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A note on uniqueness of meromorphic functions sharing two singleton sets related to a question of Chen

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Abstract. In this article we study the uniqueness problem of a special class of meromorphic functions sharing two singleton sets Our result will provide the best possible answer of a question made in [1] as well as in [2] to date, which radically improves all the results of [1] and [2] and in turn answers a question of [8] as well. We have also proposed two questions relevant to our result. Some examples have been shown by us to show that certain conditions used in the paper can not be dropped.

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1. Introduction Definitions and Results

In this paper, a meromorphic (resp. entire) function always means a meromorphic (resp. entire) function in the whole complex plane $\mathbb{C}$. It is assumed that the reader is familiar with the elementary concepts of Nevanlinna theory and in particular with its standard terms and symbols. We use $M(\mathbb{C})$ (resp. $\mathcal{E}(\mathbb{C})$) to denote the field of meromorphic (resp. entire) functions in $\mathbb{C}$. Let $S \subset \mathbb{C} \cup \{\infty\}$ be a set of distinct complex numbers and let $f \in M(\mathbb{C})$.

Definition 1.1. For a non-constant meromorphic function f and $a \in \mathbb{C}$, let $E_f(a) = \{(z,p) \in \mathbb{C} \times \mathbb{N} : f(z) = a \textit{ with multiplicity } p\}$ $\big(\overline{E}_f(a) = \{(z,1) \in \mathbb{C} \times \mathbb{N} : f(z) = a\}\big)$. Then we say f, g share the value a CM(IM) if $E_f(a) = E_g(a)\big(\overline{E}_f(a) = \overline{E}_g(a)\big)$. For $a = \infty$, we define $E_f(\infty) := E_{1/f}(0)$ $\big(\overline{E}_f(\infty) := \overline{E}_{1/f}(0)\big)$.

Definition 1.2. For a non-constant meromorphic function f and $S \subset \mathbb{C} \cup \{\infty\}$, let $E_f(S) = \bigcup_{a \in S}\{(z,p) \in \mathbb{C} \times \mathbb{N} : f(z) = \text{a with multiplicity p}\}$ $\big(\overline{E}_f(S) = \bigcup_{a \in S}\{(z,1) \in \mathbb{C} \times \mathbb{N} : f(z) = a\}\big)$. Then we say f, g share the set S CM(IM) if $E_f(S) = E_g(S)$ $\big(\overline{E}_f(S) = \overline{E}_g(S)\big)$.

Definition 1.3. We define a subset $M_1(\mathbb{C})$ of $M(\mathbb{C})$ defined by $M_1(\mathbb{C}) = \{f \in M(\mathbb{C}) \mid f$ has only finitely many poles in $\mathbb{C}\}$.

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According to Chen [2] the subsets $S_1,\ S_2,\dots,S_q$ of $\mathbb{C}\cup\{\infty\}$ are called unique range sets of meromorphic functions in $M(\mathbb{C})$ if for any two elements f and g of the family $M(\mathbb{C})$ the conditions $E_f(S_j)=E_g(S_j)$ imply $f(z)\equiv g(z)$, for each j $(j=1,2,\dots,q)$.

Actually the definition of URS was given first in [11] for $q=1$. We assume that the readers are familiar with the standard notations of Nevanlinna theory such as the Nevanlinna characteristic function $T(r,f)$, the proximity function $m(r,f)$, the counting function (reduced counting function) $N(r,\infty;f)$ $(\overline{N}(r,\infty;f))$ and so on, which are well explained in [10].

For $f\in M(\mathbb{C})$, the order and the lower order of f are defined as

$$\lambda(f)=\limsup_{r\longrightarrow\infty}\frac{\log T(r,f)}{\log r}\quad \textit{and}\quad \mu(f)=\liminf_{r\longrightarrow\infty}\frac{\log T(r,f)}{\log r},$$

respectively.

By $S(r,f)$ we mean any quantity satisfying $S(r,f)=O(\log(rT(r,f)))$ for all r possibly outside a set of finite linear measure. If f is a function of finite order, then $S(r,f)=O(\log r)$ for all r. In this paper we consider f to be a non constant meromorphic function having finitely many poles in $\mathbb{C}$ that is to say $f\in M_1(\mathbb{C})$. Clearly $\overline{N}(r,\infty;f)=O(\log r)$.

