# **On Brouwer-Heyting lattices**

#### DPRV Subba Rao<sup>1</sup>

**Abstract.** In this paper, we study the class of a BH lattices as a common frame to Brouwerian and Heyting lattices and investigate some related properties. Also, we characterize the divisibility condition in the definition of BH lattice and we obtain that the set H of all idempotent elements in a BH lattice L forms a Heyting algebra. We introduce the notion of an IBH lattice and under some specific conditions, we characterize an MV algebra, a bounded Wajsberg hoop, a Boolean Algebra, and a commutative bounded BCK algebra in terms on IBH lattices.

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## 1. Introduction

Ever since Ward and Dilworth [9] initiated the study of residuated lattices, the interest in lattice-valued logic has been rapidly growing. Several algebras playing the role of structures of true values have been introduced and axiomatised [9].

In this context, very recently in [8], Swamy KLN introduced the notion of a Brouwer-Heyting lattice (in short BH lattice) as a common abstraction to Brouwerian algebras and Heyting algebras. In fact BH lattices are a generalization of Brouwer and Heyting lattices which focuses on retaining the divisibility properties in Brouwer and Heyting lattices. Examples include commutative po groups, Boolean algebras, Heyting algebras, Brouwerian lattices, commutative BCK monoids etc. Despite the generality of BH lattices, the main result of Swamy [8] which is the decomposition of a BH lattice into a direct product of a commutative l group and a BH lattice with greatest element is of main interest.

The aim of this paper is to continue and further study of BH lattices. In what follows, in Section 2, we give the definition and preliminary results on BH lattices of Swamy [8]. In Section 3, we introduce the notion of an IBH lattice and establish some important results on an IBH lattice and we show that all idempotent elements in  $(L, \circ)$  of a BH lattice form a Heyting algebra. In Section 4, we prove that, under certain suitable conditions, an IBH lattice turns to an MV Algebra, a bounded Wajsberg hoop, a Boolean Algebra, a commutative bounded BCK algebra and vice versa.

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# 2. Definition and preliminary results on *BH* lattices:

We now recall the following definition in [8].

**Definition 2.1.** A system  $(L, \circ, e, \leq, \rightarrow)$ , where (i)  $(L, \circ, e)$  is a commutative monoid with identity 'e'. (ii)  $(L, \leq)$  is a partially ordered set and  $\rightarrow$  is a binary operation on L such that, for all  $x, a, b \in L$ ,  $(x \circ b) \leq a \Leftrightarrow x \leq a \rightarrow b$ , is called a BH monoid.

In a poset  $(L, \leq)$  we mean the join of any two elements a, b in L, denoted by  $a \vee b$ , is the supremum (least upper bound) of  $\{a, b\}$  in L, and similarly, the meet of any two elements a, b in L, denoted by  $a \wedge b$ , is the infimum (greatest lower bound) of  $\{a, b\}$  in L.

**Proposition 2.2.** [8] In any BH monoid  $(L, \circ, e, \leq, \rightarrow)$  we have the following rules of calculus: for  $a, b, c \in L$ 

- 1.  $b \leq c \Rightarrow a \circ b \leq a \circ c$ . 2.  $a \leq (a \circ b) \Rightarrow b$ .
- 3.  $(a \rightarrow b) \circ b < a$ .
- 4.  $c \to (a \circ b) = (c \to b) \to a = (c \to a) \to b$ .
- 5.  $e \rightarrow e = e$ .
- 6.  $a \rightarrow e = a$ .
- 7.  $e \leq a \rightarrow a$ .
- 8.  $e \leq a \rightarrow a$ .
- 9.  $a \leq b \Rightarrow a \rightarrow c \leq b \rightarrow c$ .
- 10.  $a \leq b \Rightarrow c \rightarrow b \leq c \rightarrow a$ .
- 11.  $b \le a \Leftrightarrow e \le a \to b$ .
- 12.  $(a \rightarrow b) \circ (b \rightarrow c) \leq a \rightarrow c$ .
- 13.  $(a \rightarrow b) \circ (b \rightarrow c) \leq a \rightarrow c$ .
- 14. If  $a \lor b$  exists, for some a, b, then  $(c \circ a) \lor (c \circ b)$  exists for any c and  $c \circ (a \lor b) = (c \circ a) \lor (c \circ b)$ .
- 15. If  $a \lor b$  exists, then  $(c \to a) \land (c \to b)$  exists and  $c \to (a \lor b) = (c \to a) \land (c \to b)$ .
- 16. If  $a \wedge b$  exists, then  $(a \rightarrow c) \wedge (b \rightarrow c)$  exists and  $a \wedge b \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c)$ .

**Definition 2.3.** A BH monoid  $(L, \circ, e, \leq, \rightarrow)$  is called a BH lattice if

- 1.  $(L, \leq, \wedge, \vee)$  is a lattice.
- 2.  $a \circ (b \wedge c) = (a \circ b) \wedge (a \circ c)$ , for all a, b, c in L.
- 3.  $((b \to a) \land e) \circ a = a \land b$ , for all a, b in L.

**Theorem 2.4.** A system  $(L, \leq, \lor, \land, \circ, e, \rightarrow)$ , where in  $(L, \circ, e)$  is a monoid and  $(L, \leq, \lor, \land)$  is lattice and  $\rightarrow$  is a binary operation on L is a BH lattice iff

- 1.  $(b \rightarrow a) \circ a \leq b$
- 2.  $a \wedge c \rightarrow b \leq a \rightarrow b$
- 3.  $a \leq a \circ b \rightarrow b$ , for all a, b in L.
- 4.  $a \circ (b \wedge c) = (a \circ b) \wedge (a \circ c)$ , for all a, b, c in L.
- 5.  $((b \rightarrow a) \land e) \circ a = a \land b$ , for all a, b in L.

The following remark is due to [8]:

*Remark* 2.5. Theorem 2.4 shows that BH lattices can be defined by means of identities alone, so that they form a variety of algebras.

