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On two-parameter $C-$groups of bounded linear operators on non-Archimedean Banach spaces

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Abstract. In this paper, we initiate the investigation of two parameter C_0 and $C-$groups of bounded linear operators on non-Archimedean Banach spaces. In contrast with the classical setting, the two-parameter of a given C_0 or $C-$group belongs to a clopen ball Ω_r^2 of the ground non-Archimedean field $\mathbb{K}^2$. We check some properties of two parameter $C-$groups of linear operators on non-Archimedean Banach spaces and we give some examples to support our work.

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1. Introduction and Preliminaries

In the classical setting, the theory of one-parameter semigroups (respectively groups) of linear operators on Banach spaces started in the first half of the last century and acquired its core in 1948 with the Hille-Yosida generation theorem [15]. Thanks to the efforts of many different schools, the theory reached a certain state of perfection, which is well represented in the monographs by A. Pazy [15]. Recently, the situation is characterized by manifold applications of this theory not only to the traditional areas such as partial differential equations or stochastic processes. Groups have become important tools for integro-differential equations and functional differential equations, in quantum mechanics or in infinite-dimensional control theory. Semigroup methods are also applied with great success to concrete equations arising, e.g., in population dynamics or transport theory. However, the semigroup theory is in competition with alternative approaches in all of these fields and that as a whole, the relevant functional-analytic toolbox now presents a highly diversified picture. M. Kostić studied the notions of generalized semigroups and cosine functions, we refer to [13]. The notions of one and two-parameter C_0-groups and $C-$groups were studied by many authors, see for examples [1], [3], [11] and [14].

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In the non-archimedean operators theory, T. Diagana [6] introduced the concept of C_0-groups of bounded linear operators on a free non-Archimedean Banach space. Also, in [10], A. El Amrani, A. Blali, J. Ettayb and M. Babahmed introduced and studied the notions of $C-$groups and cosine families of bounded linear operators on non-Archimedean Banach space. As an application of $C-$groups of linear operators, take the p-adic abstract Cauchy problem for differential equations on non-Archimedean Banach space X given by:

$$ACP(A;x)\begin{cases} \frac{du(t)}{dt} = Au(t), \quad t \in \Omega_r, \\ u(0) = Cx, \end{cases}$$

where $A : D(A) \to X$ is a linear operator and C is an invertible operator with $x \in D(A)$. For more details, we refer to [9] and [10]. The mixed C_0-groups and cosine of bounded linear operators on non-Archimedean Banach spaces over $\mathbb{K}$ were introduced by the authors, for more details, we refer to [4] and [5].

Throughout this paper, X is a non-Archimedean (n.a) Banach space over a (n.a) non trivially complete valued field $\mathbb{K}$ with valuation $|\cdot|$, $B(X)$ denotes the set of all bounded linear operators from X into X, $\mathbb{Q}_p$ is the field of p-adic numbers ($p \geq 2$ being a prime) equipped with p-adic valuation $|.|_p$, $\mathbb{Z}_p$ denotes the ring of p-adic integers of $\mathbb{Z}_p$ is the unit ball of $\mathbb{Q}_p$. For more details and related issues, we refer to [12] and [16]. We denote the completion of algebraic closure of $\mathbb{Q}_p$ under the p-adic absolute value $|\cdot|_p$ by $\mathbb{C}_p$ [12]. Let $r > 0$, Ω_r be the clopen ball of $\mathbb{K}$ centred at 0 with radius $r > 0$, that is $\Omega_r = \{t \in \mathbb{K} : |t| < r\}$. We begin with preliminaries.

Definition 1.1 ([7], Definition 2.1)**.** Let X be a vector space over $\mathbb{K}$. A non-negative real valued function $\|\cdot\| : X \to \mathbb{R}_+$ is called a non-Archimedean norm if:

(1) For all $x \in X$, $\|x\| = 0$ if and only if $x = 0$,

(2) For any $x \in X$ and $\lambda \in \mathbb{K}$, $\|\lambda x\| = |\lambda|\|x\|$,

(3) For any x , $y \in X$, $\|x + y\| \leq \max(\|x\|, \|y\|)$.

