{i=1}^{\infty} \beta_{n,i} \rho_r^*(|| \bigtriangledown f(x_n) - \bigtriangledown f(z_n^i)||)
\end{aligned}$$

$$\leq \beta_{n,0} D_f(\bar{x}, x_n) + \sum_{i=1}^{\infty} \beta_{n,i} D_f(V, x_n)$$

$$- \beta_{n,0} \sum_{i=1}^{\infty} \beta_{n,i} \rho_r^*(|| \bigtriangledown f(x_n) - \bigtriangledown f(z_n^i)||)$$

$$= D_f(\bar{x}, x_n) - \beta_{n,0} \sum_{i=1}^{\infty} \beta_{n,i} \rho_r^*(|| \bigtriangledown f(x_n) - \bigtriangledown f(z_n^i)||). \tag{3.6}$$

Using (3.2), (3.5) and (3.6), we have that

$$\begin{aligned} D_f(\bar{x}, x_{n+1}) &\leq \alpha_n D_f(\bar{x}, u) + (1-\alpha_n) D_f(\bar{x}, \overline{u_n}) \\ &\leq \alpha_n D_f(\bar{x}, u) + (1-\alpha_n) D_f(\bar{x}, x_n) \\ &\quad - (1-\alpha_n)\beta_{n,0} \sum_{i=1}^{\infty} \beta_{n,i} \rho_s^*(|| \bigtriangledown f(x_n) - \bigtriangledown f(z_n^i)||). \end{aligned} \tag{3.7}$$

The rest of the proof will be divided into two parts:
**Case 1:** Assume that there exists $n_0 \in \mathbb{N}$ such that $\{D_f(\bar{x}, x_n)\}$ is monotone decreasing for all $n \geq n_0$, then $\{D_f(\bar{x}, x_n)\}$ is convergent. Thus, we have that

$$D_f(\bar{x}, x_n) - D_f(\bar{x}, x_{n+1}) \to 0, \text{ as } n \to \infty.$$

Now, from (3.7) and condition (i), we have that

$$\lim_{n\to\infty} \beta_{n,0} \sum_{i=1}^{\infty} \beta_{n,i} \rho_r^*(|| \bigtriangledown f(x_n) - \bigtriangledown f(z_n^i)||) = 0.$$

Also, from condition (ii) and property of $\rho_r^*$, we obtain that

$$\lim_{n\to\infty} || \bigtriangledown f(x_n) - \bigtriangledown f(z_n^i)|| = 0, \text{ for all } i = 1, 2, ... \tag{3.8}$$

By Lemma 2.12, we have that $\bigtriangledown f^*$ is uniformly norm-to-norm continuous on bounded subsets, using this fact in (3.8), we obtain

$$\lim_{n\to\infty} ||x_n - z_n^i|| = 0 = \lim_{n\to\infty} d(x_n, T_i x_n). \tag{3.9}$$

From (3.1), we have

$$|| \bigtriangledown f(x_n) - \bigtriangledown f(y_n)|| = \sum_{i=1}^{\infty} \beta_{n,i} || \bigtriangledown f(x_n) - \bigtriangledown f(z_n^i)||.$$

Hence, from (3.8), we obtain that

$$\lim_{n\to\infty} || \bigtriangledown f(x_n) - \bigtriangledown f(y_n)|| = 0. \tag{3.10}$$

Since $\bigtriangledown f^*$ is uniformly continuous, we have that

$$\lim_{n\to\infty} ||x_n - y_n|| = 0. \tag{3.11}$$

Since $f$ is uniformly Fréchet differentiable on bounded subset of $E$, we have that $f$ is uniformly continuous on bounded subset of $E$. Thus we obtain from (3.11), that