We now recall the following definition from [1]:

A lattice ordered group (in short  $\ell$ -group) is a system  $G = (G, \circ, \leq)$  where (i). ( $G, \circ$ ) is a group, (ii). ( $G, \leq$ ) is a lattice, and (iii). For any  $a, b, x, y \in G$ ,  $a \leq b \Rightarrow x \circ a \circ y \leq x \circ b \circ y$ .

**Proposition 2.6.** [8] In any BH lattice L we have the following rules of calculus: for all  $a, b, c \in L$ ,

- 1.  $a \rightarrow a = e$ , for all  $a \in L$ .
- 2.  $a \land (b \lor c) = (a \land b) \lor (a \land c)$  and  $a \lor (b \land c) = (a \lor b) \land (a \lor c)$ , for all  $a, b, c \in L$ .

3. 
$$(a \lor b) \circ (a \land b) = a \circ b$$

- 4.  $a = a \circ e = (a \lor e) \circ (a \land e).$
- 5. Any  $a \in L$  with  $e \leq a$  is invertible in L with respect to the operation  $\circ$ .
- 6. For any  $a, e \lor a$  is invertible in L with respect to the operation  $\circ$ .
- 7. If  $a \in L$  is invertible with respect to the operation  $\circ$ , then  $e \to a$  is the inverse of a.
- 8. If  $b \in L$  is invertible with respect to the operation  $\circ$ , then  $a \to b = a \circ (e \to b)$ .
- 9. For any  $a \in L$ ,  $e \to a$  is invertible with respect to the operation  $\circ$ .

10. If G is the set of all invertible elements with respect to the operation  $\circ$ , then G is an  $\ell$ -group.

We now observe two more properties in BH lattices:

Remark 2.7. In a BH lattice  $(L, \leq, \lor, \land, \circ, e, \rightarrow)$ , for all  $a, b \in L$ ,  $a \circ (b \rightarrow a) = b$  iff there exist an element  $c \in L$  such that  $a \circ c = b$ .

Proof. Proof is a routine verification.

Remark 2.8. In a BH lattice  $(L, \leq, \lor, \land, \circ, e, \rightarrow)$ , for all  $a, b \in L$ ,  $(a \circ b) \rightarrow a = b$  iff there exist an element  $c \in L$  such that  $c \rightarrow a = b$ .

*Proof.* Proof is a routine verification.

# 3. IBH Lattices

In this section we introduce the notion of an IBH lattice and we will show that in an IBH lattice satisfying a certain law, the set of all idempotent elements forms an Heyting algebra.

**Definition 3.1.** A *BH* lattice  $(L, \leq, \land, \lor, \circ, e, \rightarrow)$  is called an *IBH* lattice if the identity element *e* in the monoid  $(L, \circ, e)$  is also an universal upper bound for *L* (i.e,  $x \leq e$ , for all  $x \in L$ ) and there is a universal lower bound 0 in *L* (i.e.,  $0 \leq x$ , for all  $x \in L$ ), further  $0 \neq e$ .

**Example 1**: Let  $L = [0,1] = \{a \in \mathbb{R} : 0 \le a \le 1\}$ , where in  $\le$  is the usual ordering on real numbers. Define  $\le_L$  on L as  $a \le_L b$  iff max  $\{0, a - b\} = 0$ , for all  $a, b \in L$ . Then clearly  $\le_L$  is a partial order on L. Define, for all  $a, b \in L$ ,  $a \lor b = \min\{1, \max\{0, a - b\} + b\}$  and  $a \land b = \max\{0, a + \min\{1, 1 - a + b\} - 1\}$ .  $(L, \le_L, \lor, \land)$  is a lattice. Define a binary operation  $\circ$  on L as  $a \circ b = \max\{0, a+b-1\}$ , for all  $a, b \in L$ . Then it can be easily verified that  $(L, \circ, 1)$  is a monoid. Define  $\rightarrow$  on L as  $a \rightarrow b = \min\{1, a + (1 - b)\}$ . Then  $(L, \le_L, \land, \lor, \circ, 1, \rightarrow)$  is an IBH lattice.

In the sequel, we will also refer to the IBH lattice by its universe L.

**Lemma 3.2.** Let  $(L, \leq, \land, \lor, \circ, e, \rightarrow)$  be a BH lattice and e be the unit element of  $(L, \circ)$  and  $\iota$  be an arbitrary element of L. Then (a). The following assertions are equivalent:

- i.  $\iota = e$ .
- ii.  $\iota \leq b \rightarrow a$  iff  $a \leq b$ .
- *iii.*  $a = a \rightarrow \iota$ , for all  $a \in L$ .

(b). If the BH lattice  $(L, \leq, \land, \lor, \circ, e, \rightarrow)$  contains the universal upper bound 1, then the following assertions are equivalent:

i.  $(L, \leq, \land, \lor, \circ, e, \rightarrow)$  is an IBH lattice.

ii.  $1 = b \rightarrow a$  iff  $a \leq b$ , for all  $a, b \in L$ .

*iii.*  $a = a \rightarrow 1$ , for all  $a \in L$ .

(c). Further, if the BH lattice  $(L, \leq, \land, \lor, \circ, e, \rightarrow)$  contains the universal upper bound 1 and there exists  $c \in L$  with  $1 \circ c = e$ , then 1 = e.

*Proof.* (a). Obviously  $(i) \Rightarrow (ii)$ .

Assume (ii). The fact  $a \leq a$  and the assumption (ii) will give us  $a \leq a \rightarrow \iota$ . On the other hand, by the relation  $a \rightarrow \iota \leq a \rightarrow \iota$  and by 3 of Proposition 2.2, we have  $e \leq (a \rightarrow (a \rightarrow \iota)) \rightarrow \iota$  which yields that  $\iota \leq a \rightarrow (a \rightarrow \iota)$ . Then, by our assumption, it follows that  $a \rightarrow \iota \leq a$ . Hence (ii) $\Rightarrow$ (iii).

Assume (*iii*). By Remark 2.7, we have  $a \circ (c \to a) \to a = c \to a$ , for all  $a, c \in L$ . From this it follows that  $\iota \circ e = e$ . Consequently,  $\iota$  is the unity element in  $(L, \circ)$ . Hence  $(iii) \Rightarrow (i)$ 

Assertion (b) is an immediate consequence of (a).

