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Prime coprime graph of a finite group

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Abstract. In this paper, a new graph structure called the *prime coprime graph* of a finite group G denoted by $\Theta(G)$ has been introduced. The *coprime graph* of a finite group, introduced by Ma, Wei, and Yang [*The coprime graph of a group. International Journal of Group Theory, 3(3), pp.13-23.*] is a subgraph of the *prime coprime graph* introduced in this paper. The vertex set of $\Theta(G)$ is G, and any two vertices x, y in $\Theta(G)$ are adjacent if and only if $\gcd(o(x), o(y))$ is equal to 1 or a prime number. We study how the graph properties of $\Theta(G)$ and group properties of G are related. We provide a necessary and sufficient condition for $\Theta(G)$ to be Eulerian for any finite group G. We also study $\Theta(G)$ for certain finite groups like $\mathbb{Z}_n$ and D_n and derive conditions when it is connected, complete, planar, and Hamiltonian for various $n \in \mathbb{N}$. We also study the vertex connectivity of $\Theta(\mathbb{Z}_n)$ for various $n \in \mathbb{N}$. Finally, we have computed the signless Laplacian spectrum of $\Theta(G)$ when $G = \mathbb{Z}_n$ and $G = \mathrm{D}_n$ for $n \in \{pq, p^m\}$ where p, q are distinct primes and $m \in \mathbb{N}$.

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1. Introduction

Generating graphs from various algebraic structures like groups and semigroups is nothing new. Bosak in [10] studied various kinds of graphs that were defined on semigroups. In [32], the author studied the *intersection graph* defined on a finite abelian group. A *Cayley digraph* is also an important class of directed graphs defined on finite groups, and readers may refer to [11, 23] in order to find some information about them. Kelarev and Quinn in [27] introduced the *power graph* on a semigroup S as a directed graph in which the set of vertices is S, and two distinct elements $a, b \in S$ are adjacent if and only if $b = a^m$ for some positive integer m. Motivated by the work in [27], Chakrabarty et al. studied the *undirected power graph* on semigroups in [14]. The undirected power graph on a semigroup S is the graph whose vertex set is S, and two distinct vertices $a, b \in S$ are adjacent if and only if $a = b^m$ or $b = a^n$ for some positive integers m, n. Several properties of the power graph were investigated by Cameron and Ghosh in [12] and [13]. In [26], the authors introduced a new graph known as the *order supergraph* of the power graph of

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a finite group G, whose vertex set is G and any two vertices x, y are adjacent if and only if $o(x) \mid o(y)$ or $o(y) \mid o(x)$. The automorphism group of this graph was studied in [24].

Recently, several researchers have studied spectral properties of graphs associated with algebraic structures. The spectral properties of power graph of a finite group ([29],[25],[7],[6]), Cayley graph of certain groups ([3], [1], [15]), commuting and non-commuting graph of dihedral groups ([2],[5]) etc. have been studied over the last few years.

The notion of *coprime graph* of a finite group G has existed in the literature for a long time. It was first introduced by Sattanathan and Kala as the *order prime graph* in [30]. Later on, in [28] Ma et al. reintroduced and renamed the order prime graph as the *coprime graph* and studied various properties of it. The coprime graph was studied extensively in [21] and [31]. In [4], the Laplacian spectra of the coprime graph of finite cyclic and dihedral groups were studied. In this paper, we introduce a new graph known as the *prime coprime graph* of a finite group G. We denote it by $\Theta(G)$. Clearly for a given finite group G, the *coprime graph* is a subgraph of the *prime coprime graph* introduced in this paper. We characterize some properties of $\Theta(G)$ using the algebraic properties of the group G. We study the connectedness and the diameter of the graph $\Theta(G)$. We show that $\Theta(G)$ is Eulerian if and only if G has odd order and every non-identity element of G has prime order. We also find out when $\Theta(\mathbb{Z}_n)$ is planar and Hamiltonian for various $n \in \mathbb{N}$. We also study the vertex connectivity of $\Theta(\mathbb{Z}_n)$ for various n. Finally, we find the signless Laplacian spectra of $\Theta(\mathbb{Z}_n)$ and $\Theta(\mathrm{D}_n)$ for $n \in \{pq, p^m\}$ where p, q are distinct primes and $m \in \mathbb{N}$.