$$\lim_{n\to\infty} ||f(x_n) - f(y_n)|| = 0. \tag{3.12}$$

From (3.1) and Lemma 2.7 (4), we have

$$\begin{aligned}
D_f(u_{n,j}, y_n) &\le D_f(\bar{x}, y_n) - D_f(\bar{x}, u_{n,j}) \\
&\le D_f(\bar{x}, \bigtriangledown f^*(\beta_{n,0} \bigtriangledown f(x_n) + \sum_{i=1}^{\infty} \bigtriangledown f(z_n^i))) - D_f(\bar{x}, u_{n_j}) \\
&\beta_{n,0} D_f(\bar{x}, x_n) + \sum_{i=1}^{\infty} \beta_{n,i} D_f(\bar{x}, z_n^i) - D_f(\bar{x}, u_{n_j}) \\
&\le \beta_{n,0} D_f(\bar{x}, x_n) + \sum_{i=1}^{\infty} \beta_{n,i} D_f(\bar{x}, x_n) - D_f(\bar{x}, u_{n,j}) \\
&= D_f(\bar{x}, x_n) - D_f(\bar{x}, u_{n,j}) \\
&\le D_f(\bar{x}, x_n) + \alpha_n [D_f(\bar{x}, u) - D_f(\bar{x}, u_{n,j})] - D_f(\bar{x}, x_{n+1}), \qquad (3.13)
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} D_f(u_{n,j}, y_n) = 0. \tag{3.14}$$

Hence, we have from Lemma 2.8 that

$$\lim_{n\to\infty} ||u_{n,j} - y_n|| = 0 = \lim_{n\to\infty} ||\overline{u_n} - y_n||. \tag{3.15}$$

Since $f$ is uniformly Fréchet differentiable on bounded subset of $E$, by Lemma 2.15, we have

$$\lim_{n\to\infty} || \bigtriangledown f(u_{n,j}) - \bigtriangledown f(y_n)|| = 0. \tag{3.16}$$

From (3.11) and (3.15), we obtain

$$\lim_{n\to\infty} ||\overline{u_n} - x_n|| = 0. \tag{3.17}$$

On the other hand, by the boundedness $\bigtriangledown f$ on bounded subsets of $E$, we obtain

$$\begin{aligned}
\lim_{n\to\infty} D_f(x_{n+1}, x_n) = \lim_{n\to\infty} [&D_f(x_n, p) - D_f(x_{n+1}, p) \\
&+\langle x_n - p, \bigtriangledown f(p) - \bigtriangledown f(x_{n+1})\rangle] = 0.
\end{aligned}$$

By using Lemma 2.8, we get

$$||x_{n+1} - x_n|| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.18}$$

Since $\{x_n\}$ is bounded there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup v$, by (3.15) we obtain that $u_{n_k,j} \rightharpoonup v$. Also, since $P_{T_i}$ for $i = 1, 2, ..$ are

Bregman quasi-nonexpansive, $\{z_n^i\}_{i=1}^{\infty}$ is bounded and converges weakly to $v$ by virtue of (3.9). Then, it follows that

$$\begin{aligned} d(v, T_i v) &\leq d(v, z_{n_k}^i) + d(z_{n_k}^i, x_{n_k}) + d(x_{n_k}, T_i x_{n_k}) + \mathcal{H}(T_i x_{n_k}, T_i v) \\ &\leq d(v, z_{n_k}^i) + 2d(z_{n_k}^i, x_{n_k}) + d(x_{n_k}, v), \end{aligned} \tag{3.19}$$

which implies

$$\lim_{k\to\infty} d(v, T_i v) = 0. \tag{3.20}$$

This implies that $v \in T_i v$, for each $i = 1, 2, \ldots$ Hence, $v \in \cap_{i=1}^{\infty} F(T_i)$.
Next, we show that $v \in \cap_{j=1}^{M} EP(g_j, A_j)$. From (3.1), (3.16), the fact that $\bigtriangledown f$ is uniformly continuous and $r_{n_k} > a$, we have

$$\lim_{k\to\infty} \frac{|| \bigtriangledown f(u_{n_k,j}) - \bigtriangledown f(y_{n_k})||}{r_{n_k}} = 0, \ \forall\ j = 1, 2, , .., m. \tag{3.21}$$