To prove the assertion (c): The fact  $e \leq 1$  will give us  $e \to 1 \leq e \to e = e$ Which implies that  $(e \to 1) \land e = e \to 1$ . By (*iii*) of Definition 2.3, it follows that  $(e \to 1) \circ 1 = e \longrightarrow (\star)$ . Further, from the fact  $e \leq 1$ , we have  $1 \circ e \leq 1 \circ 1$ , which causes  $1 = 1 \circ e \leq 1 \circ 1 \leq 1$ . Consequently  $1 \circ 1 = 1$ . Now multiplying with 1 on both sides of ( $\star$ ), we have  $(e \to 1) \circ 1 \circ 1 = e \circ 1$  which gives  $(e \to 1) \circ 1$ = 1. Hence e = 1.

**Lemma 3.3.** In any IBH lattice L the following relation holds:  $a \circ b \leq a \wedge b$ , for all  $a, b \in L$ .

*Proof.* For any  $a, b \in L$ , the fact  $a \leq 1$  and  $b \leq 1$  will give us  $a \circ b \leq b$  and  $a \circ b \leq a$  from which it follows that  $a \circ b \leq a \wedge b$ .

**Proposition 3.4.** Let L be an IBH lattice. Then the following are equivalent:

 $i : (a \rightarrow b) \lor (b \rightarrow a) = 1$ , for all  $a, b \in L$ . (Strong De Morgan law)

ii . 
$$(b \lor c) \to a = (b \to a) \lor (c \to a), \text{ for all } a, b, c \in L$$

iii . 
$$c \to (a \land b) = (c \to a) \lor (c \to b)$$
, for all  $a, b, c \in L$ .

*Proof.* Before proving the result we first observe the following two inequalities in L:

(a). For any  $a, b, c \in L$ ,  $(c \to b) \circ a \le (a \circ c) \to b$  -----( $\star$ )

(b). For any  $a, b, c \in L$ ,  $(c \to (a \land b)) \circ (b \to a) \leq c \to a \xrightarrow{(\star \star)} (\star \star)$ To prove (a). For any  $a, b, c \in L$ , the fact  $a \circ (c \to b) \circ b \leq a \circ c$  will result

 $(c \to b) \circ a \le (a \circ c) \to b.$ 

To prove (b). For any  $a, b, c \in L$ , by (iii) of Definition 2.3, we have  $(b \to a) \circ a = a \wedge b$  (as 1 is universal bound,  $(b \to a) \wedge 1 = (b \to a)$ ) which yields that  $(c \to (a \wedge b)) \circ (b \to a) = [c \to ((b \to a) \circ a)] \circ (b \to a) = [(c \to a) \to (b \to a)] \circ (b \to a) \leq c \to a$ .

We now prove the main result by showing the equivalence of statements  $(i) \Leftrightarrow$ (ii) and  $(i) \Leftrightarrow (iii)$  To prove  $(i) \Leftrightarrow (ii)$ : Let  $a, b, c \in L$ . Consider  $(b \lor c) \to a = ((b \lor c) \to a) \circ 1 = ((b \lor c) \to a) \circ ((c \to b) \lor (b \to c)) = (((b \lor c) \to a) \circ (c \to b)) \lor ((b \to c) \circ ((b \lor c) \to a)) \leq ((b \lor c) \circ (c \to b) \to a) \lor ((b \lor c) \circ (b \to c) \to a)$  (since by the fact (a))  $\leq (c \to a) \lor (b \to a)$ . Therefore we obtained  $(b \lor c) \to a \leq (c \to a) \lor (b \to a)$ . For obtaining the reverse of the inequality, consider  $[(c \to a) \lor (b \to a)] \circ a = [((c \to a) \circ a)] \lor [((b \to a) \circ a)] \leq b \lor c$ . Hence  $(i) \Rightarrow (ii)$  is proved. To prove  $(i) \Leftrightarrow (iii)$ : Let  $a, b, c \in L$ . Consider  $c \to (a \land b) = [(c \to (a \land b)) \circ (b \to a)] < a) \lor (c \to b)$  (by using the fact (b)). Therefore we obtained  $c \to (a \land b) \leq (c \to a) \lor (c \to b)$ . The reverse inequality is obvious. Hence we established the result.

**Lemma 3.5.** If L is an IBH lattice satisfying the strong de Morgan law then the following assertions are valid in L:

- *i.*  $a \circ b \leq (a \circ a) \lor (b \circ b)$ , for all  $a, b \in L$ .
- ii.  $(a \circ a) \land (b \circ b) \leq (a \circ b)$ , for all  $a, b \in L$ .

*Proof.* For  $a, b \in L$ , consider  $a \circ b = (a \circ b) \circ 1 = (a \circ b)[(a \to b) \lor (b \to a)]$  (By using the strong Demorgan law)  $= (a \circ b) \circ (a \to b) \lor (a \circ b) \circ (b \to a) \le (a \circ a) \lor (b \circ b)$ . The other inequality is easy to prove. Hence the result is proved.

We now give a characterization of the divisibility property in IBH lattices.

**Proposition 3.6.** In any IBH lattice L the following assertions are equivalent:

*i.* For every pair  $(a,b) \in L \times L$  with  $b \leq a$  there exists  $c \in L$  such that  $b = a \circ c$ .

*ii.* 
$$a \wedge b = a \circ (b \wedge a)$$
, for  $a, b \in L$ .

*iii.*  $b \to c = (b \to a) \circ (c \to (a \land b))$ , for  $a, b, c \in L$ .

*Proof.* Assume (i). Suppose  $a, b \in L$ . Since  $a \wedge b \leq a$ , by our supposition there is an element  $c \in L$  such that  $a \wedge b = a \circ c$ . Then we have  $c \leq b \to a$ , which causes  $a \wedge b \leq a \circ (b \to a)$ —(\*). On the other hand, we have  $a \circ (b \to a) \leq b$  (since  $b \to a \leq b \to a$ ) and  $a \circ (b \to a) \leq a$  (since  $b \to a \leq 1 = a \to a$ ) which yields that  $a \circ (b \to a) \leq a \wedge b$ —(\*\*). Now (ii) follows from (\*) and (\*\*). Assume (ii).

