∗-conformal η−Ricci solitons on ε−para Sasakian manifolds

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Abstract. The object of the present paper is to characterize ε−para Sasakian manifolds admitting ∗−conformal η−Ricci solitons. Finally, the existence of ∗−conformal η−Ricci soliton in an ε−para Sasakian manifold is proved by a concrete example.

1. Introduction

In 1993, Bejancu and Duggal [2] introduced the concept of ε−Sasakian manifolds. Also, Xufeng and Xiaoli [36] showed that every ε−Sasakian manifold must be a real hypersurface of some indefinite Kahler manifold. In 2010, Tripathi et al. [32] initiated the study of ε−almost paracontact manifolds, which is not necessarily Lorentzian. In particular, they studied ε−para Sasakian manifolds, with the structure vector field ξ is spacelike or timelike according as ε = 1 or ε = −1.

The study of Ricci solitons is a very interesting topic in differential geometry and physics. Ricci soliton is a natural generalization of Einstein metric, and is also a self-similar solution to Hamilton's Ricci flow [14, 15]. It plays a specific role in the study of singularities of the Ricci flow. A solution $g(t)$ of the non-linear evolution PDE: $\frac{\partial}{\partial t}g(t) = -2S(g(t))$, $t \in [0, I]$ is called the Ricci flow [23], where S is the Ricci tensor field associated to the metric g. A Riemannian manifold (M, g) is called a *Ricci soliton* (g, V, λ) if there are a smooth vector field V and a scalar λ such that

$$(1.1) \qquad 2S + \mathcal{L}_V g = -2\lambda g$$

on M, where S is the Ricci tensor and $\mathcal{L}_V g$ is the Lie derivative of the metric g along the vector field V. If the potential vector field V vanishes identically, then the Ricci soliton becomes trivial, and in this case the manifold is an Einstein one. The Ricci soliton is said to be shrinking, steady or expanding according to λ being negative, zero or positive, respectively.

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As a generalization of the Ricci soliton, the notion of an η-Ricci soliton was introduced by Cho and Kimura [7] and is given by the equation

$$(1.2) \qquad \mathcal{L}_V g + 2S + 2\lambda g + 2\mu \eta \otimes \eta = 0,$$

where λ and μ are constants, g is a Riemannian metric and S is the Ricci tensor associated with g. Recenty, $\eta-$Ricci solitons have been studied by various authors such as [3, 4, 5, 16, 24, 33] and many others.

In 2004, the concept of a conformal Ricci flow which is a variation of the classical Ricci flow equation was introduced by Fischer [11]. In classical Ricci flow equation the unit volume constraint plays an important role but in conformal Ricci flow equation the scalar curvature r is considered as constraint. The conformal Ricci flow on M is defined by the equations [11]

$$(1.3) \qquad \frac{\partial g}{\partial t} + 2\left(S + \frac{g}{n}\right) = -pg \ \text{ and } \ r = -1,$$

where p is a scalar non-dynamical field (time dependent scalar field), r is the scalar curvature of the manifold and n is the dimension of the manifold. Since these equations are the vector field sum of a conformal flow equation and a Ricci flow equation, they play an important role in conformal geometry. In the Riemannian setting, the notion of a conformal Ricci soliton was introduced by Basu and Bhattacharyya [1] on a Kenmotsu manifold of dimension n as

$$(1.4) \qquad \mathcal{L}_V g + 2S = -\left(2\lambda - \left(p + \frac{2}{n}\right)\right)g,$$

where λ is a constant and $\mathcal{L}_V$ is the Lie derivative along the vector field V. This notion was also studied by several authors on various kinds of almost contact metric manifolds [10, 17, 21]. Further, Siddiqi [30] introduced the notion of conformal η-Ricci soliton as

$$(1.5) \qquad \mathcal{L}_V g + 2S + \left(2\lambda - \left(p + \frac{2}{n}\right)\right)g + 2\mu \eta \otimes \eta = 0,$$

where λ and μ are constants.