The paper has been organized as follows: In Section 2, we have provided the preliminary definitions and theorems that have been used throughout the paper. In Section 3, we formally introduce the *prime coprime graph* of a finite group G, denoted by $\Theta(G)$, and study various properties of $\Theta(G)$. In Section 4, we study the vertex connectivity of $\Theta(\mathbb{Z}_n)$. In Section 5, we determine the signless Laplacian spectra of $\Theta(\mathbb{Z}_n)$ and $\Theta(\mathrm{D}_n)$ for $n \in \{pq, p^m\}$.

2. Preliminaries

In this section, for the convenience of the readers, we provide some preliminary definitions and theorems that have been used throughout the paper. We denote a graph $\mathcal{G}$ by $\mathcal{G} = (V, E)$ where V is the set of all vertices of $\mathcal{G}$ and E denotes the set of all edges of $\mathcal{G}$. A graph $\mathcal{G}$ is said to be *simple* if it has no loops or parallel edges. A graph with one vertex and no edges is called a *trivial* graph. We denote the degree of a vertex $v \in V(\mathcal{G})$ by $\deg(v)$. For a given graph $\mathcal{G}$, $\delta(\mathcal{G}) = \min\{\deg(v) : v \in \mathcal{G}\}$. A *subgraph* $\mathcal{H} = (W, F)$ of $\mathcal{G} = (V, E)$ is a graph such that $W \subseteq V$ and $F \subseteq E$. If there exists an edge between two vertices a and b, then a and b are said to be adjacent, and it is denoted by $a \sim b$. If there exists an edge between any two vertices of $\mathcal{G}$, then $\mathcal{G}$ is said to be *complete* and is denoted by K_n. A *path* P of length k in a graph G is an alternating sequence of vertices and edges $v_0, e_0, v_1, e_1, v_2, e_2, \ldots, v_{k-1}, e_{k-1}, v_k$, where $v_i's$ are distinct vertices, and e_i is the edge joining v_i and v_{i+1}. If $v_0 = v_k$, then P is said to be a *cycle* of length k. The length of the shortest cycle in $\mathcal{G}$ is known as its *girth*. A graph $\mathcal{G}$ is said to be *connected* if for any pair of vertices $u, v \in V$ there exists a path joining u and v. For a connected graph $\mathcal{G}$, the *distance* between two vertices u, v denoted by $d(u, v)$, is defined as the length of the shortest path joining u and v. The *diameter* of a connected graph $\mathcal{G}$, denoted by $\text{diam}(\mathcal{G})$, is defined as $\text{diam}(\mathcal{G}) = \max\{d(u, v) : u, v \in V\}$. A *planar* graph is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. An *isomorphism* of graphs $\mathcal{G}$ and $\mathcal{H}$ denoted by $\mathcal{G} \cong \mathcal{H}$ is a bijection f between $V(\mathcal{G})$ and $V(\mathcal{H})$ such that any two vertices $u, v \in V(\mathcal{G})$ are adjacent if and only if the vertices $f(u), f(v) \in V(\mathcal{H})$ are adjacent. An *Eulerian cycle* in a graph $\mathcal{G}$ is a cycle which visits every edge exactly once. A graph $\mathcal{G}$ is said to be *Eulerian* if it has an Eulerian cycle. A *Hamiltonian cycle* in a graph $\mathcal{G}$ is a cycle which visits every vertex exactly once. A graph $\mathcal{G}$ is said to be *Hamiltonian* if it has a Hamiltonian cycle. The *vertex connectivity* $\kappa(\mathcal{G})$ of a graph $\mathcal{G}$ is the minimum number of vertices whose removal results in a disconnected or trivial graph. We define the connectivity of a disconnected graph to be 0. Given a positive integer k, a graph $\mathcal{G}$ is said to be $k-$tough if for any integer $t > 1$, $\mathcal{G}$ cannot be split into t different connected components by the removal of fewer than kt vertices. The *toughness* of a graph $\mathcal{G}$ is defined as the largest real number t such that deletion of any s vertices from $\mathcal{G}$ results in a graph which is either connected or else has at most $\frac{s}{t}$ components. A *dominating set* of a graph $\mathcal{G}$ is a subset D of V such that for every $v \notin D$, there exists a vertex $w \in D$ for which v is adjacent to w. The *domination number* is the number of vertices in a smallest dominating set of $\mathcal{G}$. For more information on the terms used above, the readers may refer to any standard book on graph theory, say [20] or [9].