Consider  $(b \to a) \circ (c \to (a \land b)) = (b \to a) \circ ((c \to a) \to (b \to a))$  (by our assumption) =  $(b \to a) \land (c \to a) = ((b \land c) \to a)$ . Hence  $(ii) \Rightarrow (iii)$ . Assume (iii).

Suppose  $c, b \in L$  such that  $c \leq b$ . Now taking a = 1, b, c in our assumed condition  $b \to c = (b \to a) \circ (c \to (a \land b))$ , we have  $b \to c = (b \to 1) \circ (c \to (1 \land b))$ =  $(b \to 1) \circ (c \to b) = c \to 1 = c$ . Thus  $(iii) \Rightarrow (i)$ .

**Proposition 3.7.** In any IBH lattice L the following assertions holds:

- *i.*  $a \wedge b = a \circ b$ , for every idempotent a in the monoid  $(L, \circ, 1)$  and  $b \in L$ .
- *ii.*  $a \circ b \leq (a \circ a) \lor (b \circ b)$ , for all  $a, b \in L$ .

## *Proof.* To prove (i)

For  $a \wedge b \leq a$ , by property (i) of Proposition 3.6, there is an element  $c \in L$  such that  $a \wedge b = c \circ a$ . But  $c \circ a \leq (b \to a)$ . Therefore we have  $a \wedge b = c \circ a \leq a \circ (b \to a) = a \circ a \circ (b \to a)$  (since a is idempotent in  $(L, \circ)$ ) =  $a \circ (b \to a) \circ a \leq a \circ b$ . Thus  $a \wedge b \leq a \circ b$ . By Lemma 3.3, we have  $a \circ b \leq a \wedge b$ . Therefore  $a \circ b = a \wedge b$ .

## To prove (*ii*).

Let  $a, b \in L$ . By property (i) of Proposition 3.6, there exists  $c_1, c_2 \in L$  such that  $a = (a \lor b) \circ c_1$  and  $b = (a \lor b) \circ c_2$ . Now  $a \lor b = [(a \lor b) \circ c_1] \lor [(a \lor b) \circ c_2]$ =  $(a \lor b) \circ (c_1 \lor c_2)$  (by 14 of 2.2).

Now consider  $a \circ b = c_1 \circ c_2 \circ (a \lor b) \circ (a \lor b) = c_1 \circ c_2 \circ (c_1 \lor c_2) \circ (a \lor b) \circ (a \lor b) \le ((c_1 \circ c_1) \lor (c_2 \circ c_2)) \circ (a \lor b) \circ (a \lor b) = (a \circ a) \lor (b \circ b).$ 

**Corollary 3.8.** Let  $M = (L, \leq, \land, \lor, \circ, 1, \rightarrow, 0)$  be an IBH lattice and satisfying the strong DeMorgan law, then the set H of all idempotent elements in  $(L, \circ)$ forms an Heyting algebra, and the binary operation  $\rightarrow$  in H coincides with the operation  $\rightarrow$  based on  $\circ$ .

# 4. Characterization Theorems

In this section, we prove IBH lattices with some restrictions are precisely the class of MV-algebras, Wajsberg hoops, Boolean algebras, commutative bounded BCK algebras.

We first prove that a class of IBH lattices with a set of restrictions on them are precisely the class of MV-algebras.

Before seeing this, for the sake of readability of the paper, let us recall the definition and some important results on MV-algebras due to Chang [4].

Axioms of MV-algebras and some elementary consequences of [4] are as follows:

**Definition 4.1.** An *MV*-algebra is a system  $(A, +, \circ, \neg, 0, 1)$ , wherein (A, +, 0),  $(A, \circ, 1)$  are commutative monoids and  $\neg$  is a unary operation on A obeying the following axioms: for any  $a, b, c \in A$ ,

i.  $a + \neg a = 1$ . ii. a + 1 = 1. iii.  $\neg (a + b) = \neg a \circ \neg b$ . iv.  $\neg \neg (a) = a$ . v.  $\neg 0 = 1$ . i'.  $a \circ \neg a = 0$ . ii'.  $a \circ 0 = 0$ . iii'.  $\neg (a \circ b) = \neg a + \neg b$ .

In order to write the remaining axioms in the definition, a compact form of the following definition is given below:

I.  $a \lor b = (a \circ \neg b) + b$ . II.  $a \land b = (a + \neg b) \circ b$ . vi.  $a \lor b = b \lor a$ . vii.  $a \lor (b \lor c) = (a \lor b) \lor c$ . viii.  $a + (b \land c) = (a + b) \land (a + c)$ . vi<sup>'</sup>.  $a \land b = b \land a$ . vii<sup>'</sup>.  $a \land (b \land c) = (a \land b) \land c$ . viii<sup>'</sup>.  $a \circ (b \lor c) = (a \circ b) \lor (a \circ c)$ .

Let us recall here the following definition given by P. Mangani [6], which is equivalent to Definition 4.1 of C. C. Chang [4], where the same is proved by P. Mangani in [6]:

**Definition 4.1(a)**[6]: An MV algebra is an algebra  $(L, +, \circ, \neg, 0, 1)$  satisfying the following identities:

i. a + (b + c) = (a + b) + c. ii. a + 0 = a. iii. a + b = b + a. iv. a + 1 = 1. v.  $\neg \neg a = a$ . vi.  $\neg 0 = 1$ . vii.  $a + \neg a = 1$ . viii.  $\neg (\neg a + b) + b = \neg (\neg b + a) + a$ . ix.  $a \circ b = \neg (\neg a + \neg b)$ .