From (3.1), we have

$$\begin{aligned} &g(u_{n_k,j}, y) + \langle A_j u_{n_k,j}, \eta(y, u_{n_k,j})\rangle + \psi_j(y) - \psi_j(u_{n_k,j}) \\ &+ \frac{1}{r_{n_k}} \langle \bigtriangledown f(u_{n_k,j}) - \bigtriangledown f(y_{n_k}), y - u_{n_k,j}\rangle \geq 0, \ \forall\ y \in C. \end{aligned} \tag{3.22}$$

Using (L2), Lemma 2.7 (ii), it follows that

$$\begin{aligned} \frac{1}{r_{n_k}} || \bigtriangledown f(u_{n_k,j}) - \bigtriangledown f(y_{n_k})||\ ||u_{n_k,j} - y|| &\geq \langle A_j u_{n_k,j}, \eta(u_{n_k,j}, y)\rangle + \psi_j(u_{n_k,j}) \\ &\quad - \psi_j(y) - g_j(u_{n_k,j}, y)\ \forall\ y \in C \\ &\geq \langle A_j u_{n_k,j}, \eta(u_{n_k,j}, y)\rangle + \psi_j(u_{n_k,j}) \\ &\quad - \psi_j(y) + g_j(y, u_{n_k,j})\ \forall\ y \in C. \end{aligned} \tag{3.23}$$

Using (3.21), the fact that $u_{n_k,j} \rightharpoonup v$ and taking $\liminf$ as $n \to \infty$ in the above inequality, we get

$$0 \geq \langle A_j v, \eta(v, y)\rangle + \psi_j(v) - \psi_j(y) + g_j(v, y), \ \forall\ y \in C \text{ and } j = 1, 2, .., M. \tag{3.24}$$

Now for any $t \in (0, 1)$ and $y \in C$, let $y_t = ty + (1 - t)v$. Then $y_t \in C$ and so

$$0 \geq \langle A_j v, \eta(v, y_t)\rangle + \psi_j(v) - \psi_j(y_t) + g_j(v, y_t), \ \forall\ y \in C \text{ and } j = 1, 2, .., M. \tag{3.25}$$

Therefore by $L1$, $L2$, Lemma 2.7 (ii), (iii) and (3.25), we have

$$\begin{aligned} 0 &= g_j(y_t, y_t) + \langle A_j v, \eta(y_t, y_t)\rangle + \psi_j(y_t) - \psi_j(y_t) \\ &= g_j(y_t, ty + (1 - t)v) + \langle A_j v, \eta(ty + (1 - t)v, y_t)\rangle \\ &\quad + \psi_j(ty + (1 - t)v) - \psi_j(y_t) \end{aligned}$$

$$
\begin{aligned}
&\leq t[g_j(y_t,y)+\langle A_jv,\eta(y,y_t)\rangle+\psi_j(y)-\psi_j(y_t)]\\
&+(1-t)[g_j(y_t,v)+\langle A_jv,\eta(v,y_t)\rangle+\psi_j(v)-\psi_j(y_t)]\\
&\leq t[g_j(y_t,y)+\langle A_jv,\eta(y,y_t)\rangle+\psi_j(y)-\psi_j(y_t)].
\end{aligned}
$$

That is,

$$g_j(y_t,y)+\langle A_jv,\eta(y,y_t)\rangle+\psi_j(y)-\psi_j(y_t)\geq 0. \tag{3.26}$$

Since $y_t=ty+(1-t)v$, we have

$$g_j(ty+(1-t)v,y)+\langle A_jv,\eta(y,ty+(1-tv)\rangle+\psi_j(y)-\psi_j(ty+(1-t)v)\geq 0. \tag{3.27}$$

By using (L3) and the lower semicontinuity of $\psi$, we obtain by allowing $t\to 0$ that

$$g_j(v,y)+\langle A_jv,\eta(y,v)\rangle+\psi_j(y)-\psi_j(v)\geq 0,\ \forall\ y\in C.$$