Let $\mathcal{G}$ be a finite simple undirected graph with $V(\mathcal{G}) = \{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* of $\mathcal{G}$, denoted by $A(\mathcal{G}) = (a_{ij})$ is defined as $a_{ij} = 1$ if $v_i \sim v_j$ and $a_{ij} = 0$ otherwise. The *degree matrix* of $\mathcal{G}$, denoted by $D(\mathcal{G}) = (d_{ii})$ is a diagonal matrix, where d_{ii} denotes the degree of the i^{th} vertex of $\mathcal{G}$. The *Laplacian matrix* $L(\mathcal{G})$ is defined as $L(\mathcal{G}) = D(\mathcal{G}) - A(\mathcal{G})$. The *signless Laplacian matrix* $Q(\mathcal{G})$ is defined as $Q(\mathcal{G}) = D(\mathcal{G}) + A(\mathcal{G})$. The matrix $Q(\mathcal{G})$ is a real and symmetric matrix and hence all its eigenvalues are real. Also, $Q(\mathcal{G})$ is a positive semi-definite matrix and hence all its eigenvalues are non-negative. For more information on $Q(\mathcal{G})$, readers may refer to [17], [18] and [19]. We arrange the eigenvalues of $Q(\mathcal{G})$ as $\lambda_1(\mathcal{G}) \geq \lambda_2(\mathcal{G}) \geq \cdots \geq \lambda_n(\mathcal{G})$ in non-increasing order, and repeated according to their multiplicities.

For $n \in \mathbb{N}$, the number of positive integers that are less than or equal to n and are relatively prime to n is denoted by $\varphi(n)$. The function φ is known as *Euler's phi function*. We know that a finite cyclic group of order n is isomorphic to $(\mathbb{Z}_n, +)$, where $\mathbb{Z}_n = \{0, 1, 2 \ldots, n-2, n-1\}$, and hence we prove our results for $\mathbb{Z}_n$ instead of an arbitrary cyclic group. An element $a \in \mathbb{Z}_n$ is said to be a *generator* of $\mathbb{Z}_n$ if $\gcd(a, n) = 1$. An element which is not a generator is known as a *non-generator*. We denote the *dihedral group* of order $2n$ by D_n. The *order* of an element $g \in G$, denoted by $o(g)$, is the least positive integer n such that

$g^n = e$, where e is the identity element of G. The number of elements in a set S is denoted by $|S|$. For basic definitions and notations on group theory, the readers are referred to [22].

The theorems used in the paper have been listed below. The proof of Theorems 2.1 to 2.4 can be found in [9] or [20], while the proof of Theorem 2.6 can be found in [8].

Theorem 2.1. *A connected graph $\mathcal{G}$ has an Eulerian cycle if and only if* $\deg(v)$ *is even for all $v \in \mathcal{G}$.*

Theorem 2.2. *For any graph $\mathcal{G}$, $\kappa(\mathcal{G}) \leq \delta(\mathcal{G})$.*

Theorem 2.3. *The complete graph K_5 and the complete bipartite graph $K_{3,3}$ are non-planar.*

Theorem 2.4 (Ore)**.** *Let $\mathcal{G}$ be a finite and simple graph with n vertices where $n \geq 3$. If* $\deg(v) + \deg(w) \geq n$ *for every pair of distinct non-adjacent vertices v and w of $\mathcal{G}$, then $\mathcal{G}$ is Hamiltonian.*

Theorem 2.5. *[16] If $\mathcal{G}$ is Hamiltonian, then $\mathcal{G}$ is* 1*-tough.*

Theorem 2.6. *If J denotes the square matrix of order n with all entries equal to one and I denotes the identity matrix of order n then the eigenvalues of $aI + bJ$ are a with multiplicity $n - 1$ and $a + nb$ with multiplicity* 1*.*

3. Prime Coprime Graph of a Finite Group

Let G be a finite group such that $|G| > 2$. The prime coprime graph $\Theta(G) = (V, E)$ is defined as follows: The vertex set V is the set G, and any two distinct vertices x, y are adjacent if and only if $\gcd(o(x), o(y))$ is equal to 1 or a prime number. We now study some basic properties of $\Theta(G)$.