The following theorem is due to C. C. Chang [4]:

**Theorem 4.2.** In any MV-algebra A, for all  $a, b, c \in A$ , we have,

*i*. *a* ∨ 0 = *a* = *a* ∧ 1, *a* ∧ 0 = 0, and *a* ∨ 1 = 1. *ii*. *a* ∨ *a* = *a* = *a* ∧ *a*. *iii*. ¬(*a* ∨ *b*) = ¬*a* ∧ ¬*b* and ¬(*a* ∧ *b*) = ¬*a* ∨ ¬*b*. *iv*. *a* ∧ (*a* ∨ *b*) = *a* = *a* ∨ (*a* ∧ *b*). *v*. If *a* + *b* = 0, then *a* = *b* = 0. *vi*. If *a* ∘ *b* = 1, then *a* = *b* = 1.

- vii. If  $a \lor b = 0$ , then a = b = 0.
- viii. If  $a \wedge b = 1$ , then a = b = 1.

The following three theorems will characterize MV-algebras as a class of certain IBH lattices:

**Theorem 4.3.** Let  $(L, \leq, \land, \lor, \circ, \rightarrow, 0, 1)$  be an IBH lattice satisfying the following conditions:

- i. For all  $a \in L$ ,  $0 \to (0 \to a) = a$ .
- *ii.* For all  $a, b \in A$ ,  $a \circ (b \to a) = b \circ (a \to b)$ .

Set  $\neg a = 0 \rightarrow a$ , for any  $a \in A$ . Then  $(L, +, \circ, \neg, 0, 1)$  is an MV-algebra.

*Proof.* Let  $(L, \leq, \land, \lor, \circ, \rightarrow, 0, 1)$  be an IBH lattice satisfying conditions (i) and (ii) in the hypothesis. To obtain the MV-algebra structure for L, let us define a unary operation  $\neg$  on L and a binary operation + on L as follows: Define, a unary operation  $\neg$  on L by  $\neg a = 0 \rightarrow a$ , for every  $a \in L$  and a binary operation + on L by  $a + b = \neg(\neg a \circ \neg b)$ , for all  $a, b \in L$ . We now prove that  $(L, +, \circ,$  $\neg$ , 0, 1) is an *MV*-algebra using Definition 4. We first observe  $\neg \neg a = a$ , for all  $a \in L$ . For this, let  $a \in L$ . Then, by (i) of the hypothesis,  $0 \to (0 \to a) = a$ , which is same as  $\neg \neg a = a$  (by the definition of  $\neg$  on L). Therefore (v.) of Definition 4 is proved. We now verify all the remaining conditions of Definition 4. Let  $a, b, c \in L$ . Consider  $a + (b + c) = \neg(\neg a \circ \neg (b + c))$  (by the definition of + on L) =  $\neg(\neg a \circ (\neg b \circ \neg c))$  (by the definition of + on L) =  $\neg((\neg a \circ \neg b) \circ \neg c)$ (since  $\circ$  is associative on L) =  $\neg(\neg(a+b) \circ \neg c)$  (by the definition of + on L) = (a+b)+c. Therefore (i.) of Definition 4 is proved. Consider  $a+0 = \neg(\neg a \circ \neg 0)$  $= \neg(\neg a \circ 1)$  (by 6 of Proposition 2.2  $= \neg(\neg a)$  (since 1 is identity element of the monoid  $(L, \circ, 1) = a$ . Therefore (ii.) of Definition 4 is proved. For  $a, b \in L$ , consider  $a + b = \neg(\neg a \circ \neg b) = \neg(\neg b \circ \neg a)$  (since  $\circ$  is commutative) = b + a. Therefore (iii.) of Definition 4 is proved. For  $a \in L$ , consider  $a+1 = \neg(\neg a \circ \neg 1)$  $= \neg(\neg a \circ 0) = \neg 0$  (since  $x \circ 0 = x \land 0 = 0$ , for all  $x \in L$ ) = 1. Therefore (iv.) of Definition 4 is proved. The condition (vi.) of Definition 4 follows from 6 of Proposition 2.2. For  $a \in L$ , consider  $a + \neg a = \neg(\neg \circ \neg \neg a) = \neg(\neg \circ a) = \neg(0)$ (since  $(0 \rightarrow a) \circ a = 0$ , for all  $a \in L$ ) = 1. Therefore (vii.) of Definition 4 is proved. For  $a, b \in L$  we now see that,  $\neg(\neg a \circ b) \circ b = \neg(a \circ \neg b) \circ a$ . To see this, consider  $\neg(\neg a \circ b) \circ b = [0 \rightarrow ((0 \rightarrow a) \circ b)] \circ b = [(0 \rightarrow (0 \rightarrow a)) \rightarrow b] \circ b =$  $(a \to b) \circ b = (b \to a) \circ a = \neg (\neg b \circ a) \circ a = \neg (a \circ \neg b) \circ a$ . Therefore (viii.) of Definition 4 is proved. Condition (ix.) follows from the definition of + on L. Therefore  $(L, +, \circ, \neg, 0, 1)$  is an *MV*-algebra. 

#### **Theorem 4.4.** Any MV algebra $(L, +, \circ, \neg, 0, 1)$ is an IBH lattice.

*Proof.* Let  $(L, +, \circ, \neg, 0, 1)$  be an MV algebra. Define a relation  $\leq$  on L by  $a \leq b$  iff  $(a \circ \neg b) + b = b$ , for all  $a, b \in L$ . By (i), (ii) and (iv) of Theorem 4.2 it follows that  $\leq$  is a partial order on L. Clearly  $(L, \leq, \land, \lor, 0, 1)$  forms a bounded lattice. To see L is an IBH lattice, we define a binary operation  $\rightarrow$ 

on L as  $a \to b = a + \neg b$ , for all  $a, b \in L$  and will see that  $a \to b$  satisfies the BH monoid condition. i.e.,  $x \circ b \leq a \Leftrightarrow x \leq a \to b$ . For this, we use the following result in MV algebras: In any MV algebra L,  $a \leq b \Leftrightarrow \neg a + b = 1$ , for any  $a, b \in L$ . Now for any  $x, a, b \in L$ , from the above cited result,  $x \circ b \leq a$  if and only if  $\neg(x \circ b) + a = 1$  if and only if  $\neg x + \neg b + a = 1$  if and only if  $x \leq a + \neg b$  if and only if  $x \leq a + \neg b = a \to b$ . Thus L is an IBH lattice.

