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α-almost Ricci solitons on $(k, \mu)'$-almost Kenmotsu manifolds

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Abstract. We consider α-almost Ricci solitons on $(k, \mu)'$-almost Kenmotsu manifolds with an η-parallel Ricci tensor. Then we study α-almost Ricci solitons on $(k, \mu)'$-almost Kenmotsu manifolds satisfying the curvature conditions $P.\phi = 0$, $Q.P = 0$ and $Q.R = 0$ respectively. Finally, we construct an example of a 3-dimensional $(k, \mu)'$-almost Kenmotsu manifold which admits an α-almost Ricci soliton.

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1. Introduction

In 1982, the notion of Ricci flow was introduced by Hamilton [16] to find the canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold M defined as follows:

$$\frac{\partial}{\partial t} g = -2S, \tag{1.1}$$

where S denotes the Ricci tensor and g is a metric tensor. Ricci solitons are special solutions of the Ricci flow equation (1.1) of the form $g = \sigma(t) f_t^* g$ with the initial condition $g(0) = g$, where f_t, $t \in \{\mathbb{R}\}$ is a family of diffeomorphisms on M and $\sigma(t)$ is the scaling function. A Ricci soliton is a generalization of an Einstein metric. We recall the notion of Ricci solitons according to [6]. On the manifold M, a Ricci soliton is a triplet (g, V, λ) with g a Riemannian metric, V a vector field (called the soliton vector field) and λ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.2}$$

where $\pounds$ is the Lie derivative. Metrics satisfying (1.2) are interesting and useful in physics and are often referred to as quasi-Einstein metrics([5],[4]).

The Ricci soliton is said to be shrinking, steady or expanding whenever λ is negative, zero or positive, respectively. Ricci solitons have been studied by

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several authors such as ([10], [9], [8], [17],[16],[22],[24],[23],[25],[30]) and many others.

Ricci solitons have several generalizations, such as almost Ricci solitons ([12],[13],[15]), η-Ricci solitons ([1],[2],[7]), generalized Ricci solitons and many others.

As a generalization of Ricci soliton, Pigola et. al [21] introduced the notion of an almost Ricci soliton by considering the constant λ as a smooth function. Recently, Gomes et. al [15] extended the notion of an almost Ricci soliton to α-almost Ricci soliton (briefly, α-ARS) on a complete Riemannian manifold by

$$\frac{\alpha}{2}\pounds_V g + S + \lambda g = 0, \tag{1.3}$$

where $\alpha : M \to \mathbb{R}$ is a smooth function. In particular, a Ricci soliton is the 1-ARS with constant λ. In [14], Ghosh and Patra studied α-ARS on K-contact metric manifolds.

The projective curvature tensor P [29] in a manifold (M^{2n+1}, g) is defined by

$$P(U,V)W = R(U,V)W - \frac{1}{2n}[g(V,W)QU - g(U,W)QV], \tag{1.4}$$

where Q is the Ricci tensor operator defined by $S(U,V) = g(QU,V)$ and $U,V,W \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields of M.

The paper is organized as follows:

After preliminaries in Section 2, we consider α-almost Ricci solitons on $(k,\mu)'$-almost Kenmotsu manifolds with an η-parallel Ricci tensor in Section 3. Next, in Section 4 we study α-almost Ricci solitons on $(k,\mu)'$-almost Kenmotsu manifolds satisfying the curvature condition $P.\phi = 0$. Section 5 is devoted to the study off α-almost Ricci solitons on $(k,\mu)'$-almost Kenmotsu manifolds satisfying the curvature condition $Q.P = 0$. In Section 6, we investigate α-almost Ricci solitons on $(k,\mu)'$-almost Kenmotsu manifolds satisfying the curvature condition $Q.R = 0$. Finally, in Section 7 we construct an example of a 3-dimensional $(k,\mu)'$-almost Kenmotsu manifold admitting an α-almost Ricci soliton.

2. Preliminaries

Let (M^{2n+1}, g) be a smooth Riemannian manifold of dimension $2n+1$. On this manifold if there exist a (1,1)-type tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\phi^2 U = -U + \eta(U)\xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi U, \phi V) = g(U,V) - \eta(U)\eta(V) \tag{2.2}$$

for any vector fields U, V, then (ϕ, ξ, η, g) is called an almost contact metric structure and M^{2n+1} is called an almost contact metric manifold (see [3]).