Hence, we obtain that $v\in EP(g_j,A_j)$, for each $j=1,2,...M$.
We now show that $\{x_n\}$ converges strongly to $z=Proj_\Gamma^f u$. In view of Lemma 2.6 and (3.4), we have that

$$
\begin{aligned}
D_f(z,x_{n+1})&=D_f(z,\triangledown f^*[\alpha_n\triangledown f(u)+(1-\alpha_n)\triangledown f(\overline{u_n})])\\
&=V(z,\alpha_n\triangledown f(u)+(1-\alpha_n)\triangledown f(\overline{u_n}))\\
&\leq V(z,\alpha_n\triangledown f(u)+(1-\alpha_n)\triangledown f(y_n)-\alpha_n(\triangledown f(u)-\triangledown f(u))\\
&-\langle\triangledown f^*[\alpha_n\triangledown f(u)+(1-\alpha_n)\triangledown f(\overline{u_n})]-z,-\alpha_n(\triangledown f(u)-\triangledown f(z)\rangle)\\
&=V(z,\alpha_n\triangledown f(z)+(1-\alpha_n)\triangledown f(\overline{u_n})\\
&+\alpha_n\langle x_{n+1}-z,\triangledown f(u)-\triangledown f(z)\rangle\\
&=D_f(z,\triangledown f^*[\alpha_n\triangledown f(u)+(1-\alpha_n)\triangledown f(\overline{u_n})])\\
&+\alpha_n\langle x_{n+1}-z,\triangledown f(u)-\triangledown f(z)\rangle\\
&\leq\alpha_nD_f(z,z)+(1-\alpha_n)D_f(z,\overline{u_n})+\alpha_n\langle x_{n+1}-z,\triangledown f(u)-\triangledown f(z)\rangle\\
&=(1-\alpha_n)D_f(z,x_n)+\alpha_n\langle x_{n+1}-z,\triangledown f(u)-\triangledown f(z)\rangle. && (3.28)
\end{aligned}
$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup q$ and

$$\limsup_{n\to\infty}\langle x_{n+1}-z,\triangledown f(u)-\triangledown f(z)\rangle=\lim_{k\to\infty}\langle x_{n_k+1}-z,\triangledown f(u)-\triangledown f(z)\rangle.$$

By (3.18) and $x_{n_k}\rightharpoonup q$, we get that $x_{n_k+1}\rightharpoonup q$. Using this and Lemma 2.14 (i), we have

$$
\begin{aligned}
\limsup_{n\to\infty}\langle x_{n+1}-z,\triangledown f(u)-\triangledown f(z)\rangle&=\lim_{k\to\infty}\langle x_{n_k+1}-z,\triangledown f(u)-\triangledown f(z)\rangle\\
&=\langle q-z,\triangledown f(u)-\triangledown f(z)\rangle\\
&\leq 0. && (3.29)
\end{aligned}
$$

Applying Lemma 2.17 and (3.29) in (3.28), we have that $\{x_n\}$ converges strongly to $z$.

**CASE 2:** Assume that $\{D_f(z, x_n)\}$ is not monotone decreasing. Then there exists a subsequence $\{n_k\}_{k\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$ such that

$$D_f(z, x_{n_k}) < D_f(z, x_{n_k+1}),$$

for $k \in \mathbb{N}$, then by Lemma 2.16, there exists a nondecreasing sequence $\{m_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $m_k \to \infty$,

$$D_f(z, x_{m_k}) < D_f(z, x_{m_k+1}) \text{ and } D_f(z, x_k) \leq D_f(z, x_{k+1}),$$

for all $k \in \mathbb{N}$. This together with (3.7), conditions (i) and (ii) implies that

$$\lim_{k\to\infty} \rho_r^*(|| \bigtriangledown f(x_{n_k}) - \bigtriangledown f(z^i_{n_k})||) = 0.$$

Following the same argument as in Case 1, we arrive at

$$\limsup_{k\to\infty} \langle x_{m_{k+1}} - z, \bigtriangledown f(u) - \bigtriangledown f(z)\rangle = \lim_{k\to\infty} \langle x_{m_k} - z, \bigtriangledown f(u) - \bigtriangledown f(z)\rangle.$$

It follows from (3.28) that

(3.30)
$$D_f(z, x_{m_{k+1}}) \leq (1 - \alpha_{m_k}) D_f(z, x_{n_k}) + \alpha_{m_k} \langle x_{m+1} - z, \bigtriangledown f(u) - \bigtriangledown f(z)\rangle.$$