In 1959, the notion of a $*-$Ricci tensor on almost Hermitian manifolds was introduced by Tachibana [31] and further studied by Hamada [13] on real hypersurfaces of non-flat complex space forms. A Riemannian metric g on a smooth manifold M is called a $*$-Ricci soliton if there exists a smooth vector field V and a real number λ, such that [18]

$$(1.6) \qquad (\mathcal{L}_V g)(X, Y) + 2S^*(X, Y) + 2\lambda g(X, Y) = 0,$$

where

$$(1.7) \qquad S^*(X, Y) = g(Q^* X, Y) = \text{Trace} \{\phi \circ R(X, \phi Y)\}$$

for all vector fields X, Y on M [6]. Here, ϕ is a tensor field of type $(0, 2)$ and Q^* is the $(1, 1)$ $*-$Ricci operator.

If $S^*(X, Y) = \lambda g(X, Y) + \mu \eta(X) \eta(Y)$ for all vector fields X, Y and λ, μ are smooth functions, then the manifold is called a $*-\eta$-Einstein manifold. Further if $\mu = 0$, that is, $S^*(X, Y) = \lambda g(X, Y)$ for all vector fields X, Y, then the manifold becomes $*$-Einstein. Recently, the $*-$Ricci solitons have been studied by various authors in several ways to a different extent such as [8, 12, 19, 25, 34] and many others. Recently, Roy et al. [27] introduced and studied the notion of a $*-$conformal $\eta-$Ricci soliton on an n-dimensional Sasakian manifold. A Riemannian metric g on M is called $*-$conformal $\eta-$Ricci soliton, if

$$(1.8) \qquad \pounds_\xi g + 2S^* + (2\lambda - (p + \frac{2}{n}))g + 2\mu \eta \otimes \eta = 0,$$

where $\pounds_\xi$ is the Lie derivative along the vector field ξ, S^* is the $*-$Ricci tensor and λ, μ are constants.

2. Preliminaries

An n-dimensional manifold M admits an almost paracontact structure (ϕ, ξ, η), where ϕ is a $(1, 1)$ tensor field, ξ is a structure vector field, η is a 1-form if [29]

$$(2.1) \qquad \phi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0.$$

Let g be a pseudo-Riemannian metric such that

$$(2.2) \qquad g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X) \eta(Y),$$

where $\epsilon = \pm 1$. Then M is called an $\epsilon-$almost paracontact metric manifold equipped with an $\epsilon-$almost paracontact metric structure $(\phi, \xi, \eta, \xi, \epsilon)$ [32]. In particular, if index(g) $= 1$, i.e., g is a Lorentzian metric, then an ϵ-almost paracontact metric manifold is said to be a Lorentzian almost paracontact manifold.
From (2.2) we have

$$(2.3) \qquad g(\phi X, Y) = g(X, \phi Y),$$

$$(2.4) \qquad \eta(X) = \epsilon g(X, \xi)$$

for all vector fields $X, Y \in \chi(M)$; $\chi(M)$ is a set of all smooth vector fields on M. From (2.4) it follows that

$$(2.5) \qquad g(\xi, \xi) = \epsilon,$$

that is, the structure vector field ξ is never lightlike.

Definition 2.1. An $\epsilon-$almost paracontact metric manifold is called $\epsilon-$para Sasakian manifold if [32]

$$(2.6) \qquad (\nabla_X \phi) Y = -g(\phi X, \phi Y)\xi - \epsilon \eta(Y) \phi^2 X,$$

where ∇ denotes the Levi-Civita connection with respect to g.

For $\epsilon = 1$ and g Riemannian, M is the usual para Sasakian manifold [28, 29]. For $\epsilon = -1$, g Lorentzian and ξ replaced by $-\xi$, M becomes a Lorentzian para Sasakian manifold [20].

In an $\epsilon-$para Sasakian manifold, we have [5, 32]

$$(2.7) \qquad \nabla_X \xi = \epsilon \phi X,$$

$$(2.8) \qquad (\nabla_X \eta)Y = \epsilon g(Y, \phi X),$$

$$(2.9) \qquad R(X,Y)\xi = \eta(X)Y - \eta(Y)X,$$

$$(2.10) \qquad R(\xi, X)Y = -\epsilon g(X,Y)\xi + \eta(Y)X,$$

$$(2.11) \qquad \eta(R(X,Y)Z) = \epsilon(g(X,Z)\eta(Y) - g(Y,Z)\eta(X)),$$

$$(2.12) \qquad S(X,\xi) = -(n-1)\eta(X),$$

$$(2.13) \qquad Q\xi = -\epsilon(n-1)\xi$$

for all $X, Y, Z \in \chi(M)$; where ∇, R, S and Q denotes the Levi-Civita connection, the curvature tensor, the Ricci tensor and the Ricci operator, respectively, on M.