Property (3) of Definition 1.1 is referred to as the ultrametric or strong triangle inequality.

Definition 1.2 ([7], Definition 2.2)**.** A non-Archimedean normed space is a pair $(X; \|\cdot\|)$ where X is a vector space over $\mathbb{K}$ and $\|\cdot\|$ is a non-Archimedean norm on X.

Definition 1.3 ([6], Definition 2.2)**.** A non-Archimedean Banach space is a vector space endowed with non-Archimedean norm, which is complete.

For more details on non-Archimedean Banach spaces and related issues, see for example [7].

Proposition 1.4 ([7], Proposition 2.16)**.** *(1) A closed subspace of a non-Archimedean Banach space is a non-Archimedean Banach space.*

(2) The direct sum of two non-Archimedean Banach spaces is a non-Archimedean Banach space.

Example 1.5 ([7], Example 2.20)**.** Let $c_0(\mathbb{K})$ denote the set of all sequences $(x_i)_{i\in\mathbb{N}}$ in $\mathbb{K}$ such that $\lim\limits_{i\to\infty} x_i = 0$. Then, $c_0(\mathbb{K})$ is a vector space over $\mathbb{K}$ and

$$\| (x_i)_{i\in\mathbb{N}} \| = \sup_{i\in\mathbb{N}} |x_i|$$

is a non-Archimedean norm for which $(c_0(\mathbb{K}), \|\cdot\|)$ a non-Archimedean Banach space.

In this section, we define and discuss properties of non-Archimedean Banach spaces which have bases.

Definition 1.6 ([6], Definition 2.5)**.** A non-Archimedean Banach space $(X, \|\cdot\|)$ over a non-Archimedean valued field (complete) $(K, |\ |)$ is said to be a free non-Archimedean Banach space if there exists a family $(x_i)_{i\in I}$ of elements of X indexed by a set I such that each element $x \in X$ can be written uniquely like a pointwise convergent series defined by $x = \sum\limits_{i\in I} \lambda_i x_i$ and $\|x\| = \sup\limits_{i\in I} |\lambda_i| \|x_i\|$.

Definition 1.7. [7] Let $(X, \|\cdot\|)$ be a non-Archimedean Banach space. The non-Archimedean Banach space $(B(X), \|\cdot\|)$ is the collection of all bounded linear operators from X into itself equipped with the operator-norm defined by

$$(\forall A \in B(X)) \ \ \|A\| = \sup_{x\in X\setminus\{0\}} \frac{\|A(x)\|}{\|x\|}.$$

Throughout this paper, X is a $(n.a)$ Banach space over a $(n.a)$ non trivially complete valued field $\mathbb{K}$ of characteristic zero with valuation $|.|$, $B(X)$ is equipped with the norm of Definition 1.7 and for all $r > 0$, $\Omega_r^* = \Omega_r\setminus\{0\}$, denotes the clopen ball of center 0 with radius r deprived of zero.

Definition 1.8 ([6], Definition 3.1)**.** Let $r > 0$ be a chosen real number such that $(T(t))_{t\in\Omega_r}$ are well defined. A one-parameter family $(T(t))_{t\in\Omega_r}$ of bounded linear operators from X into X is a group of bounded linear operators on X if

(i) $T(0) = I$ where I is the unit operator of X,

(ii) For all $t, s \in \Omega_r$ $T(t+s) = T(t)T(s)$.

The group $(T(t))_{t\in\Omega_r}$ will be called of class C_0 or strongly continuous if the following condition holds:

- For each $x \in X$, $\lim\limits_{t\to 0} \|T(t)x - x\| = 0$.

A group of bounded linear operators $(T(t))_{t\in\Omega_r}$ is uniformly continuous if and only if $\lim\limits_{t\to 0}\|T(t)-I\|=0$.
The linear operator A defined by

$$D(A)=\{x\in X:\lim_{t\to 0}\frac{T(t)x-x}{t}\ exists\},$$

and

$$Ax=\lim_{t\to 0}\frac{T(t)x-x}{t}\ for\ each\ x\in D(A),$$

is called the infinitesimal generator of the group $(T(t))_{t\in\Omega_r}$.

We have the following definition.