Since $D_f(z, x_{m_k}) \leq D_f(z, x_{m_{k+1}})$, we have that

$$\alpha_{m_k} D_f(z, x_{m_k}) \leq D_f(z, x_{m_k}) - D_f(z, x_{m_{k+1}}) + \alpha_{m_k} \langle x_{m_{k+1}} - z, \bigtriangledown f(u) - \bigtriangledown f(z)\rangle \tag{3.31}$$

$$\leq \alpha_{m_k} \langle x_{m_{k+1}} - z, \bigtriangledown f(u) - \bigtriangledown f(z)\rangle. \tag{3.32}$$

In particular, since $\alpha_{m_k} > 0$, we obtain

$$D_f(z, x_{m_k}) \leq \langle x_{m_{k+1}} - z, \bigtriangledown f(u) - \bigtriangledown f(z)\rangle.$$

In view of (3.30), we deduce that

$$\lim_{k\to\infty} D_f(z, x_{m_k}) = 0.$$

This together with (3.31) implies that

$$D_f(z, x_{m_{k+1}}) = 0.$$

On the other hand, we have $D_f(z, x_k) \leq D_f(z, x_{k+1})$ for all $k \in \mathbb{N}$ which implies that $\{x_k\} \to z$ as $k \to \infty$. Thus, we obtain that $x_n \to z$ as $n \to \infty$. $\square$

We obtain the following consequences of our main result.

Suppose in Theorem 3.1, we choose $i = j = 1$, then we obtain the following result:

**Corollary 3.2.** *Let $E$ be a real reflexive Banach space with $E^*$ its dual and $C$ be a nonempty closed convex subset of $\operatorname{int}\operatorname{dom} f$. Let $f : E \to (-\infty, +\infty]$ be a strongly coercive, Legendre, uniformly Fréchet differentiable and totally convex function which is bounded on bounded subsets of $E$. Let $T : C \to P(C)$ be a multivalued nonexpansive mapping such that $P_T : C \to P(C)$ are Bregman quasi-nonexpansive. Let $A : C \to E^*$ be $\eta$-hemicontinuous and relaxed $\eta$-$\alpha$ monotone mapping satisfying the assumptions of Lemma 2.7, $g : C \times C \to \mathbb{R}$ be bifunction satisfying $(L1)-(L4)$, $\psi : C \to \mathbb{R}$ be proper, convex and lower semi-continuous functions. Suppose $\Gamma := F(T) \cap EP(g, A) \neq \emptyset$. For arbitrary $u, x_1 \in C$, let $\{x_n\}$ be the sequence generated by*

$$
\begin{cases}
y_n = \bigtriangledown f^*[\beta_n \bigtriangledown f(x_n) + (1-\beta_n) \bigtriangledown f(z_n)], \;\; z_n \in P_T x_n; \\
u_n \in C, \quad \text{such that}, \\
g(u_n, y) + \langle Au_n, \eta(y, u_n)\rangle + \psi(y) - \psi(u_n) \\
+\frac{1}{r_n}\langle \bigtriangledown f(u_n) - \bigtriangledown f(y_n), y - u_n\rangle \geq 0, \;\; \forall\, y \in C; \\
x_{n+1} = \bigtriangledown f^*\big[\alpha_n \bigtriangledown f(u) + (1-\alpha_n) \bigtriangledown f(u_n)\big], n \in \mathbb{N};
\end{cases}
\tag{3.33}
$$

*where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$, and $\{r_n\} \subset [a, \infty)$ for some $a > 0$ satisfying the following conditions:*

*(i)* $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, *and* $\liminf\limits_{n\to\infty} \alpha_n = 0$;

*(ii)* $\liminf\limits_{n\to\infty} \beta_n > 0$.