Definition 2.2. An $\epsilon-$para Sasakian manifold M is said to be a generalized $\eta-$Einstein manifold if the following condition [37]

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) + cg(\phi X, Y)$$

holds on M, where a, b and c are smooth functions on M. If $c = 0$, $b = c = 0$, and $a = c = 0$, then the manifold is called an η-Einstein, an Einstein and a special type of $\eta-$Einstein manifold, respectively.

Lemma 2.3. *In an n-dimensional $\epsilon-$para Sasakian manifold, we have [8]*

$$
\begin{aligned}
(2.14) \quad \bar{R}&(X, Y, \phi Z, \phi W) \\
= \quad & \bar{R}(X, Y, Z, W) + 2g(Y,Z)\eta(X)\eta(W) \\
& -2g(X,Z)\eta(Y)\eta(W) - \epsilon\{g(Y,Z)g(X,W) \\
& -g(X,Z)g(Y,W) - g(X,\phi W)g(Y,\phi Z) \\
& +g(Y,\phi W)g(X,\phi Z)\} + 2g(X,W)\eta(Y)\eta(Z) \\
& -2g(Y,W)\eta(X)\eta(Z)
\end{aligned}
$$

for any X, Y, Z, W on M, where $\bar{R}(X, Y, Z, W) = g(R(X,Y)Z, W)$.

Lemma 2.4. *In an n-dimensional ε−para Sasakian manifold the ∗−Ricci tensor is given by*

(2.15) $S^*(Y,Z) = S(Y,Z) - \epsilon(n-2)g(Y,Z) - \epsilon\psi g(Y,\phi Z) + (2n-3)\eta(Y)\eta(Z),$

where S and S^* are the Ricci tensor and the ∗−Ricci tensor of type $(0,2)$, respectively, and $\psi = trace\ \phi$.

Proof. Let $\{e_i\}, i = 1,2,3..........n$ be an orthonormal basis of the tangent space at each point of the manifold. Therefore from the equations (2.14) and (1.7), we have

$$
\begin{aligned}
S^*(Y,Z) &= \sum_{i=1}^{n} \bar{R}(e_i, Y, \phi Z, \phi e_i) \\
&= \sum_{i=1}^{n} [\bar{R}(e_i, Y, Z, e_i) + 2g(Y,Z)\eta(e_i)\eta(e_i) \\
&\quad -2g(e_i,Z)\eta(Y)\eta(e_i) - \epsilon\{g(Y,Z)g(e_i,e_i) \\
&\quad -g(e_i,Z)g(Y,e_i) - g(e_i,\phi e_i)g(Y,\phi Z) \\
&\quad +g(Y,\phi e_i)g(e_i,\phi Z)\} + 2g(e_i,e_i)\eta(Y)\eta(Z) \\
&\quad -2g(Y,e_i)\eta(e_i)\eta(Z)]. \\
&= S(Y,Z) - \epsilon(n-2)g(Y,Z) - \epsilon\psi g(Y,\phi Z) + (2n-3)\eta(Y)\eta(Z).
\end{aligned}
$$

This completes the proof. $\square$

3. ∗−conformal η−Ricci solitons on ε−para Sasakian manifolds

Suppose that an $n-$dimensional ε−para Sasakian manifold admits ∗−conformal η−Ricci solitons. Then (1.8) holds and thus we have

(3.1) $(£_\xi g)(Y,Z) + 2S^*(Y,Z) + [2\lambda - (p+\frac{2}{n})]g(Y,Z) + 2\mu\eta(Y)\eta(Z) = 0.$

In an ε−para Sasakian manifold, we have

(3.2) $(£_\xi g)(Y,Z) = g(\nabla_Y\xi, Z) + g(Y, \nabla_Z\xi) = 2\epsilon g(\phi Y, Z).$

Therefore, from (3.1) and (3.2), we find

(3.3) $S^*(Y,Z) = -[\lambda - \frac{1}{2}(p+\frac{2}{n})]g(Y,Z) - \mu\eta(Y)\eta(Z) - \epsilon g(\phi Y, Z).$