Usually ξ and η are called the Reeb or characteristic vector field and an almost contact 1-form respectively.

The fundamental 2-form Φ on an almost contact metric manifold M^{2n+1} is defined by $\Phi(U,V)=g(U,\phi V)$ for any vector fields U and V.

Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost contact manifold. We define on the product $M^{2n+1}\times\mathbb{R}$ an almost complex structure J by

$$J(U,f\tfrac{d}{dt})=(\phi U-f\xi,\eta(U)\tfrac{d}{dt}),$$

where U denotes a vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$ and f is a C^∞-function on $M^{2n+1}\times\mathbb{R}$. We denote by $[\phi,\phi]$ the Nijenhuis tensor of ϕ (see [3]), if $[\phi,\phi]=-2d\eta\otimes\xi$ (or equivalently, the almost complex structure J is integrable), then the almost contact metric structure is said to be normal.

On an almost contact metric manifold if there hold $d\eta=0$ and $d\Phi=2\eta\wedge\Phi$, then the manifold is said to be an almost Kenmotsu manifold (see [26]). A normal almost Kenmotsu manifold is said to be a Kenmotsu manifold (see [18]) and this is also equivalent to

$$(\nabla_U\phi)V=g(\phi U,V)\xi-\eta(V)\phi U \tag{2.3}$$

for any vector fields U,V, where ∇ denotes the Levi-Civita connection of the metric g. On an almost Kenmotsu manifold, we set $h=\frac{1}{2}\pounds_\xi\phi$ and $h'=h\circ\phi$, where $\pounds$ is the Lie derivative. It is easily seen that the above two operators are both symmetric. The following formulas can be seen in ([12],[11])

$$h\xi=h'\xi=0,\quad tr(h)=tr(h')=0,\quad h\phi+\phi h=0, \tag{2.4}$$

$$\nabla_U\xi=U-\eta(U)\xi+h'U. \tag{2.5}$$

According to Pastore and Saltarelli [20], on an almost Kenmotsu manifold $(M^{2n+1},\phi,\xi,\eta,g)$, if ξ belongs to the generalized $(k,\mu)'$-nullity distribution, that is,

$$R(U,V)\xi=k(\eta(V)U-\eta(U)V)+\mu(\eta(V)h'U-\eta(U)h'V) \tag{2.6}$$

holds for certain smooth functions k and μ and any vector field $U,V\in\chi(M)$, then we say that M^{2n+1} is a generalized $(k,\mu)'$-almost Kenmotsu manifold. In particular, if both k and μ in relation (2.6) are constants then M^{2n+1} is called a $(k,\mu)'$-almost Kenmotsu manifold [12]. If on M^{2n+1} there holds

$$R(U,V)\xi=k(\eta(V)U-\eta(U)V)+\mu(\eta(V)hU-\eta(U)hV)$$

for any vector fields U,V and $k,\mu\in\mathbb{R}$, then M^{2n+1} is said to be a (k,μ)-almost Kenmotsu manifold. Dileo and Pastore in [12] proved that a (k,μ)-almost Kenmotsu manifold satisfies $k=-1$ and $h=0$. Therefore, we regard (k,μ)-almost Kenmotsu manifolds as a special case of $(k,\mu)'$-almost Kenmotsu manifolds. Following Dileo and Pastore [12], on any $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} we have

$$h'^2U=-(k+1)U+(k+1)\eta(U)\xi \tag{2.7}$$

for any vector field $U \in \chi(M)$ and $\mu = -2$. From (2.7) we know that $h' = 0$ identically if and only if $k = -1$ and $h' \neq 0$ everywhere if and only if $k < -1$. It follows from (2.6) that

$$R(\xi, U)V = k(g(U, V)\xi - \eta(V)U) - 2(g(h'U, V)\xi - \eta(V)h'U) \tag{2.8}$$

for any vector fields U, V.