In 1976, Gross [4] stated the following question:

Question 1.4. (*See [8]*) *Are there two finite sets S_1 and S_2 such that any two non-constant entire functions f and g must be identical if $E_f(Sj)=E_g(Sj)$ for $j=1,2$?*

Inspired by the above mentioned question, Chen [1] proposed the generalized and extended form of *Question 1.1* as follows:

Question 1.5. *For a family $\mathcal{G}\subseteq M(\mathbb{C})$, determine subsets $S_1,S_2,\dots,S_q$ of $\mathbb{C}\cup\{\infty\}$ in which the cardinality of every S_i $(i=1,2,\dots,q)$ is as small as possible, and minimize the number q such that any two elements f and g of $\mathcal{G}$ are algebraically dependent if $E_f(S_i)=E_g(S_i)$ for every i $(i=1,2,\dots,q)$, that is, if f and g share every S_i $(i=1,2,\dots,q)$ CM (counting multiplicity).*

Several papers (see, e.g., [4, 5, 6, 7]) dealt with the problems of URS for meromorphic functions in $M(\mathbb{C})$. In 2017, choosing the family $\mathcal{G}=M_1(\mathbb{C})$, Chen [1] solved *Question 1.2* and obtained the following result.

Theorem A. [1] Let $S_1=\{\alpha_1\}$ and $S_2=\{\beta_1,\beta_2\}$, where α_1,β_1,β_2 are three distinct finite complex numbers satisfying $(\beta_1-\alpha_1)^2\neq(\beta_2-\alpha_1)^2$. If two non constant meromorphic functions $f(z)$ and $g(z)$ in $M_1(\mathbb{C})$ share S_1 CM, S_2 IM, and if the order of $f(z)$ is neither an integer nor infinite, then $f(z)\equiv g(z)$.

Recently Chen [2] also proved *Theorem A* for the sharing of the sets S_1 IM and S_2 CM and obtained the following result.

Theorem B. [2] Let S_1, S_2 be defined same as in *Theorem A* where where α_1,β_1,β_2 are three distinct finite complex numbers satisfying $(\beta_1-\alpha_1)^2\neq(\beta_2-\alpha_1)^2$. If two non-constant meromorphic functions $f(z)$ and $g(z)$ in $M_1(\mathbb{C})$ share S_1 IM, S_2 CM, and if the order of $f(z)$ is neither an integer nor infinite, then $f(z)\equiv g(z)$.

Remark 1.6. Let $\#(S)$ denote the cardinality of the set S. Clearly in *Theorem A* and *Theorem B*, $\max\{\#(S_i)\} = 2$, for $i = 1, 2$. Therefore it will be interesting to investigate the validity of the theorems for $\max\{\#(S_i)\} = 1$, i.e., when both sets contain only one element which is in fact, the least number of elements a nonvoid can possess. Also, note that in this case the condition $(\beta_1 - \alpha_1)^2 \neq (\beta_2 - \alpha_1)^2$ becomes meaningless.

In view of *Question 1.5* and *Remark 1.6* it will be justifiable to test the validity of an analogous result corresponding to *Theorems A* and *B* when, like S_1, S_2, also contains only one element, as that will be the best possible result ever for URS in $M_1(\mathbb{C})$.

Our main result in this paper is considering the minimum cardinality of URS S_i, $i = 1, 2$ in $M_1(\mathbb{C})$ which indeed surpass both results of Chen ([1], [2]) as far as the possible answer of *Question 1.5* is concerned in $M_1(\mathbb{C})$.

Theorem 1.7. *Let $S_1 = \{\alpha\}$ and $S_2 = \{\beta\}$, where α, β are two distinct finite complex numbers. If two non constant meromorphic functions $f(z)$ and $g(z)$ in $M_1(\mathbb{C})$ share S_1 CM and S_2 IM, and if the order of $f(z)$ is neither an integer nor infinite, then $f(z) \equiv g(z)$.*

Note 1.8. *In the above theorem we do not require any sufficient conditions like Theorems A or B. So it will be the best result to date.*

Let us consider two functions $f(z) = \frac{e^{2z}-1}{2e^z}$ and $g = e^z$. Clearly f and g share the set $\{i\}$ and $\{-i\}$ IM but $f(z) \not\equiv g(z)$, where the order of f is an integer. Similarly if $f(z) = \frac{e^{2e^z}-1}{2e^{e^z}}$ and $g = e^{e^z}$, then it is easy to see that f and g share the set $\{i\}$ and $\{-i\}$ IM, where the order of f is infinite but $f(z) \not\equiv g(z)$. Hence the following question is inevitable:

Question 1.9. *Keeping all the other conditions intact as in* Theorem 1.7 *if the sharing of the set S_1 CM is replaced by S_1 IM, does* Theorem 1.7 *still hold?*

The following examples show that in *Theorem 1.7* the condition "the order of $f(z)$ is not an integer" can not be removed.