*Then $\{x_n\}$ converges strongly to $Proj^f_\Gamma u$, where $Proj^f_\Gamma$ is the Bregman projection of $C$ onto $\Gamma$.*

For approximating the common solution of a finite family of Variational-Like Inequality Problem and common fixed point of a countable family of multivalued Bregman nonexpansive mappings we have the following as an application of our main result:

**Corollary 3.3.** *Let $E$ be a real reflexive Banach space with $E^*$ its dual and $C$ be a nonempty closed convex subset of $\operatorname{int}\operatorname{dom} f$. Let $f : E \to (-\infty, +\infty]$ be a strongly coercive, Legendre, uniformly Fréchet differentiable and totally convex function which is bounded on bounded subsets of $E$. Let $T_i : C \to P(C), i = 1, 2, ...$ be multivalued nonexpansive mappings such that $P_{T_i} : C \to P(C)$ are Bregman quasi-nonexpansive. For each $j = 1, 2, 3, ..M$, let $A_j : C \to E^*$ be $\eta$-hemicontinuous and relaxed $\eta$-$\alpha$ monotone mappings satisfying the assumptions of Lemma 2.7, $\psi_j : C \to \mathbb{R}$ be proper, convex and lower semi-continuous functions. Suppose $\Gamma := \big[\cap_{i=1}^{\infty} F(T_i) \bigcap \cap_{j=1}^{M} VLIP(C, A_j)\big] \neq \emptyset$.*

*For arbitrary $u, x_1 \in C$, let $\{x_n\}$ be the sequence generated by*

$$(3.34) \qquad \begin{cases} y_n = \bigtriangledown f^*[\beta_{n,0} \bigtriangledown f(x_n) + \sum\limits_{i=1}^{\infty} \beta_{n,i} \bigtriangledown f(z_n^i)], \;\; z_n^i \in P_{T_i} x_n; \\ u_{n_j} \in C, \quad \textit{such that}, \\ \langle A_j u_{n,j}, \eta(y, u_{n_j})\rangle + \psi_j(y) - \psi_j(u_{n,j}) \\ +\frac{1}{r_n}\langle \nabla f(u_{n,j}) - \nabla f(y_n), y - u_{n_j}\rangle \geq 0, \;\; \forall\; y \in C; \\ j_n \in Argmax\{D_f(u_{n_j}, y_n), \;\; j = 1, 2, ..., M\}, \overline{u_n} = u_{n_j}; \\ x_{n+1} = \bigtriangledown f^*\big[\alpha_n \bigtriangledown f(u) + (1-\alpha_n) \bigtriangledown f(\overline{u_n})\big], n \in \mathbb{N}; \end{cases}$$

*where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, and $\{r_n\} \subset [a, \infty)$ for some $a > 0$ satisfying the following conditions:*

*(i)* $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, *and* $\liminf\limits_{n\to\infty} \alpha_n = 0$;

*(ii)* $\liminf\limits_{n\to\infty} \beta_{n,i} > 0$ *and* $\sum\limits_{i=0}^{\infty} \beta_{n,i} = 1$.

*Then $\{x_n\}$ converges strongly to $Proj_\Gamma^f u$, where $Proj_\Gamma^f$ is the Bregman projection of $C$ onto $\Gamma$.*

Suppose in Theorem 3.1, we set $E = H$ a real Hilbert space, then we obtain the following as a consequence of our main result:

**Corollary 3.4.** *Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$. Let $T_i : C \to P(C), i = 1, 2, ...$ be multivalued nonexpansive mappings such that $P_{T_i} : C \to P(C)$ are Bregman quasi-nonexpansive. For each $j = 1, 2, 3, ..M$, let $A_j : C \to E^*$ be $\eta$-hemicontinuous and relaxed $\eta - \alpha$ monotone mappings satisfying the assumptions of Lemma 2.7 and $g_j : C \times C \to \mathbb{R}$ be bifunctions satisfying $(L1) - (L4)$ and $\psi_j : C \to \mathbb{R}$ be proper, convex and lower semi-continuous functions. Suppose $\Gamma := \big[\cap_{i=1}^{\infty} F(T_i) \bigcap \cap_{j=1}^{M} EP(g_j, A_j)\big] \neq \emptyset$. For arbitrary $u, x_1 \in C$, let $\{x_n\}$ be the sequence generated by*