Definition 2.1. A $(k, \mu)'$-almost Kenmotsu manifold M is said to be an η-parallel Ricci tensor [27] if

$$g((\nabla_U Q)V, W) = 0$$

for arbitrary vector fields U, V, W.

Proposition 2.1([28]). On a $(k, \mu)'$-almost Kenmotsu manifold with $k < -1$ the Ricci operator is given by

$$QU = -2nU + 2n(k + 1)\eta(U)\xi - 2nh'U, \tag{2.9}$$

where the Ricci operator is defined by $S(U, V) = g(QU, V)$.

Proposition 2.2. For an α-almost Ricci soliton on a $(k, \mu)'$-almost Kenmotsu manifold we have $\lambda = -2nk$.

Proof: Let $(g, \xi, \alpha, \lambda)$ be an α-ARS on a $(k, \mu)'$-almost Kenmotsu manifold M^{2n+1}. Then we have

$$\frac{\alpha}{2}(\pounds_\xi g)(U, V) + S(U, V) + \lambda g(U, V) = 0. \tag{2.10}$$

Now

$$(\pounds_\xi g)(U, V) = g(\nabla_U \xi, V) + g(U, \nabla_V \xi). \tag{2.11}$$

Using (2.5) in the above equation, we get

$$(\pounds_\xi g)(U, V) = 2[g(U, V) - \eta(U)\eta(V) + g(h'U, V)]. \tag{2.12}$$

Substituting the value of $(\pounds_\xi g)(U, V)$ from (2.12) in (2.10), we have

$$S(U, V) = -(\alpha + \lambda)g(U, V) + \alpha\eta(U)\eta(V) - \alpha g(h'U, V). \tag{2.13}$$

Now replacing U and V by ξ in the above equation and using (2.9), we get

$$\lambda = -2nk. \tag{2.14}$$

This completes the proof.

Again putting $W = \xi$ in (1.4), we get

$$P(U, V)\xi = R(U, V)\xi - \frac{1}{2n}[\eta(V)QU - \eta(U)QV]. \tag{2.15}$$

Using (2.6) and (2.13) in the above equation, we get

$$\begin{aligned} P(U, V)\xi &= (k + \frac{\alpha + \lambda}{2n})(\eta(V)U - \eta(U)V) \\ &\quad +(\mu + \frac{\alpha}{2n})(\eta(V)h'U - \eta(U)h'V). \end{aligned} \tag{2.16}$$

3. α-ARS on $(k,\mu)'$-almost Kenmotsu manifolds with η-parallel Ricci tensor

In this section we consider α-ARS on $(k,\mu)'$-almost Kenmotsu manifolds with η-parallel Ricci tensor. Then from Definition 2.1., we get

$$g((\nabla_U Q)V, W) = 0. \tag{3.1}$$

From (2.13) we have

$$QV = -(\alpha+\lambda)V + \alpha\eta(V)\xi - \alpha h'V. \tag{3.2}$$

Taking covariant differentiation of (3.2) with respect to U, we obtain

$$\begin{aligned}(\nabla_U Q)V &= -(U\alpha)V - (U\lambda)V + (U\alpha)\eta(V)\xi - (U\alpha)h'V \\ &\quad +\alpha[((\nabla_U\eta)V)\xi + \eta(V)\nabla_U\xi] - \alpha\nabla_U h'V + \alpha h'\nabla_U V.\end{aligned} \tag{3.3}$$

Using (2.5) in the foregoing equation, we obtain

$$\begin{aligned}(\nabla_U Q)V &= -(U\alpha)V - (U\lambda)V + (U\alpha)\eta(V)\xi - (U\alpha)h'V \\ &\quad +\alpha[g(U,V)\xi - 2\eta(U)\eta(V)\xi + g(h'U,V)\xi + \eta(V)U + \eta(V)h'U] \\ &\quad -\alpha\nabla_U h'V + \alpha h'\nabla_U V.\end{aligned} \tag{3.4}$$