By considering (3.3), (2.15) takes the form

(3.4) $S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z) + cg(\phi Y, Z),$

where $a = \epsilon(n-2) - \lambda + \frac{1}{2}(p + \frac{2}{n})$, $b = -(2n - 3 + \mu)$ and $c = \epsilon(\psi - 1)$.
Taking $Z = \xi$ in (3.4), we find

$$(3.5) \qquad\qquad S(Y, \xi) = (a\epsilon + b)\eta(Y).$$

It yields

$$(3.6) \qquad\qquad Q\xi = (a + b\epsilon)\xi.$$

Thus from (2.12) and (3.5), we obtain

$$(3.7) \qquad\qquad \lambda + \epsilon\mu = \frac{1}{2}(p + \frac{2}{n}).$$

Thus we have the following:

Theorem 3.1. *Let M be an n-dimensional $\epsilon-$para Sasakian manifold. If the manifold admits a $*-$conformal $\eta-$Ricci soliton, then M is a generalized η-Einstein manifold of the form (3.4), and the scalars λ and μ are related by $\lambda + \epsilon\mu = \frac{1}{2}(p + \frac{2}{n})$.*

Now, let (g, V, λ, μ) be a $*-$conformal $\eta-$Ricci soliton on an $\epsilon-$para Sasakian manifold such that V is pointwise collinear with ξ, i.e., $V = h\xi$, where h is a function. Then (1.8) holds and thus we have

$$(3.8) \qquad \mathcal{L}_{h\xi}g + 2S^* + (2\lambda - (p + \frac{2}{n}))g + 2\mu\eta \otimes \eta = 0.$$

Applying the property of the Lie derivative and Levi-Civita connection in (3.8), we have

$$hg(\nabla_X\xi, Y) + \epsilon(Xh)\eta(Y) + hg(X, \nabla_Y\xi) + \epsilon(Yh)\eta(X)$$

$$+2S^*(X, Y) + (2\lambda - (p + \frac{2}{n}))g(X, Y) + 2\mu\eta(X)\eta(Y) = 0$$

which by using (2.7) and (2.15) takes the form

$$(3.9) \qquad 2h\epsilon g(\phi X, Y) + \epsilon(Xh)\eta(Y) + \epsilon(Yh)\eta(X)$$

$$+2S(X, Y) + [2\lambda - 2\epsilon(n-2) - (p + \frac{2}{n})]g(X, Y)$$

$$-2\epsilon\psi g(X, \phi Y) + (4n + 2\mu - 6)\eta(X)\eta(Y) = 0.$$

Putting $Y = \xi$ and using (2.1) and (2.12), (3.9) reduces to

$$(3.10) \qquad (Xh) + [(\xi h) + 2\lambda - (p + \frac{2}{n}) + 2\epsilon\mu]\eta(X) = 0.$$

Again putting $X = \xi$ in (3.10) and using (2.1), we get

$$(3.11) \qquad (\xi h) = -[\lambda - \frac{1}{2}(p + \frac{2}{n}) + 2\epsilon\mu], \qquad \epsilon \neq 0.$$

Combining the equations (3.10) and (3.11), we find

$$(3.12) \qquad (Xh) = -[\lambda + \epsilon\mu - \frac{1}{2}(p + \frac{2}{n})]\eta(X).$$

It is clear that h is constant if $\lambda + \epsilon\mu = \frac{1}{2}(p + \frac{2}{n})$. Thereore from (3.9) we obtain

$$S(X,Y) = -[\lambda - \epsilon(n-2) - \frac{1}{2}(p + \frac{2}{n})]g(X,Y) + \epsilon(\psi - h)g(\phi X, Y)$$

$$-(2n + \mu - 3)\eta(X)\eta(Y).$$

Thus we have the following theorem:

Theorem 3.2. *If (g, V, λ, μ) is a *−conformal η−Ricci soliton on an ε−para Sasakian manifold such that V is pointwise collinear with ξ, then V is a constant multiple of ξ and the manifold is a generalized η−Einstein manifold, provided $\lambda + \epsilon\mu = \frac{1}{2}(p + \frac{2}{n})$.*

4.   Ricci recurrent ε−para Sasakian manifolds admitting *−conformal η−Ricci solitons

Definition 4.1. An n-dimensional ε−para Sasakian manifold is said to be Ricci recurrent if there exists a non-zero 1-form ω such that [22]

$$(4.1) \qquad (\nabla_X S)(Z, W) = \omega(X)S(Z, W)$$

for all X, Z and W on M.