Definition 1.9 ([10], Definition 2.21)**.** Let $r>0$ and $C\in\ B(X)$ invertible. A one-parameter family $(T(t))_{t\in\Omega_r}$ of bounded linear operators from X into X is called a $C-$group if the following conditions hold:

(i) $T(0)=C$,

(ii) For all $t,\ s\in\Omega_r$, $CT(t+s)=T(t)T(s)$,

(iii) For all $x\in\ X$, $T(\cdot)x:\Omega_r\longrightarrow\ X$ is continuous.

The linear operator A defined by

$$D(A)=\{x\in\ X:\ \lim_{t\to 0}\frac{T(t)x-Cx}{t}\ exists\ \},$$

and

$$Ax=C^{-1}\lim_{t\to 0}\frac{T\left(t\right)x-Cx}{t},\ for\ each\ x\in D(A),$$

is called the infinitesimal generator of the group $(T(t))_{t\in\Omega_r}$.

Remark 1.10 ([10], Definition 2.21). Let $(T(t))_{t\in\Omega_r}$ be a C_0-group of infinitesimal generator A, and let $C\in B(X)$ invertible such that for all $t\in\Omega_r$, $CT(t)=T(t)C$. Define for each $t\in\Omega_r$, $S(t)=T(t)C$, then $(S(t))_{t\in\Omega_r}$ is a $C-$group of infinitesimal generator A. In this sense, Definition 1.9 generalizes the Definition of C_0- group.

We have the following theorem.

Theorem 1.11 ([10], Definition 2.23)**.** *Let $(T(t))_{t\in\Omega_r}$ be a $C-$group satisfying: there exists $M>0$ such that for each $t\in\Omega_r$, $\|T(t)\|\leq M$, and let A be its infinitesimal generator. Then, for every $x\in D(A)$, $t\in\Omega_r$, $T(t)x\in D(A)$. Furthermore,*

$$\frac{dT(t)}{dt}x=AT(t)x=T(t)Ax.$$

The classical two parameter semigroups and $C-$semigroups of linear operators were studied by many authors, we refer to [1], [2], [3], and [11]. For more details of non-Archimedean operators theory, we refer to [4], [7], [8], [10], [16] and [17].

2. Main results

Throughout this paper, $\mathbb{K}$ is a non-Archimedean complete valued field of characteristic zero with valuation $|\cdot|$. We start with the following definitions.

Definition 2.1. Let $r > 0$ be a real number. A two-parameter family $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ of bounded linear operators on X is said to be a two-parameter group of bounded linear operators from X into X if

(i) $T(0,0) = I$,

(ii) For every $t_1,\ t_2,\ s_1$ *and* $s_2 \in \Omega_r,\ T(s_1+s_2, t_1+t_2) = T(s_1,t_1)T(s_2,t_2)$.

Definition 2.2. A two-parameter group $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ on X is a strongly continuous or two parameter C_0-group if

$$\Big(\forall x \in X\Big) \lim_{(s,t)\longrightarrow(0,0)} \|T(s,t)x - x\| = 0. \tag{2.1}$$

Definition 2.3. A two-parameter group $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ is a uniformly continuous group if

$$\lim_{(s,t)\longrightarrow(0,0)} \|T(s,t) - I\| = 0. \tag{2.2}$$

Definition 2.4. A two-parameter C_0-group $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ is uniformly bounded if there exists $M \geq 1$ such that for all $(s,t) \in \Omega_r^2,\ \|T(s,t)\| \leq M$.

Definition 2.5. A two-parameter C_0-group $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ is a C_0-group of contractions if

$$\Big(\forall (s,t) \in \Omega_r^2\Big)\ \|T(s,t)\| \leq 1.$$

Remark 2.6. Let X be a non-Archimedean Banach space over $\mathbb{K}$.

(i) Let $(T(t))_{t\in\Omega_r}$ and $(S(s))_{s\in\Omega_r}$ be one-parameter groups of linear operators on X such that for all $t,s \in \Omega_r$, $T(t)S(s) = S(s)T(t)$. Set for all $t,s \in \Omega_r$, $U(s,t) = T(t)S(s)$. Then $(U(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ is a two parameter group of linear operators on X.