Using the above value of $(\nabla_U Q)V$ from (3.4) in (3.1), we get

$$\begin{aligned}&-(U\alpha)g(V,W) - (U\lambda)g(V,W) + (U\alpha)\eta(V)\eta(W) - (U\alpha)g(h'V,W) \\ &+\alpha[g(U,V)\eta(W) - 2\eta(U)\eta(V)\eta(W) + g(h'U,V)\eta(W) + g(U,W)\eta(V) \\ &+g(h'U,W)\eta(V)] - \alpha g(\nabla_U h'V, W) + \alpha g(h'\nabla_U V, W) = 0.\end{aligned} \tag{3.5}$$

Putting $V=\xi$ in the above equation and using (2.1), (2.5) and (2.7), we obtain

$$(U\lambda)\eta(W) + \alpha[k\{g(U,W) - \eta(U)\eta(W)\} - 2g(h'U,W)] = 0. \tag{3.6}$$

Putting $W=\xi$ in the above equation, we get

$$(U\lambda) = 0, \tag{3.7}$$

which implies $\lambda =$ constant.

Using (3.7) in (3.6), we obtain

$$\alpha[k\{g(U,W) - \eta(U)\eta(W)\} - 2g(h'U,W)] = 0. \tag{3.8}$$

Putting $U = W = e_i$ in (3.8), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i(1 \leq i \leq 2n+1)$, we get

$$2nk\alpha = 0,$$

which implies $\alpha = 0$, since $k < -1$.

Using this values of α in (1.3), we get

$$S(U,V) = -\lambda g(U,V).$$

Thus we can state the following:

Theorem 3.1. *If a $(k,\mu)'$-almost Kenmotsu manifold admits an α-almost Ricci soliton with η-parallel Ricci tensor, then it becomes an Einstein manifold.*

4. α-ARS on $(k,\mu)'$-almost Kenmotsu manifolds satisfying the curvature condition $P.\phi = 0$

In this section we study α-ARS on $(k,\mu)'$-almost Kenmotsu manifolds with curvature condition $P.\phi = 0$, that is,

$$(P.\phi)(U,V)W = 0, \tag{4.1}$$

from which it follows that

$$P(U,V)\phi W - \phi(P(U,V)W) = 0. \tag{4.2}$$

Putting $W = \xi$ in (4.2), we get

$$\phi(P(U,V)\xi) = 0. \tag{4.3}$$

Using (2.16) in (4.3), we get

$$\begin{aligned}&(k+\frac{\alpha+\lambda}{2n})(\eta(V)\phi U - \eta(U)\phi V)\\ &+(\mu+\frac{\alpha}{2n})(\eta(V)\phi h'U - \eta(U)\phi h'V)) = 0.\end{aligned} \tag{4.4}$$

Replacing U by ϕU in the foregoing equation and using (2.1) and (2.4), we get

$$\begin{aligned}&(k+\frac{\alpha+\lambda}{2n})(-U+\eta(U)\xi)\eta(V)\\ &-(\mu+\frac{\alpha}{2n})\eta(V)\phi hU = 0.\end{aligned} \tag{4.5}$$

Again replacing U by ϕU and putting $V = \xi$ in the above equation, we get

$$(k+\frac{\alpha+\lambda}{2n})\phi U + (\mu+\frac{\alpha}{2n})hU = 0. \tag{4.6}$$

Taking inner product of (4.6) with respect to Z, we get

$$(k+\frac{\alpha+\lambda}{2n})g(\phi U,Z) + (\mu+\frac{\alpha}{2n})g(hU,Z) = 0. \tag{4.7}$$

Interchanging U and Z in the above equation, we get

$$(k+\frac{\alpha+\lambda}{2n})g(U,\phi Z) + (\mu+\frac{\alpha}{2n})g(U,hZ) = 0. \tag{4.8}$$

Subtracting (4.8) from (4.7), we obtain

$$(k+\frac{\alpha+\lambda}{2n})g(\phi U,Z) = 0.$$

It follows that

$$\alpha+\lambda = -2nk. \tag{4.9}$$

Using (2.14) in the above equation, we get

$$\alpha = 0.$$

Using this value of α in (1.3), we get

$$S(U,V) = -\lambda g(U,V).$$

Thus we can state the following:

Theorem 4.1. *If a $(k,\mu)'$-almost Kenmotsu manifold admits an α-almost Ricci soliton and satisfies the curvature condition $P.\phi = 0$, then the manifold is an Einstein manifold.*