Assume that M is a Ricci recurrent ε−para Sasakian manifold admitting *−conformal η−Ricci solitons. Therefore, the Ricci tensor of the manifold satisfies

$$(4.2) \qquad (\nabla_X S)(Z, W) = \omega(X)S(Z, W).$$

This implies that

$$(4.3) \qquad \nabla_X S(Z, W) - S(\nabla_X Z, W) - S(Z, \nabla_X W) = \omega(X)S(Z, W).$$

Taking the covariant derivative of (3.4) with respect to X and using (2.6), (2.8), we have

$$(4.4) \qquad \nabla_X S(Z, W) = a[g(\nabla_X Z, W) + g(Z, \nabla_X W)] + b[-\epsilon g(\phi X, \phi Z)\eta(W)$$

$$-\epsilon\eta(Z)g(\phi^2 X, W) + g(\phi\nabla_X Z, W) + g(\phi Z, \nabla_X W)] + c[\epsilon g(Z, \phi X)\eta(W)$$

$$+\eta(\nabla_X Z)\eta(W) + \epsilon\eta(Z)g(W, \phi X) + \eta(Z)\eta(\nabla_X W)].$$

By using (3.4) and (4.4), (4.3) gives

$$-\epsilon b[g(\phi X, \phi Z)\eta(W) + g(\phi^2 X, W)\eta(Z)]$$

$$+\epsilon c[g(Z,\phi X)\eta(W) + g(\phi X, W)\eta(Z)] = \omega(X)S(Z,W)$$

which by putting $W = \xi$ then using (2.1) and (2.12) yields

(4.5) $-b\epsilon g(X,Z) + b\eta(X)\eta(Z) + c\epsilon g(Z,\phi X) = -(n-1)\omega(X)\eta(Z).$

Suppose the associated 1-form ω is equal to the associated 1-form η, then from (4.5) we have

(4.6) $-b\epsilon g(X,Z) + (n-1+b)\eta(X)\eta(Z) + c\epsilon g(Z,\phi X) = 0.$

Now replacing Z by ϕZ in (4.6), we get

(4.7) $-bg(X,\phi Z) + cg(\phi X, \phi Z) = 0, \quad \epsilon \neq 0.$

Replacing X by ϕX and using (2.1), (4.7) becomes

(4.8) $-bg(\phi X, \phi Z) + cg(X, \phi Z) = 0.$

By adding (4.7) and (4.8), we obtain $c = b$ which gives $\mu = -2n + 3 + \epsilon(1-\psi)$ and hence from (3.7) we get $\lambda = \epsilon(2n-3) - 1 + \psi + \frac{1}{2}(p + \frac{2}{n})$. Thus we can state the following:

Theorem 4.2. *If (g, ξ, λ, μ) is a $*-$conformal $\eta-$Ricci soliton in an n-dimensional Ricci recurrent $\epsilon-$para Sasakian manifold, then $\lambda = \epsilon(2n-3) - 1 + \psi + \frac{1}{2}(p + \frac{2}{n})$ and $\mu = -2n + 3 + \epsilon(1-\psi)$.*

Now by using the values of λ and μ in (3.4), we obtain

$$S(X,Z) = (-n\epsilon + \epsilon + 1 - \psi)g(X,Z) + \epsilon(\psi - 1)g(\phi X, Z) - \epsilon(1-\psi)\eta(X)\eta(Z).$$

Thus we have

Corollary 4.3. *An n-dimensional Ricci recurrent $\epsilon-$para Sasakian manifold admitting a $*-$conformal $\eta-$Ricci soliton (g, ξ, λ, μ) is a generalized $\eta-$Einstein manifold.*

5. $\eta-$parallel $\phi-$tensor in $\epsilon-$para Sasakian manifolds admitting $*-$conformal $\eta-$Ricci solitons

In this section we study the $\eta-$parallel $\phi-$tensor in an $\epsilon-$para Sasakian manifold admitting a $*-$conformal $\eta-$Ricci soliton. If the $(1,1)$ tensor ϕ is $\eta-$parallel, then we have [6]