**Theorem 4.5.** In any MV algebra  $(L, +, \circ, \neg, 0, 1)$  the following two conditions hold:

- i. For all  $a \in L$ ,  $0 \to (0 \to a) = a$ .
- *ii.* For all  $a, b \in A$ ,  $a \circ (b \to a) = b \circ (a \to b)$ .

*Proof.* Let  $(L, +, \circ, \neg, 0, 1)$  be an MV algebra. Define the binary operation  $\rightarrow$  on L as  $a \rightarrow b = a + \neg b$ , for all  $a, b \in L$ . Then clearly  $(L, \circ, 1, \leq, \rightarrow)$  is a BH monoid.

To prove (i.), let  $a \in L$ , consider  $0 \to (0 \to a) = 0 \to (0 + \neg a) = 0 + \neg (0 + \neg a) = a$ .

To prove (ii.), let  $a, b \in L$ , consider  $b \circ (a \to b) = b \circ (a + \neg b) = a \wedge b$  (by ii of Definition 4.1) =  $a \circ (b + \neg a) = a \circ (b \to a)$ . Hence the theorem is established.

We now see that an IBH lattice under certain conditions becomes a hoop algebra.

Let us recall the definition of a hoop [2].

**Definition 4.6.** [2] A hoop is an algebra  $(A, \rightarrow, \circ, 1)$ , where in,

- i.  $(A, \circ, 1)$  is a commutative monoid and
- ii.  $\rightarrow$  is a binary operation on A satisfying, for all  $a, b \in A$ ,
  - a.  $a \to a = 1$ . b.  $a \circ (a \to b) = b \circ (b \to a)$ . c.  $a \to (b \to c) = (a \circ b) \to c$ .

The following lemma is due to W.J. Blok et.al [2]:

**Lemma 4.7.** In any hoop, we have the following properties: for any  $a, b, c \in A$ ,

- i.  $b \leq a \rightarrow b$ .
- *ii.*  $a \leq (a \rightarrow b) \rightarrow b$ .
- *iii.*  $a \to (b \to c) = b \to (a \to c)$ .
- iv.  $a \to b \le (b \to c) \to (a \to c)$ .
- v. If a is an idempotent element in A with respect to  $\circ$ , then  $a \to (b \to c) = (a \to b) \to (a \to c)$ .

- vi.  $a \rightarrow 1 = 1$ .
- vii.  $1 \rightarrow a = 1$ .

viii.  $a \leq b \Rightarrow b \rightarrow z \leq a \rightarrow z \text{ and } z \rightarrow a \leq z \rightarrow b.$ 

**Definition 4.8.** [2] A Wajsberg hoop is a hoop  $(A, \rightarrow, \circ, 0, 1)$  satisfying  $(a \rightarrow b) \rightarrow b = (b \rightarrow a) \rightarrow a$ , for all  $a, b, \in A$ .

**Definition 4.9.** [2] A bounded hoop is an algebra  $(A, \rightarrow, \circ, 0, 1)$  such that  $(A, \rightarrow, \circ, 1)$  is a hoop and  $0 \le a$ , for all  $a \in A$ .

**Definition 4.10.** [2] A bounded Wajsberg hoop is a Wajsberg hoop which contains a least element 0.

We now give characterization of a bounded Weisberg hoop in terms of IBH lattices:

**Theorem 4.11.** Every bounded Wajsberg hoop  $(A, \rightarrow, \circ, 0, 1)$  is an IBH lattice satisfying  $b \rightarrow (b \rightarrow a) = a \rightarrow (a \rightarrow b)$ , for all  $a, b \in A$  and vice versa.

*Proof.* Let  $(A, \rightarrow, \circ, 1)$  be a Wajsberg hoop. Define an order relation  $\leq$  on A such that  $a \leq b$  iff  $a \rightarrow b = 1$ , for all  $a, b \in A$ . To see  $\leq$  is a partial order on A. The properties of  $\leq$  namely reflexive and antisymmetric on A follows from (ii)(a), (ii)(b) of Definition 4.6. Now to establish the transitive property of  $\leq$  on A, we use the following property, given by (13) of [3], in any hoop: For  $x, a, b \in A, a \circ x = b$  iff  $b \rightarrow a = 1$  — (\*)

Now, for  $a, b, c \in A$ , suppose  $a \leq b$  and  $b \leq c$ , which implies  $a \to b = 1$  and  $b \to c = 1$  (by definition of  $\leq$ ). Then, by  $(\star)$ ,  $b \circ x = a$  and  $c \circ y = b$ , for some  $x, y \in A$ . Therefore  $a = b \circ x = (c \circ y) \circ x = c \circ (y \circ x)$  which results in  $a \to c = 1$ . Hence  $a \leq c$ . Therefore  $\leq$  is a partial order on A.

To obtain IBH lattice structure on A, we first see that, for any  $x, a, b \in A$ ,  $(a \circ x) \leq b$  iff  $x \leq a \rightarrow b$ .—(\*\*)

For  $a, b \in A$ , we have  $a \circ x \leq b$  if and only if  $(a \circ x) \to b = 1$  if and only if  $(a \circ x) \to b = x \to (a \to b) = 1$  (by (c.) of Definition 4.6) if and only if  $x \leq a \to b$ . Hence the binary operation  $\to$  on A satisfying the condition  $(a \circ x) \leq b$  if and only if  $x \leq (a \to b)$ .

Now define a new binary operation \* on A (which will play the role of  $\rightarrow$  in a BH monoid) such that  $b * a = a \rightarrow b$ . For  $x, a, b \in A$ , suppose  $x \circ b \leq a$  if and only if  $x \leq b \rightarrow a$  if and only if  $x \leq a * b$ . Therefore  $(A, \circ, 1, \leq, *)$  is a BH monoid.

We now see that  $(A, \leq, *, \circ, 1)$  is an *IBH* lattice.