5. α-ARS on $(k,\mu)'$-almost Kenmotsu manifolds satisfying the curvature condition $Q.P = 0$

In this section we study α-ARS on $(k,\mu)'$-almost Kenmotsu manifolds with curvature condition $Q.P = 0$, which implies

$$(Q.P)(U,V)W = 0. \tag{5.1}$$

From (5.1), we get

$$Q(P(U,V)W) - P(QU,V)W - P(U,QV)W - P(U,V)QW = 0. \tag{5.2}$$

Using (3.2) in (5.2), we get

$$\begin{aligned} &2(\alpha+\lambda)P(U,V)W + \alpha\eta(P(U,V)W)\xi - \alpha h'P(U,V)W \\ &-\alpha\eta(U)P(\xi,V)W + \alpha P(h'U,V)W - \alpha\eta(V)P(U,\xi)W \\ &+\alpha P(U,h'V)W - \alpha\eta(W)P(U,V)\xi + \alpha P(U,V)h'W = 0. \end{aligned} \tag{5.3}$$

Putting $W = \xi$ in the above equation, we get

$$\begin{aligned} &\{2(\alpha+\lambda) - \alpha\}P(U,V)\xi - \alpha h'P(U,V)\xi - \alpha\eta(U)P(\xi,V)\xi \\ &+\alpha P(h'U,V)\xi - \alpha\eta(V)P(U,\xi)\xi + \alpha P(U,h'V)\xi = 0. \end{aligned} \tag{5.4}$$

Using (2.16) in (5.4), we obtain

$$\begin{aligned} &\lambda[(k + \frac{\alpha+\lambda}{2n}(\eta(V)U - \eta(U)V) \\ &+(\mu + \frac{\alpha}{2n})(\eta(V)h'U - \eta(U)h'V)] = 0. \end{aligned} \tag{5.5}$$

Replacing U by ϕU in the above equation, we get

$$\lambda[(k + \frac{\alpha+\lambda}{2n})\phi U - (\mu + \frac{\alpha}{2n})hU] = 0. \tag{5.6}$$

Taking the inner product of (5.6) with respect to Z, we get

$$\lambda[(k+\frac{\alpha+\lambda}{2n})g(\phi U,Z)-(\mu+\frac{\alpha}{2n})g(hU,Z)]=0. \tag{5.7}$$

Interchanging U and Z in the above equation, we get

$$\lambda[(k+\frac{\alpha+\lambda}{2n})g(\phi Z,U)-(\mu+\frac{\alpha}{2n})g(hZ,U)]=0. \tag{5.8}$$

Subtracting (5.8) from (5.7), we get

$$\lambda(k+\frac{\alpha+\lambda}{2n})g(\phi U,Z)=0. \tag{5.9}$$

It follows that

$$\lambda(k+\frac{\alpha+\lambda}{2n})=0,$$

which implies either $\lambda=0$ or $\alpha+\lambda=-2nk$. Again using (2.14), $\alpha+\lambda=-2nk$ implies $\alpha=0$.

Using this value of α in (1.3), we get

$$S(U,V)=-\lambda g(U,V).$$

Hence we conclude the following:

Theorem 5.1. *If a $(k,\mu)'$-almost Kenmotsu manifold admits an α-almost Ricci soliton and satisfies the curvature condition $Q.P=0$, then either the Ricci soliton is steady or the manifold is an Einstein manifold.*

6. α-ARS on $(k,\mu)'$-almost Kenmotsu manifolds satisfying the curvature condition $Q.R=0$

In this section we study α-ARS on $(k,\mu)'$-almost Kenmotsu manifolds with curvature condition $Q.R=0$. Therefore

$$(Q.R)(U,V)W=0 \tag{6.1}$$

for all smooth vector fields U,V,W. The explicit form of the above equation is

$$Q(R(U,V)W)-R(QU,V)W-R(U,QV)W-R(U,V)QW=0. \tag{6.2}$$

Using (3.2) in (6.2), we obtain

$$\begin{aligned}&2(\alpha+\lambda)R(U,V)W+\alpha\eta(R(U,V)W)\xi-\alpha h'R(U,V)W-\alpha\eta(U)R(\xi,V)W\\&+\alpha R(h'U,V)W-\eta(V)R(U,\xi)W+\alpha R(U,h'V)W+\alpha R(U,V)h'W=0.\end{aligned} \tag{6.3}$$