Example 1.10. Let $f = (e^z - 1)^2 + 1$ and $g(z) = e^z$. Then it is easy to verify that f, g share 2 CM and 1 IM, but $f \not\equiv g$.

Example 1.11. Let $f = (e^z - 1)^2$, $g = (e^z - 1)$. Clearly f, g share 1 CM, 0 IM, but $f \not\equiv g$.

Example 1.12. Let $f = a - 3b(e^{-z} - e^{-2z})$ and $g = a - b(1 - e^z)^3$. Clearly f, g share $a - b$ CM and a IM, where a and $b(\neq 0)$ are constants, but $f \not\equiv g$.

Example 1.13. Let $f(z) = a - 3b(e^z + e^{2z})$ and $g(z) = a + b(1 + e^{-z})^3$. Then it is easy to verify that f, g share $a + b$ CM and a IM, where a and $b(\neq 0)$ are constants, but $f \not\equiv g$.

Note 1.14. *In the above examples if we replace e^z and e^{-z} by e^{e^z} and e^{-e^z} respectively then the examples remain valid. So from the above examples we can infer that in Theorem 1.7 the condition 'the order of $f(z)$ is not infinite" can not also be dropped.*

The assumption "non-constant meromorphic functions $f(z)$ and $g(z)$ in $M_1(\mathbb{C})$" in *Theorems 1.1* cannot be relaxed to "non-constant meromorphic functions $f(z)$ and $g(z)$ in $M(\mathbb{C})$", as shown by the following example.

Example 1.15. Let $f(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^{3n}}$ and $g(z)=\frac{1}{f(z)}$, $S_1=\{1\}$, $S_2=\{-1\}$. Then using *Lemma 2.1* in Section 2 we have $\lambda(f)=\frac{1}{3}$. Also by *Lemma 2.2* in *Section* 2 we see that $g(z)$ has infinitely many poles in $\mathbb{C}$. Moreover, $f(z)$ and $g(z)$ share S_1, S_2 CM. But $f(z)\not\equiv g(z)$.

To deal with te L-function, Han [9] obtained a similar type of result like *Theorem 1.7*, but the method adopted in the present paper is different as well as quite simpler than the method adopted in [9]

Considering two functions $f(z)=\sqrt{2}\,sinz\ \left(\text{respct. } \frac{e^{2z}+1}{2e^{z}}\right)$ and $g=\sqrt{2}\,cosz$ ((respct.) e^{z}), it is easy to see that f and g share the set $\{1,-1\}$ CM (IM) but $f(z)\not\equiv g(z)$. So we see that *Theorem 1.7* is not in general true for a set consisting of two elements for CM and IM sharing respectively when the order of the function is an integer. Similarly considering $f(z)=\sqrt{2}\ \frac{e^{e^{iz}}-e^{-e^{iz}}}{2i}$ (respct. $\frac{e^{2e^{z}}+1}{2e^{e^{z}}}$) and $g=\sqrt{2}\ \frac{e^{e^{iz}}+e^{-e^{iz}}}{2}$ $\left(\text{respct. } -e^{e^{z}}\right)$, it is easy to see that when the order of f is infinite, f and g can share the set $\{1,-1\}$ CM (IM), yet $f(z)\not\equiv g(z)$. Hence we can also propose the following question:

Question 1.16. *Does* Theorem 1.7 *hold good if two non constant meromorphic functions $f(z)$ and $g(z)$ in $M_1(\mathbb{C})$ share S_1 CM or even IM, where $\{\#(S_1)\}=2$ and if the order of $f(z)$ is neither an integer nor infinite?*

2. Lemmas

In this section we present some important lemmas which will be needed in the sequel.

Lemma 2.1. (p.288, [3]) Let $f(z)=\sum_{n=0}^{\infty}a_nz^n\in\mathcal{E}(\mathbb{C})$ be non-constant and of finite order. Then

$$\lambda(f)=\frac{1}{\liminf_{n\longrightarrow\infty}\frac{-\log|a_n|}{n\log n}}.$$

Lemma 2.2. (p.293, [3]) Let $f(z)\in\mathcal{E}(\mathbb{C})$. If the order of $f(z)$ is neither an integer nor infinite, then $f(z)$ assumes every finite value infinitely often.