For this, we show that  $a \circ (a \to b)$  is the glb of  $\{a, b\}$  in A, for all  $a, b \in A$ , under the partial order  $\leq$ .

i.e.,  $a \wedge b = a \circ (a \rightarrow b)$ , for all  $a, b \in A$ 

Let  $a, b \in A$ . Consider  $a \circ (a \to b) \to a = (a \to b) \to (a \to a) = (a \to b) \to 1$ = 1. Consequently  $a \circ (a \to b) \le a$  (by the definition of  $\le$ ).

Consider  $a \circ (a \to b) \to b = b \circ (b \to a) \to b = (b \to a) \to (b \to b) = 1$  which yields that  $a \circ (a \to b) \le b$ .

Therefore  $a \circ (a \to b) \to b$  is a lower bound of  $\{a, b\}$ .

Now suppose  $c \in A$  such that  $c \leq a$  and  $c \leq b$ . Then we have  $c = u \circ a$  and  $c = v \circ b$  for some  $u, v \in A$ . Since  $(A, \leq, \circ)$  is a partially ordered monoid, we have  $c = u \circ a \leq a \circ (a \rightarrow b)$ . Hence  $a \wedge b = a \circ (a \rightarrow b)$ , for all  $a, b \in A$ .

We now show that  $(a \to b) \to b$  is the lub of  $\{a, b\}$  in A, for all  $a, b \in A$ . i.e.,  $a \lor b = (a \to b) \to b$ , for all  $a, b \in A$ . By (ii) of Lemma 4.7 we have  $a \leq (a \to b) \to b$  and  $b \leq (b \to a) \to a = a \leq (a \to b) \to b$  (since A is a Wajsberg hoop), which shows that  $a \leq (a \to b) \to b$  is an upper bound of  $\{a, b\}$ . Suppose  $a \leq z$  and  $b \leq z$ , then we have  $a \to z = 1$  and  $b \to z = 1$ . We show that  $(a \to b) \to b \leq z$ . For this, by (ii) of Lemma 4.7 we have  $a \leq (a \to b) \to b$ . Then, by (viii) of Lemma 4.7, it implies that  $[(a \to b) \to b] \to z \leq a \to z = 1$ . Again by the use of (viii) of Lemma 4.7 we have  $1 \to 1 \leq ([(a \to b) \to b] \to z) \to 1$  which yields that  $[(a \to b) \to b] \to z = 1$ . Hence  $(a \to b) \to b \leq z$ . Therefore  $(a \to b) \to b$  is lub of  $\{a, b\}$  in A. Therefore  $(A \leq \land, \lor, \lor, \ast, \circ, 1)$  is a lattice. Since A is a Wajsberg hoop we have  $a \to 1 = 1$ , for all  $a \in A$ .

We now verify  $(A, \leq, \wedge, \vee, \circ, 1, *)$  satisfies all the conditions of an IBH lattice.

- i. Let  $a, b \in A$ . Consider  $(b * a) \circ a = (a \to b) \circ a = a \land b \leq b$ . Hence  $(b * a) \circ a \leq b$ .
- ii.  $(a \land b) * b = b \rightarrow (a \land b) \le b \rightarrow a = a * b$ . Therefore  $(a \land b) * b \le a * b$ .
- iii. Since  $a \circ b \leq a \circ b$ , by the definition of  $\rightarrow$ , we have  $a \leq b \rightarrow (a \circ b)$ . Therefore  $a \leq (a \circ b) * b$ .
- iv. Since  $a \circ (a \to b) = a \wedge b$ , for  $a, b \in A$ , it follows that  $a \circ ((b * a) \wedge 1) = a \wedge b$ .

Hence  $(A, \leq, \land, \lor, \circ, 1, *)$  satisfies all the conditions of an *IBH* lattice. Therefore A is an *IBH* lattice satisfying  $b \to (b \to a) = a \to (a \to b)$ , for all  $a, b \in A$ .

Conversely, Suppose  $A = (A, \leq, \land, \lor, \rightarrow, \circ, 0, 1)$  is an IBH attice, satisfying  $b \to (b \to a) = a \to (a \to b)$ , for all  $a, b \in A$ . Then obviously  $A = (A, \to, \circ, 0, 1)$  is a bounded Wajsberg hoop.

We now see IBH lattices under certain conditions are precisely Boolean algebras.

To see this we recall some of the following notions and properties of orthomodular lattices given in [7]:

**Definition 4.12.** [7] An ortholattice is a lattice  $(L, \land, \lor, ', 0, 1)$  with universal bounds 0,1 and a unary operation  $\prime$  on L satisfying the following properties:

i. 
$$x'' = x$$
 for every  $x \in L$ ,

ii. 
$$x \leq y$$
 implies  $y' \leq x'$  for every  $x, y \in L$ ,

iii.  $x \wedge x' = 0, x \vee x' = 1$  for every  $x \in L$ .

**Definition 4.13.** [7] An ortho lattice L is called an ortho modular lattice if it satisfies the orthomodular law:  $y = x \vee (x' \wedge y)$  for every  $x, y \in L$  with  $x \leq y$ .

**Definition 4.14.** [7] Elements x, y of an orthomodular lattice L are called orthogonal (denoted by  $x \perp y$ ) if  $x \leq y'$ .

**Definition 4.15.** [7] Let L be an orthomodular lattice and let  $x, y \in L$ . We say x and y are compatible if and only if there exist in L mutually orthogonal elements  $x_1, y_1, z$ :  $x_1 \perp y_1, x_1 \perp z, y_1 \perp z$  ( $x_1 \leq y'_1, x_1 \leq z', y_1 \leq z'$ ) such that  $x = x_1 \lor z$  and  $y = y_1 \lor z$ .

Note 4.16. [7] In an orthomodular lattice  $L, x, y \in L$  are compatible if and only if  $x \wedge y = x \wedge (x' \vee y) = y \wedge (y' \vee x)$ .