Putting $W=\xi$ in the above equation and using (2.6), we get

$$\lambda[k(\eta(V)U-\eta(U)V)+\mu(\eta(V)h'U-\eta(U)h'V)]=0. \tag{6.4}$$

Replacing U by ϕU in (6.4), we obtain

$$\lambda[k\eta(V)\phi U - \mu\eta(V)hU] = 0. \tag{6.5}$$

Taking the inner product of (6.5) with respect to Z, we get

$$\lambda[k\eta(V)g(\phi U, Z) - \mu\eta(V)g(hU, Z)] = 0. \tag{6.6}$$

Interchanging U and Z in the above equation, we get

$$\lambda[k\eta(V)g(\phi Z, U) - \mu\eta(V)g(hZ, U)] = 0. \tag{6.7}$$

Subtracting (6.7) from (6.6), we infer

$$\lambda k g(\phi U, Z) = 0.$$

This implies

$$\lambda k = 0.$$

Since $k < -1$, then the above equation implies $\lambda = 0$.

Thus we can state the following:

Theorem 6.1. *If a $(k,\mu)'$-almost Kenmotsu manifold admits an α-almost Ricci soliton and satisfies the curvature condition $Q.R = 0$, then the Ricci soliton is steady.*

7. Example

We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. Let ξ, e_2, e_3 be three vector fields in $\mathbb{R}^3$ which satisfy [12]

$$[\xi, e_2] = -e_2 - e_3, \quad [\xi, e_3] = -e_2 - e_3, \quad [e_2, e_3] = 0.$$

Let g be the Riemannian metric defined by

$$g(\xi,\xi) = g(e_2, e_2) = g(e_3, e_3) = 1 \quad \textit{and} \quad g(\xi, e_2) = g(\xi, e_3) = g(e_2, e_3) = 0.$$

Let η be the 1-form defined by $\eta(W) = g(W, \xi)$, for any $W \in \chi(M)$.

Let ϕ be the (1,1)-tensor field defined by

$$\phi\xi = 0, \quad \phi e_2 = e_3, \quad \phi e_3 = -e_2.$$

Then using the linearity of ϕ and g, we have

$$\eta(\xi) = 1,$$

$$\phi^2 U = -U + \eta(U)\xi,$$

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V)$$

for any $U, V \in \chi(M)$. Thus the structure (ϕ, ξ, η, g) is an almost contact structure.

Moreover, $h'\xi = 0, \quad h'e_2 = e_3 \quad and \quad h'e_3 = e_2.$

In [19] the authors obtained the expression for the curvature tensor and the Ricci tensor as follows:

$$R(\xi, e_2)\xi = 2(e_2 + e_3), \quad R(\xi, e_2)e_2 = -2\xi, \quad R(\xi, e_2)e_3 = -2\xi,$$

$$R(e_2, e_3)\xi = R(e_2, e_3)e_2 = R(e_2, e_3)e_3 = 0,$$

$$R(\xi, e_3)\xi = 2(e_2 + e_3), \quad R(\xi, e_3)e_2 = -2\xi, \quad R(\xi, e_3)e_3 = -2\xi.$$

With the help of the expressions of the curvature tensor, we conclude that the characteristic vector field ξ belongs to the $(k, \mu)'$-nullity distribution with $k = -2$ and $\mu = -2$.

Using the expression of the curvature tensor, we find the values of the Ricci tensor as follows:

$$S(\xi, \xi) = -4, \quad S(e_2, e_2) = S(e_3, e_3) = -2.$$

From (2.13) we obtain

$$S(\xi, \xi) = -\lambda, S(e_2, e_2) = -(\alpha + \lambda) \text{ and } S(e_3, e_3) = -(\alpha + \lambda).$$

Therefore $\alpha = -2$ and $\lambda = 4$.

Hence it is α-ARS on $(k, \mu)'$-almost Kenmotsu manifolds.

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