Lemma 2.3. (Theorem 1.44, [3], [12]) Let $h(z)\in\mathcal{E}(\mathbb{C})$, and let $f(z)=e^{h(z)}$. Then
(i) if $h(z)$ is a polynomial of degree k, then $\lambda(f)=\mu(f)=k=degh$;
(ii) if $h(z)$ is a transcendental entire function, then $\lambda(f)=\mu(f)=\infty$.

Lemma 2.4. *[12] Let $T_1(r)$ and $T_2(r)$ be two nonnegative, nondecreasing real functions defined in $r>r_0>0$. If $T_1(r)=O(T_2(r))$ $(r\longrightarrow\infty\ ,r\notin E)$, where E is a set with finite linear measure, then*

$$\limsup_{r\longrightarrow\infty}\frac{\log^+T_1(r)}{\log r}\le\limsup_{r\longrightarrow\infty}\frac{\log^+T_2(r)}{\log r}$$

and

$$\liminf_{r\longrightarrow\infty}\frac{\log^+ T_1(r)}{\log r}\leq\liminf_{r\longrightarrow\infty}\frac{\log^+ T_2(r)}{\log r}$$

imply that the order and the lower order of $T_1(r)$ *are not greater than the order and the lower order of* $T_2(r)$*, respectively.*

Lemma 2.5. (Theorem 1.42, [12]). Let $f(z)\in M(\mathbb{C})$. If 0 and ∞ are two Picard exceptional values of $f(z)$, then $f(z)=e^{h(z)}$, where $h(z)\in\mathcal{E}(\mathbb{C})$.

3. Proof of the theorem

Proof of the Theorem 1.7. At first let us assume $f(z)\not\equiv g(z)$. Now define the following two functions:

$$F(z)=f(z)-\beta$$

$$G(z)=g(z)-\beta,$$

clearly $F(z)\not\equiv G(z)$. Since f and g share S_1 CM, S_2 IM, therefore $F(z)$ and $G(z)$ share ξ $(=\alpha-\beta)$ CM and 0 IM. First we consider the following auxiliary function :

$$\hat{H}=\frac{\mathcal{U}(G-\xi)}{(F-\xi)},\tag{3.1}$$

where $\mathcal{U}$ is a rational function such that $\hat{H}$ has neither a pole nor a zero in $\mathbb{C}$. It is evident that such a function $\mathcal{U}$ does exist since F and G have finitely many poles and in view of the condition that f and g share the set S_1 CM, a possible zero or pole of $\hat{H}$ may only come from a pole of F or G. Since $\hat{H}$ is an entire function with no zero and pole then from *Lemma 2.5* we can write

$$\hat{H}=\frac{\mathcal{U}(G-\xi)}{(F-\xi)}=e^{\psi},\tag{3.2}$$

for some entire function ψ. Noting that $f(z)$ and $g(z)$ have only finitely many poles, we have $\overline{N}(r;F)=O(\log r)=\overline{N}(r;G)$.

Again

$$\begin{aligned}T(r,G)&\leq\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{G-\xi}\right)+\overline{N}(r,G)+S(r,G)\\&\leq\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-\xi}\right)+O(\log r)+S(r,G)\\&\leq 2T(r,F)+O(\log r)+S(r,G),\end{aligned}\tag{3.3}$$

as $r\longrightarrow\infty$, $r\notin E$, where E is a set with finite linear measure. Then by (3.3) and *Lemma 2.4* we have

$$\lambda(G)\leq\lambda(F),$$

and proceeding similarly we get

$$\lambda(F) \leq \lambda(G).$$

Hence we have

$$\lambda(G) = \lambda(F). \tag{3.4}$$

By the first fundamental theorem it is easy to verify $\lambda(F) = \lambda(f)$ and $\lambda(G) = \lambda(g)$. Since $\mathcal{U}$ is a rational function therefore

$$\lambda(e^{\psi}) \leq \lambda(F). \tag{3.5}$$

In view of the fact that $\mathcal{U}$ is a rational function, we have $T(r, \mathcal{U}) = O(\log r)$. So it follows that

$$\overline{N}\left(r, \frac{1}{F}\right) \leq \overline{N}\left(r, \frac{1}{(e^{\psi}/\mathcal{U} - 1)}\right) \leq T(r, e^{\psi}) + O(\log r). \tag{3.6}$$