*Note* 4.17. [7] Properties of orthomodular lattices:

- i. Let us denote  $y \odot x = y \wedge x'$  for  $x \leq y$ . Then for every  $x \leq y$  we have  $x \perp (y \odot x)$  and, according to the orthomodular law,  $y = x \lor (y \odot x)$ . For every pair x, y of elements of an orthomodular lattice L we have  $x - (x \wedge y) \wedge y = x \wedge (x \wedge y)' \wedge y = (x \wedge y) \wedge (x \wedge y)' = ((x \wedge y)' \lor (x \wedge y))' = 1' = 0.$
- ii. An orthomodular lattice L is a Boolean algebra iff every pair x, y of elements of L is compatible, i.e.,  $x \ominus (x \land y)$  and  $y \ominus (x \land y)$  are orthogonal.

The following two lemmas are necessary for obtaining a characterization theorem of a Boolean algebra in terms of IBH lattices:

**Lemma 4.18.** If *L* is an orthomodular IBH lattice then the following property holds in *L*: For all  $a, b \in L$ , if  $a \wedge b = 0$  and  $a \wedge b' = 0$ , then a = 0.

*Proof.* Suppose L is an orthomodular IBH lattice and suppose  $a \wedge b = 0$  and  $a \wedge b' = 0$ , for  $a, b \in L$ . Since L is an IBH lattice we have  $a \circ b \leq a \wedge b$  and  $a \circ b' \leq a \wedge b'$  from which it follows that  $a \circ b = 0$  and  $a \circ b' = 0$ , which implies  $b \leq 0 \rightarrow a$  and  $b' \leq 0 \rightarrow a$  consequently  $1 \leq 0 \rightarrow a$  (since  $b \lor b' = 1$ ). Hence a = 0.

**Lemma 4.19.** If L is an orthomodular IBH lattice then every pair  $a, b \in L$  is compatible.

*Proof.* Suppose L is an orthomodular IBH lattice.

We now see that for  $a, b \in L$ ,  $a \odot (a \land b)$  and  $b \odot (a \land b)$  are orthogonal. Let  $c = a \land b$  and  $u = (a \odot c) \land b'$ . Then  $((a \odot c) \odot u) \land b' = 0$  and, since  $(a \odot c) \land b = 0$ , also  $[(a \odot c) \odot u] \land b = 0$ . Then by Lemma 4.13, it follows that  $(a \odot c) \odot u = 0$  and therefore a - c = u. Since  $u \perp b$ , we have  $u \perp (b \odot c)$  and therefore  $a \odot (a \land b)$  and  $b \odot (a \land b)$  are orthogonal. Hence the pair a, b is compatible.

#### Theorem 4.20. An IBH lattice is orthomodular iff it a Boolean Algebra.

*Proof.* Suppose  $(L, \leq, \land, \lor, ', 0, 1)$  is a Boolean algebra. Then  $(L, \circ, 1)$  is commutative monoid, where in  $\circ$  is defined by  $a \circ b = a \land b$ , for all  $a, b \in L$ . Define  $\rightarrow$  on L as  $a \rightarrow b = a \lor b'$ . Then it is routine verification to see L is an orthomodular IBH lattice.

Conversely, suppose L is an orthomodular IBH lattice. Then by Lemma 4.19 and by note 4.17(ii) it follows that L is a Boolean algebra.

We now see the equivalence of IBH lattices and commutative bounded BCK algebras.

The following notions and properties are due to [5]:

**Definition 4.21.** [5] A BCK-algebra is an algebra denoted as  $(A, \rightarrow, 1)$ , where in  $\rightarrow$  is a binary operation on A and 1 is a constant element in A, is an algebra satisfying the following identities: for all  $a, b \in A$ ,

- i.  $a \rightarrow b = b \rightarrow a = 1$  then a = b.
- ii.  $a \rightarrow 1 = 1$ .
- iii.  $1 \to a = a$ .
- iv.  $(a \rightarrow b) \rightarrow ((b \rightarrow c) \rightarrow (a \rightarrow c)) = 1.$

**Definition 4.22.** [5] If A is a BCK-algebra, then the relation  $\leq$  defined by  $a \leq b$  iff  $a \rightarrow b$  is a partial order on A, with respect to this order 1 is the largest element of A. In this case A will be called bounded BCK algebra if A has the smallest element 0.

**Definition 4.23.** [5] In BCK algebra  $(A, \rightarrow, 1)$ , if we have  $(a \rightarrow b) \rightarrow b = (b \rightarrow a) \rightarrow a$ , for all  $a, b \in A$ , then A is called commutative.

Now we have the following characterization theorem:

**Theorem 4.24.** Every IBH lattice  $(A, \leq, \land, \lor, \circ, \rightarrow, 0, 1)$  satisfying the following conditions:

- i. For all  $a \in A$ ,  $0 \to (0 \to a) = a$ .
- *ii.* For all  $a, b \in A$ ,  $a \circ (b \to a) = b \circ (a \to b)$ .

is a commutative bounded BCK algebra. Conversely every commutative bounded BCK algebra  $(A, \rightarrow, 0, 1)$  is an IBH lattice satisfying

- i. For all  $a \in A$ ,  $0 \to (0 \to a) = a$ .
- *ii.* For all  $a, b \in A$ ,  $a \circ (b \to a) = b \circ (a \to b)$ .

*Proof.* Suppose  $(A, \rightarrow, 0, 1)$  is any commutative bounded BCK algebra. Then by Theorem 1.7.1 of [5] we have A is an MV-algebra. Again applying Theorem 4.5 we have A is an IBH lattice satisfying

- i. For all  $a \in A$ ,  $0 \to (0 \to a) = a$ .
- ii. For all  $a, b \in A$ ,  $a \circ (b \to a) = b \circ (a \to b)$ .

Conversely, If  $(A, \leq, \land, \lor, \circ, \rightarrow, 0, 1)$  is an IBH lattice satisfying the following conditions:

- i. For all  $a \in A$ ,  $0 \to (0 \to a) = a$ .
- ii. For all  $a, b \in A$ ,  $a \circ (b \to a) = b \circ (a \to b)$ .

then by Theorem 4.3 we have A is an MV-algebra. Again by using Theorem 1.7.1 of [5], we have any MV algebra is a commutative bounded BCK-algebra. Consequently A is a commutative bounded BCK algebra. Hence the theorem is proved.

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