$$(3.35) \qquad \begin{cases} y_n = \beta_{n,0} x_n + \sum\limits_{i=1}^{\infty} \beta_{n,i} z_n^i, \;\; z_n^i \in P_{T_i} x_n; \\ u_{n_j} \in C, \quad \textit{such that}, \\ g(u_{n,j}, y) + \langle A_j u_{n,j}, \eta(y, u_{n_j})\rangle + \psi_j(y) - \psi_j(u_{n,j}) \\ +\frac{1}{r_n}\langle u_{n,j} - y_n, y - u_{n_j}\rangle \geq 0, \;\; \forall\; y \in C; \\ j_n \in Argmax\{||y_n - u_{n_j}||^2, \;\; j = 1, 2, ..., M\}, \overline{u_n} = u_{n_j}; \\ x_{n+1} = \alpha_n u + (1-\alpha_n)\overline{u_n}, n \in \mathbb{N}; \end{cases}$$

*where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, and $\{r_n\} \subset [a, \infty)$ for some $a > 0$ satisfying the following conditions:*

*(i)* $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, *and* $\liminf\limits_{n\to\infty} \alpha_n = 0$;

*(ii)* $\liminf_{n\to\infty} \beta_{n,i} > 0$ *and* $\sum_{i=0}^{\infty} \beta_{n,i} = 1$.

*Then* $\{x_n\}$ *converges strongly to* $P_\Gamma^f u$*, where* $P_\Gamma^f$ *is the metric projection of* $C$ *onto* $\Gamma$.

# 4 Numerical Example

In this section we give a numerical example to show the efficiency of our main result.
Let $E = \mathbb{R}\times\mathbb{R}$ and $C = [-1, 1]\times[-1, 1]$. Define a mapping $A(x_1, x_2) = (x_1, x_2)$ for all $x = (x_1, x_2) \in C$, $\alpha : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $\alpha(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ for all $x = (x_1, x_2) \in E$ and $\eta : C \times C \to \mathbb{R}\times\mathbb{R}$ by $\eta((x_1, x_2), (y_1, y_2)) = (x_1 - y_1, x_2 - y_2)$ for all $(x, y) \in C\times C$ with $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then, the mapping $A$ is a relaxed $\eta$-$\alpha$ monotone mapping. Indeed, for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in C$, we have

$$\begin{aligned} \langle Ax - Ay, \eta(x, y)\rangle &= ((x_1 - y_1), (x_2 - y_2))((x_1 - y_1), (x_2 - y_2)) \\ &= [(x_1 - y_1)^2 + (x_2 - y_2)^2] \\ &\geq \frac{1}{2}[(x_1 - y_1)^2 + (x_2 - y_2)^2] = \alpha(x - y). \end{aligned}$$

Hence, $A$ is a relaxed $\eta$-$\alpha$ monotone mapping.
Let $u_n = (u_n^1, u_n^2)$, $y_n = (y_n^1, y_n^2)$, $z_n = (z_n^1, z_n^2)$ and $y = (y_1, y_2)$. Define the bifunction $g : C \times C \to \mathbb{R}$ by $g : (x, y) = -5x^2 + 3xy + 2y^2$, $A(x) = x$, $\eta(x, y) = 2(x - y)$ and $\psi(x) = x^2$. By using Lemma 2.7, we have that

$$g(u_n, y) + \langle Au_n, \eta(y, u_n)\rangle + \phi(y) - \phi(u_n) + \frac{1}{r_n}\langle u_n - y_n, y - u_n\rangle \geq 0, \quad \forall y \in \mathbb{R}^2$$

$$\iff -5u_n^2 + 3u_n y + 2y^2 + u_n(2(y - u_n)) + y^2 - u_n^2 + \frac{1}{r_n}(y - u_n)(u_n - y_n) \geq 0.$$

By simple calculations, we obtain

$$u_n = T_{r_n}(y_n) = \frac{y_n}{11r_n + 1}.$$

That is,

$$u_n^1 = T_{r_n}(y_n^1) = \frac{y_n^1}{11r_n + 1} \quad \text{and} \quad u_n^2 = T_{r_n}(y_n^2) = \frac{y_n^2}{11r_n + 1}.$$