(5.1) $g((\nabla_X \phi)Y, Z) = 0$

for all $X, Y, Z \in \chi(M)$. From (2.6) and (5.1), we obtain

(5.2) $g(X,Y)\eta(Z) + \eta(Y)g(X,Z) - 2\epsilon\eta(X)\eta(Y)\eta(Z) = 0.$

Putting $Z = \xi$ in (5.2), we find

$$g(X, Y) = \epsilon\eta(X)\eta(Y)$$

which by replacing X by QX and using (3.5) yields

(5.3) $$S(X, Y) = (1 - n - \epsilon\lambda - \mu + \frac{\epsilon}{2}(p + \frac{2}{n})\eta(X)\eta(Y).$$

By using (3.7) in (5.3), it follows that

$$S(X, Y) = (1 - n)\eta(X)\eta(Y).$$

Thus we have the following theorem:

Theorem 5.1. *If (g, ξ, λ, μ) is a ∗−conformal η−Ricci soliton in an n-dimensional ε−para Sasakian manifold and if the tensor ϕ is η−parallel, then the manifold is a special type of η−Einstein manifold.*

6. Conformal curvature tensor on ε−para Sasakian manifolds admitting ∗−conformal η−Ricci solitons

If the Riemannian metric g on a manifold M is conformally related with a flat Euclidean metric, then g is called conformally flat. A Riemannian manifold equipped with a conformally flat Riemannian metric is named a conformally flat manifold. By using a conformal transformation, Weyl [35] introduced a generalized curvature tensor which vanishes whenever the metric is conformally flat. Due to this reason it is called the confomal curvature tensor. It is well-known that a Riemannian manifold M of dimension n is conformally flat if and only if the Weyl conformal curvature tensor field C vanishes for $n > 3$. The conformal curvature tensor C in an $n−$dimensional ε−para Sasakian manifold is defined by

(6.1) $$C(X, Y)Z = R(X, Y)Z - \frac{1}{(n - 2)}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX$$

$$-g(X, Z)QY] + \frac{r}{(n - 1)(n - 2)}[g(Y, Z)X - g(X, Z)Y],$$

where $R(X, Y)Z$, $S(X, Y)$ and r are the curvature tensor, the Ricci tensor and the scalar curvature of M, respectively, and Q is the Ricci operator defined as $S(X, Y) = g(QX, Y)$.

First, let us consider an $n−$dimensional ε−para Sasakian manifold admitting ∗−conformal η−Ricci soliton, which is conformally flat, i.e., $C(X, Y)Z = 0$. Then, from (6.1) it follows that

$$R(X, Y)Z = \frac{1}{(n - 2)}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]$$

(6.2) $$-\frac{r}{(n - 1)(n - 2)}[g(Y, Z)X - g(X, Z)Y].$$

Taking the inner product of (6.2) with ξ, we have

$$g(X,Z)\eta(Y) - g(Y,Z)\eta(X)$$
$$= \frac{\epsilon}{(n-2)}[S(Y,Z)\eta(X) - S(X,Z)\eta(Y) + g(Y,Z)\eta(QX) - g(X,Z)\eta(QY)]$$
$$- \frac{\epsilon r}{(n-1)(n-2)}[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]$$

which by putting $X = \xi$ and using (2.1), (2.4), (3.5) and (3.6) leads to

$$(6.3) \qquad S(Y,Z) = [\frac{r}{(n-1)} - (a+b\epsilon) - \epsilon(n-2)]g(Y,Z)$$

$$- [\frac{\epsilon r}{(n-1)} - 2(a\epsilon+b) - (n-2)]\eta(Y)\eta(Z).$$

By using the values of $a + b\epsilon = -\epsilon(n-1)$, $r = -1$ and the equation (3.7), (6.3) turns to

$$S(Y,Z) = [\epsilon - \frac{1}{(n-1)}]g(Y,Z) - [n - \frac{\epsilon}{(n-1)}]\eta(Y)\eta(Z).$$

Thus we have the following:

Theorem 6.1. *Let M be an $n-$dimensional $\epsilon-$para Sasakian manifold admitting a $*-$conformal $\eta-$Ricci soliton. If the manifold M is conformally flat, then it is an $\eta-$Einstein manifold.*