(ii) Let $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ be an uniformly bounded group on X, if $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ is uniformly continuous on X, then the map $T : \Omega_r^2 \longrightarrow B(X)$ is continuous on Ω_r^2.

We have the following statements.

Theorem 2.7. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ be a two-parameter uniformly bounded group on X. Then the following statements hold:*

(i) $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ is uniformly continuous, if and only if $(T(0,t))_{t\in\Omega_r}$ and $(T(s,0))_{s\in\Omega_r}$ are uniformly continuous.

(ii) $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ *is strongly continuous, if and only if* $(T(0,t))_{t\in\Omega_r}$ *and* $(T(s,0))_{s\in\Omega_r}$ *are uniformly continuous.*

Proof.

(i) If $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ is uniformly continuous, then it is easy to see that $(T(0,t))_{t\in\Omega_r}$ and $(T(s,0))_{s\in\Omega_r}$ are uniformly continuous.
Conversely, assume that $(T(0,t))_{t\in\Omega_r}$ and $(T(s,0))_{s\in\Omega_r}$ are uniformly continuous on X. Then,

$$\begin{aligned} \|T(s,t)-I\| &= \|T(s,0)T(0,t)-I\| \\ &= \|T(s,0)T(0,t)-T(s,0)+T(s,0)-I\| \\ &\leq \max\Big\{\|T(s,0)T(0,t)-T(s,0)\|;\|T(s,0)-I\|\Big\} \\ &\leq \max\Big\{\|T(s,0)\|\|T(0,t)-I\|;\|T(s,0)-I\|\Big\}. \end{aligned}$$

Since $(T(s,0))_{s\in\Omega_r}$ is uniformly bounded on X, then $\lim\limits_{(s,t)\longrightarrow(0,0)}\|T(s,t)-I\| = 0$.

(ii) Since $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ is uniformly bounded, then if $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ is strongly continuous, then it is easy to see that $(T(0,t))_{t\in\Omega_r}$ and $(T(s,0))_{s\in\Omega_r}$ are strongly continuous.
Conversely, assume that $(T(0,t))_{t\in\Omega_r}$ and $(T(s,0))_{s\in\Omega_r}$ are strongly continuous on X. Then for all $x\in X$,

$$\begin{aligned} \|T(s,t)x-x\| &= \|T(s,0)T(0,t)x-x\| \\ &= \|T(s,0)T(0,t)x-T(s,0)x+T(s,0)x-x\| \\ &\leq \max\Big\{\|T(s,0)T(0,t)x-T(s,0)x\|;\|T(s,0)x-x\|\Big\} \\ &\leq \max\Big\{\|T(s,0)\|\|T(0,t)x-x\|;\|T(s,0)x-x\|\Big\}. \end{aligned}$$

Since $(T(s,0))_{s\in\Omega_r}$ is uniformly bounded on X, then for all $x\in X$,

$$\lim_{(s,t)\longrightarrow(0,0)}\|T(s,t)x-x\| = 0.$$

□

We have the following proposition.

Proposition 2.8. *Let* X *be a non-Archimedean Banach space over* $\mathbb{K}$*, let* $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ *be a two-parameter group on* X*. Then the following statements hold:*

(i) $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ *is uniformly bounded, if and only if* $(T(0,t))_{t\in\Omega_r}$ *and* $(T(s,0))_{s\in\Omega_r}$ *are uniformly bounded.*

(ii) $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ *is a* C_0- *group of contractions, if and only if* $(T(0,t))_{t\in\Omega_r}$ *and* $(T(s,0))_{s\in\Omega_r}$ *are* C_0-*groups of contractions.*

Proof. Obvious. □

We have the following example.