Let $f : E \to (-\infty, +\infty)$ be defined by $f(x) = \frac{x^4}{4}$, then $\nabla f(x) = x^3$, $f(x^*) = \frac{3}{4}x^{*\frac{4}{3}}$ and $\nabla f^*(x^*) = x^{*\frac{1}{3}}$. Choose the sequences $r_n = \frac{2n}{n+1}$, $\alpha_n = \dfrac{1}{8(n+1)}$ and $\beta_n = \dfrac{n+1}{5n}$. Then, we use iteration (3.33) of Corollary 3.2.
We also consider the following cases for our numerical results.

**Case 1:** $x_1 = (0.5, -0.25)^T$ and $u = (0.5, -0.25)^T$,
**Case 2:** $x_1 = (2, 25)^T$ and $u = (0.5, -0.25)^T$,
**Case 3:** $x_1 = (2, 25)^T$ and $u = (-100, 30)^T$,
**Case 4:** $x_1 = (-1, 3)^T$ and $u = (2, -40)^T$.

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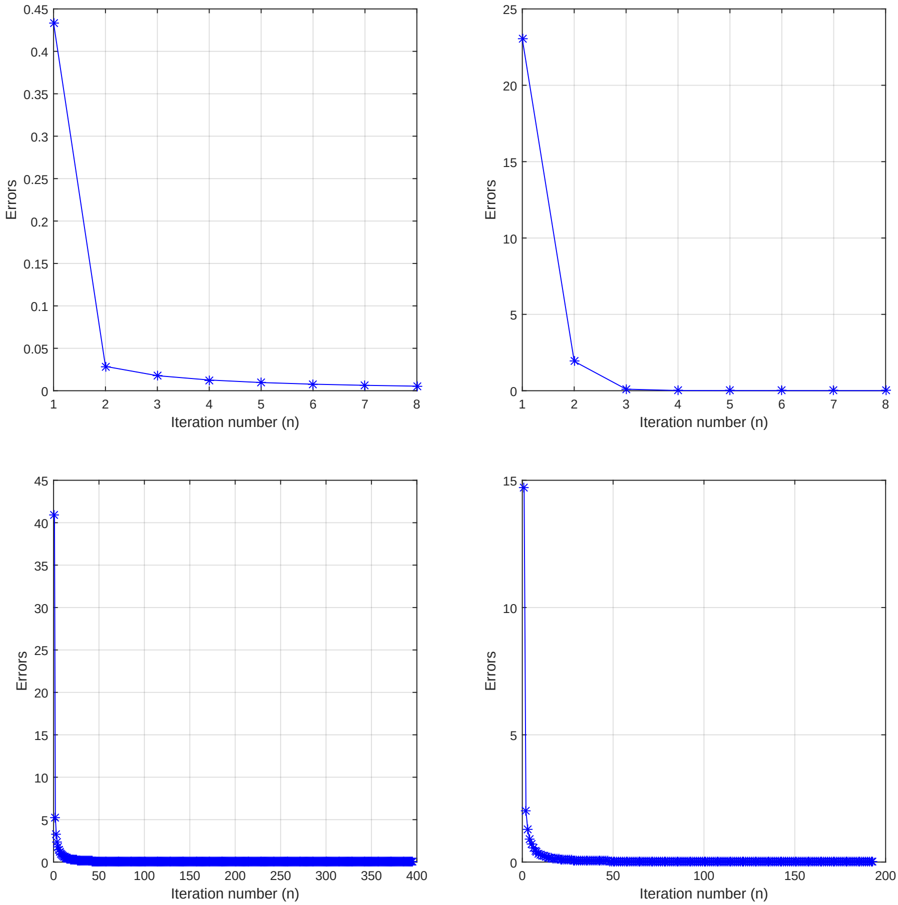


Figure 1: Errors vs Iteration numbers for **Example 5.1**: **Case 1** (top left); **Case 2** (top right); **Case 3** (bottom left); **Case 4** (bottom right).

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