Next, we consider an $n-$dimensional $\epsilon-$para Sasakian manifold admitting a $*-$conformal $\eta-$Ricci soliton, which is $\phi-$conformally semisymmetric [9, 26], i.e., $C \cdot \phi = 0$. This implies that

$$(6.4) \qquad (C(X,Y) \cdot \phi)Z = C(X,Y)\phi Z - \phi C(X,Y)Z = 0.$$

By using (6.1), (6.4) takes the form
(6.5)
$$R(X,Y)\phi Z - \phi R(X,Y)Z + \frac{1}{(n-2)}[S(Y,Z)\phi X - S(X,Z)\phi Y - S(Y,\phi Z)X$$

$$+ S(X,\phi Z)Y + g(Y,Z)\phi QX - g(X,Z)\phi QY - g(Y,\phi Z)QX + g(X,\phi Z)QY]$$

$$+ \frac{r}{(n-1)(n-2)}[g(Y,\phi Z)X - g(X,\phi Z)Y - g(Y,Z)\phi X + g(X,Z)\phi Y] = 0.$$

Putting $Y = \xi$ in (6.5) and using (2.1), (2.4), (2.10), (3.5), (3.6), we find

$$[1 - \frac{\epsilon r}{(n-1)(n-2)}](\epsilon g(X,\phi Z)\xi + \eta(Z)\phi X) + \frac{1}{n-2}[S(X,\phi Z)\xi$$

$$+ (a+b\epsilon)(g(X,\phi Z)\xi + \epsilon\eta(Z)\phi X) + \epsilon\eta(Z)\phi QX] = 0$$

which by taking the inner product with ξ gives

$$(6.6) \qquad S(X,\phi Z) = -\epsilon[n - 2 - \frac{\epsilon r}{(n-1)} + (a\epsilon+b)]g(X,\phi Z).$$

Replacing Z by ϕZ in (6.6), then using (2.1) and (3.5), we obtain

$$(6.7) \quad S(X,Z) \;=\; -\epsilon[n-2-\frac{\epsilon r}{(n-1)}+(a\epsilon+b)]g(X,Z)$$
$$+[n-2-\frac{\epsilon r}{(n-1)}+2(a\epsilon+b)]\eta(X)\eta(Z).$$

By using the values of $a+b\epsilon = -\epsilon(n-1)$, $r=-1$ and the equation (3.7), (6.7) turns to

$$S(X,Z) = [\epsilon - \frac{1}{(n-1)}]g(X,Z) - [n - \frac{\epsilon}{(n-1)}]\eta(X)\eta(Z).$$

Thus we have the following:

Theorem 6.2. *Let M be an $n−$dimensional $\epsilon−$para Sasakian manifold admitting a $*−$conformal $\eta−$Ricci soliton. If the manifold M is $\phi−$conformally semisymmetric, then it is an $\eta−$Einstein manifold.*

Example. We consider the 5-dimensional manifold $M = \{(y,z,u,v,w) \in \mathbb{R}^5\}$, where (y,z,u,v,w) are the standard coordinates in $\mathbb{R}^5$. Let e_1, e_2, e_3, e_4 and e_5 be the vector fields on M given by

$$e_1 = \epsilon e^{-w}\frac{\partial}{\partial y}, \;\; e_2 = \epsilon e^{-w}\frac{\partial}{\partial z}, \;\;\; e_3 = \epsilon e^{-w}\frac{\partial}{\partial u}, \;\; e_4 = \epsilon e^{-w}\frac{\partial}{\partial v}, \;\;\; e_5 = \epsilon\frac{\partial}{\partial w} = \xi.$$

Let g be the indefinite Riemannian metric defined by

$$g(e_i, e_j) = 0, \;\; i \neq j, \;\; i,j = 1,2,3,4,5$$

and

$$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = g(e_4,e_4) = 1, \;\; g(e_5,e_5) = \epsilon.$$

Let η be the 1-form on M defined by $\eta(Z) = \epsilon g(Z,e_5) = \epsilon g(Z,\xi)$ for all $Z \in \chi(M)$. Let ϕ be the $(1,1)$ tensor field on M defined by

$$\phi e_1 = e_1, \;\; \phi e_2 = e_2, \;\; \phi e_3 = e_2, \;\; \phi e_4 = e_4, \;\; \phi e_5 = 0.$$