Example 2.9. Let X be a non-Archimedean Banach space over $\mathbb{Q}_p$, let $A,\ B\in B(X)$ such that $AB=BA$ and $\max\{\|A\|,\|B\|\}<r$ with $r=p^{\frac{-1}{p-1}}$, then for all $(s,\ t)\in\Omega_r\times\Omega_r$, $T(s,t)=e^{sA}e^{tB}$ is well-defined. Using, Example 2.1 of [10], $(T(s,0))_{s\in\Omega_r}$ and $(T(0,t))_{t\in\Omega_r}$ are C_0-groups of contractions on X. Furthermore, for all $(s,\ t)\in\Omega_r\times\Omega_r$,

$$\begin{aligned}\|T(s,t)\| &= \|e^{sA}e^{tB}\|\\ &\leq \|e^{sA}\|\|e^{tB}\|\\ &\leq 1.\end{aligned}$$

Thus, $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ is a two-parameter C_0-group of contractions on X.

We have the following definition.

Definition 2.10. Let $r>0$ be a real number and $C\in B(X)$ be an invertible operator. A two-parameter family $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ of bounded linear operators from X into X is said to be a two-parameter $C-$group of bounded linear operators on X if

(i) $T(0,0)=C$,

(ii) For every $t_1,\ t_2,\ s_1$ and $s_2\in\Omega_r$, $CT(s_1+s_2,t_1+t_2)=T(s_1,t_1)T(s_2,t_2)$,

(iii) For all $x\in\ X$, $T(\cdot,\cdot)x:\Omega_r^2\longrightarrow\ X$ is continuous on Ω_r^2.

Remark 2.11. Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ be a two-parameter $C-$group on X. Setting for all $s,t\in\Omega_r$, $u(s)=T(s,0)$ and $v(t)=T(0,t)$, we have:

(i) It is easy to see that these families are two commuting one-parameter $C-$groups such that for all $s,t\in\Omega_r$, $CT(s,t)=u(s)v(t)$, also $(u(s))_{s\in\Omega_r}$ and $(u(t))_{t\in\Omega_r}$ commute with C and if A_1 and A_2 are their generators, respectively, then we will think of (A_1,A_2) as the generator of $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$.

(ii) If $(u(s))_{s\in\Omega_r}$ and $(u(t)_{t\in\Omega_r}$ are two commuting one-parameter $C-$groups (as above), then one can see that for all $s,t\in\Omega_r$, $T(s,t)=u(s)v(t)$ is a two-parameter C^2-group of linear operators on X.

In the following theorem we can see some elementary properties of a two-parameter $C-$group.

Theorem 2.12. *Suppose that $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ is a two-parameter uniformly bounded $C-$group of infinitesimal generator (A_1,A_2). Then for all $x\in D(A_1)\cap$*

$D(A_2)$, and each $s,\ t \in \Omega_r$, $u(s)x$ and $v(t)x$ are in $D(A_1) \cap D(A_2)$. Furthermore, for $x \in D(A_1) \cap D(A_2)$, and $i = 1, 2$,

$$\frac{\partial}{\partial t_i} T(s,t)x = A_i T(s,t)x = T(s,t)A_i x,$$

where $t_1 = s$ and $t_2 = t$.

Proof. Set for all $s, t \in \Omega_r$, $u(s) = T(s,0)$ and $v(t) = T(0,t)$, let $x \in D(A_1) \cap D(A_2)$, then there is $y \in X$ such that

$$\lim_{s \to 0} \frac{u(s)x - Cx}{s} = Cy.$$

Thus,

$$\lim_{s \to 0} \frac{u(s)v(t)x - Cv(t)x}{s} = Cv(t)y.$$

which is in X, and this implies that $v(t)x$ is in $D(A_1)$, similarly it is in $D(A_2)$.

By Theorem 1.11, $\frac{\partial}{\partial s} T(s,t)x$, $\frac{\partial}{\partial t} T(s,t)x$ exists, and

$$\frac{\partial}{\partial s} CT(s,t)x = \frac{d}{ds}(u(s)(v(t)x)) \tag{2.3}$$

$$= A_1 u(s)(v(t)x) \tag{2.4}$$

$$= A_1 T(s,t)Cx \tag{2.5}$$

$$= CA_1 T(s,t)x. \tag{2.6}$$

Hence from continuity of C, we have

$$C\frac{\partial}{\partial s} T(s,t)x = \frac{\partial}{\partial s} CT(s,t)x = CA_1 T(s,t)x.$$

But C is invertible so,

$$\frac{\partial}{\partial s} T(s,t)x = A_1 T(s,t)x = A_1 T(s,t).$$

Similarly, we obtain

$$\frac{\partial}{\partial t} T(s,t)x = A_2 T(s,t)x = T(s,t)A_2 x.$$

□

Let H and K be generators of $(T(s,0))_{s \in \Omega_r}$ and $(T(0,t))_{t \in \Omega_r}$ respectively. From Remark 2.11, we will think of (H,K) as the generator of $(T(s,t))_{(s,t) \in \Omega_r \times \Omega_r}$. We have the following theorem.