Next, introduce the following auxiliary function

$$\Delta = \left(\frac{F'}{F(F-\xi)} - \frac{G'}{G(G-\xi)}\right)(F-G). \tag{3.7}$$

Since $F(z)$ and $G(z)$ share ξ CM and 0 IM then it is easy to verify that

$$\overline{N}\left(r, \frac{1}{F-\xi}\right) \leq \overline{N}\left(r, \frac{1}{\Delta}\right) \leq T(r, \Delta) + O(1). \tag{3.8}$$

Also Δ is analytic at every 0 point of F (*or* G). The only possible poles of Δ come from the poles of F and G, which are finitely many. Therefore

$$N(r, \Delta) = O(\log r). \tag{3.9}$$

Now in view of the fact that the order of $F(z)$ is finite and $\mathcal{U}(z)$ is rational, from (3.2), by the logarithmic derivative lemma and the first fundamental theorem we obtain

$$\begin{aligned} & m(r, \Delta) \\ \leq\ & m\left(r, \frac{F'}{F}\right) + m\left(r, \frac{F-G}{F-\xi}\right) + m\left(r, \frac{G'}{G}\right) + m\left(r, \frac{F-G}{G-\xi}\right) \\ \leq\ & m\left(r, 1 - \frac{G-\xi}{F-\xi}\right) + m\left(r, \frac{F-\xi}{G-\xi} - 1\right) + O(\log r) \\ \leq\ & 2T(r, e^{\psi}) + O(\log r). \end{aligned} \tag{3.10}$$

Hence using (3.9) and (3.10) from (3.8) we obtain

$$\overline{N}\left(r, \frac{1}{F-\xi}\right) \leq 2T(r, e^{\psi}) + O(\log r). \tag{3.11}$$

Using the second fundamental theorem and in view of (3.6) and (3.11) we have

$$\begin{aligned} T(r,F) &\leq \overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-\xi}\right)+\overline{N}(r,F)+S(r,F) \\ &\leq O(T(r,e^{\psi}))+O(\log r)+S(r,F), \end{aligned} \tag{3.12}$$

as $r \longrightarrow \infty$, $r \notin E$, which together with *Lemma 2.4* means

$$\lambda(F) \leq \lambda(e^{\psi}). \tag{3.13}$$

Hence from (3.5) and (3.13) we have

$$\lambda(F) = \lambda(e^{\psi}), \tag{3.14}$$

which contradicts *Lemma 2.3*, since order of $F(z)$ is neither integer nor infinite and $\psi(z)$ is entire function.

Therefore $\Delta(z) \equiv 0$. We are now ready to get back to our original task of showing that $f(z) \equiv g(z)$. As at first we have assumed $f(z) \not\equiv g(z)$, we have $F(z) \not\equiv G(z)$. So from $\Delta(z) \equiv 0$ we have

$$\frac{F'}{F(F-\xi)} = \frac{G'}{G(G-\xi)}. \tag{3.15}$$

Clearly from (3.15), we have $F(z)$ and $G(z)$ share 0 CM. That is to say, $g(z)$ and $f(z)$ must share β CM.

Recall from (3.6) that

$$\overline{N}\left(r,\frac{1}{F}\right) \leq T(r,e^{\psi})+O(\log r)+S(r,F).$$

We have got the above inequality by using only the facts that F and G share ξ CM and 0 IM. Now considering F and G share 0 CM, by symmetry or doing exactly the same way as done in (3.2), we deduce that

$$\overline{N}\left(r,\frac{1}{F-\xi}\right) \leq T(r,e^{\phi})+O(\log r)+S(r,F),$$

for some $\phi \in \mathcal{E}(\mathbb{C})$. Finally, we use the second fundamental theorem again to deduce that

$$\begin{aligned} T(r,F) &\leq \overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-\xi}\right)+\overline{N}(r,F) \\ &\leq O(T(r,e^{\mu}))+O(\log r)+S(r,F), \end{aligned}$$

as $r \longrightarrow \infty$, $r \notin E$, where $T(r,e^{\mu}) = \max\{T(r,e^{\phi}),T(r,e^{\psi})\}$ which imply

$$\lambda(F) \leq \lambda(e^{\mu}).$$

So by the same argument as in (3.5) we will get $\lambda(F) = \lambda(e^{\mu})$, again a contradiction. Hence our assumption is wrong. Therefore from $\Delta \equiv 0$ we get $F \equiv G \implies f \equiv g$. $\square$

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