The linearity of ϕ and g yields

$$\eta(e_5) = 1, \;\; \phi^2 Z = Z - \eta(X)\xi, \;\; g(\phi Z, \phi U) = g(Z,U) - \epsilon\eta(Z)\eta(U)$$

for all $Z, U \in \chi(M)$. Thus for $e_5 = \xi$, the structure $(\phi, \xi, \eta, g, \epsilon)$ defines an indefinite almost contact metric structure on M. Then we have

$$[e_1,e_2] = [e_1,e_3] = [e_1,e_4] = [e_2,e_3] = [e_2,e_4] = [e_3,e_4] = 0,$$

$$[e_1,e_5] = \epsilon e_1, [e_2,e_5] = \epsilon e_2, [e_3,e_5] = \epsilon e_3, [e_4,e_5] = \epsilon e_4.$$

The Riemannian connection ∇ of the metric g is given by

$$2g(\nabla_Y Z, U) = Yg(Z,U) + Zg(U,Y) - Ug(Y,Z)$$
$$- g(Y,[Z,U]) + g(Z,[U,Y]) + g(U,[Y,Z]),$$

which is known as Koszul's formula. Using Koszul's formula, we can easily calculate

$$\nabla_{e_1}e_1 = -e_5, \quad \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_3 = 0, \quad \nabla_{e_1}e_4 = 0, \quad \nabla_{e_1}e_5 = \epsilon e_1,$$

$$\nabla_{e_2}e_1 = 0, \quad \nabla_{e_2}e_2 = -e_5, \quad \nabla_{e_2}e_3 = 0, \quad \nabla_{e_2}e_4 = 0, \quad \nabla_{e_2}e_5 = \epsilon e_2,$$

$$\nabla_{e_3}e_1 = 0, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_3 = -e_5, \quad \nabla_{e_3}e_4 = 0, \quad \nabla_{e_3}e_5 = \epsilon e_3,$$

$$\nabla_{e_4}e_1 = 0, \quad \nabla_{e_4}e_2 = 0, \quad \nabla_{e_4}e_3 = 0, \quad \nabla_{e_4}e_4 = -e_5, \quad \nabla_{e_4}e_5 = \epsilon e_4,$$

$$\nabla_{e_5}e_1 = 0, \quad \nabla_{e_5}e_2 = 0, \quad \nabla_{e_5}e_3 = 0, \quad \nabla_{e_5}e_4 = 0, \quad \nabla_{e_5}e_5 = 0.$$

It follows that $\nabla_Z\xi = \epsilon\phi Z$. Hence the manifold is an $\epsilon-$para Sasakian manifold. It is known that

$$R(Y, Z)U = \nabla_Y\nabla_Z U - \nabla_Z\nabla_Y U - \nabla_{[Y,Z]}U.$$

By the above results, we can easily obtain the non-vanishing components of the curvature tensors as follows:

$$R(e_1, e_2)e_2 = R(e_1, e_3)e_3 = R(e_1, e_4)e_4 = -\epsilon e_1, \quad R(e_1, e_5)e_5 = -e_1,$$

$$R(e_1, e_2)e_1 = R(e_3, e_2)e_3 = R(e_4, e_2)e_4 = \epsilon e_2, \quad R(e_2, e_5)e_5 = -e_2,$$

$$R(e_1, e_3)e_1 = R(e_4, e_3)e_4 = R(e_2, e_3)e_2 = \epsilon e_3, \quad R(e_3, e_5)e_5 = -e_3,$$

$$R(e_1, e_5)e_1 = R(e_2, e_5)e_2 = R(e_3, e_5)e_3 = R(e_4, e_5)e_4 = \epsilon e_5,$$

$$R(e_1, e_4)e_1 = R(e_2, e_4)e_2 = R(e_3, e_4)e_3 = \epsilon e_4, \quad R(e_5, e_4)e_5 = e_4.$$

With the help of the above results we get the components of the Ricci tensor as follows:

$$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = S(e_4, e_4) = -4\epsilon, \quad S(e_5, e_5) = -4.$$

From (3.4), we have $S(e_5, e_5) = -4 - \epsilon\lambda - \mu + \frac{\epsilon}{2}(p + \frac{2}{5})$. By equating both the values of $S(e_5, e_5)$, we obtain

$$\lambda + \epsilon\mu = \frac{1}{2}(p + \frac{2}{5}).$$

Hence λ and μ satisfies the equation (3.7) and so g defines a $*-$conformal $\eta-$Ricci soliton on the 5-dimensional $\epsilon-$para Sasakian manifold.

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