Theorem 2.13. *Assume that H and K are closed linear operators on X, let (H,K) be the generator of an uniformly bounded two-parameter $C-$group $(T(s,t))_{(s,t) \in \Omega_r \times \Omega_r}$, then $D(H) \cap D(HK) \subseteq D(KH)$, and for $x \in D(H) \cap D(HK)$,*

$$HKx = KHx.$$

Proof. Let $x \in D(H) \cap D(KH)$; from the strong continuity of $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ and the fact that K is closed, we have

$$\begin{aligned}
C^2HKx &= C\lim_{s\to 0}\frac{T(s,0)Kx - CKx}{s}\\
&= \lim_{s\to 0}\frac{1}{s}\Big(T(s,0)\Big(\lim_{t\to 0}\frac{T(0,t)x - Cx}{t}\Big) - C\lim_{t\to 0}\frac{T(0,t)x - Cx}{t}\Big)\\
&= \lim_{s\to 0}\lim_{t\to 0}\frac{1}{st}\Big(T(s,0)T(0,t)x - T(s,0)Cx - CT(0,t)x + C^2x\Big)\\
&= \lim_{s\to 0}\lim_{t\to 0}\frac{1}{st}\Big(T(0,t)T(s,0)x - T(s,0)Cx - CT(0,t)x + C^2x\Big)\\
&= \lim_{s\to 0}\lim_{t\to 0}\frac{1}{t}\Big(T(0,t)\Big(\frac{T(s,0)x - Cx}{s}\Big) - C\frac{T(s,0)x - Cx}{s}\Big)\\
&= CK\Big(\lim_{s\to 0}\frac{T(s,0)x - Cx}{s}\Big)\\
&= C^2KHx.
\end{aligned}$$

Since, C is invertible, this completes the proof. □

We finish with some examples of two-parameter $C-$groups of bounded linear operators on non-Archimedean Banach space X.

Example 2.14. Assume that $\mathbb{K} = \mathbb{Q}_p$ and $r = p^{\frac{-1}{p-1}}$, let X be a free non-Archimedean Banach space over $\mathbb{Q}_p$ and $(e_i)_{i\in\mathbb{N}}$ an orthogonal base of X. Define for each $s, t \in \Omega_r$, $x \in X$,

$$T(s,t)x = \sum_{i\in\mathbb{N}}(1-\alpha_i)e^{s\mu_i + t\beta_i}x_ie_i,$$

where $(\alpha_i)_{i\in\mathbb{N}}, (\mu_i)_{i\in\mathbb{N}}, (\beta_i)_{i\in\mathbb{N}} \subset \Omega_r$. It is easy to check that the family $(T(s,t))_{t\in\Omega_r\times\Omega_r}$ is well defined and define a two-parameter $C-$group of bounded linear operators X.

Example 2.15. Let X be a non-Archimedean Banach space over $\mathbb{Q}_p$, let $A, B \in B(X)$ such that $AB = BA$ and $\max\{\|A\|, \|B\|\} < r$ with $r = p^{\frac{-1}{p-1}}$, then for all $t \in \Omega_r$, $T(s,t) = (I-A)e^{sA+tB}$ is a two-parameter $C-$group of bounded linear operators on X. In fact

(i) $T(0,0) = (I-A)$.

(ii) For all $t, s, u, v \in \Omega_r, T(s,u)T(t,v) = (I-A)e^{sA+uB}(I-A)e^{tA+vB} = (I-A)T(s+t, u+v)$.

(iii) It is easy to check that for all $x \in X$, $T(\cdot,\cdot)x : \Omega_r\times\Omega_r \to X$ is continuous on Ω_r^2.

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