Theorem 3.1. *The graph $\Theta(G)$ satisfies the following properties:*

(a). The domination number of $\Theta(G)$ is 1 *and $\{e\}$ is a dominating set of $\Theta(G)$.*

(b). The set $\{x\}$ is a dominating set of $\Theta(G)$ if and only if $o(x)$ is equal to 1 *or a prime number.*

Proof. (a). Since $\gcd(o(a), o(e)) = o(e) = 1$, we find that e is adjacent to a for all $a \in G$. Hence, the set $\{e\}$ is a dominating set of $\Theta(G)$, which implies that the domination number of $\Theta(G)$ is 1.

(b). Let $x \in G$. If $o(x)$ is equal to 1 or a prime number, then $gcd(o(a), o(x))$ is equal to 1 or a prime number for all $a \in G$. Thus, x is adjacent to a for all $a \in G$, which implies that $\{x\}$ is a dominating set of $\Theta(G)$.
Conversely, let $\{x\}$ be a dominating set of $\Theta(G)$. Assume the contrary that $o(x)$ is neither 1 nor a prime number. Then, $o(x)$ is composite which implies that $x \neq x^{-1}$. Thus, $\gcd(o(x), o(x^{-1})) = o(x)$, which is

composite. Hence, x is not adjacent to x^{-1}. Since $x \neq x^{-1}$, it contradicts the fact that $\{x\}$ is a dominating set of $\Theta(G)$. Hence, $o(x)$ is either equal to 1 or a prime number.

□

Theorem 3.2. *The graph $\Theta(G)$ is connected and the diameter of $\Theta(G)$ is at most 2.*

Proof. Let $x, y \in \Theta(G)$. If $\gcd(o(x), o(y))$ is equal to 1 or a prime number, then x is adjacent to y, and we are done. If $\gcd(o(x), o(y))$ is composite then x and y are not adjacent. Consider the identity element e of G. Since $o(e) = 1$, so x and y are both adjacent to e. Thus, we find that there always exists a path of length 2 between any two non-adjacent vertices $x, y \in \Theta(G)$. Thus, $\Theta(G)$ is connected and the diameter of $(\Theta(G))$ is at most 2. □

Theorem 3.3. *If the girth of $\Theta(G)$ is finite, then it equals 3.*

Proof. The proof follows from the simple fact that for any two distinct vertices $x, y \in G$ where $x, y \neq e$, there exists a path of length 2 given by $x \sim e \sim y$ from x to y. If x, y are adjacent for some x and y, then the girth of $\Theta(G)$ is finite and it equals 3. □

Theorem 3.4. *The graph $\Theta(G)$ is Eulerian if and only if G is an odd-order group, and every non-identity element has prime order.*

Proof. Suppose the graph $\Theta(G)$ is Eulerian. Using Theorem 3.2, we find that $\Theta(G)$ is connected. Since $\Theta(G)$ is Eulerian, using Theorem 2.1 we find that every vertex in $\Theta(G)$ has an even degree. Since the identity element e of G is connected to every other vertex in $\Theta(G)$, $\deg(e) = |G| - 1$. Since $\deg(e)$ must be even, the order of G must be odd. Thus, G has no elements of order 2. Now let a be any non-identity element of G. We claim that a has prime order. Assume that the order of a is composite. Let us consider the set

$$E_a = \{b \in G : \gcd(o(a), o(b)) \text{ is equal to 1 or a prime number}\}.$$

We notice that $b \in E_a$ if and only if $b^{-1} \in E_a$. Thus, the number of non-identity elements present in E_a (if any) are even. Also, the identity element e of G is in E_a. Thus, E_a has an odd number of elements. Thus, $|E_a|$ is an odd number. Let $E_a^* = E_a \setminus \{a\}$. Since the order of a is composite, so $a \notin E_a$. Thus, $E_a = E_a^*$. We further note that the elements of E_a^* are those vertices of $\Theta(G)$ which are adjacent to the vertex a of $\Theta(G)$. Thus, $|E_a| = |E_a^*| = \deg(a)$. Since $\Theta(G)$ is Eulerian, $\deg(a)$ must be even, but we have proved that $\deg(a)$ is odd, which is contradictory. Hence, our initial assumption that order of a is composite, is false. Thus, the order of a must be a prime number. Thus, we find that if $\Theta(G)$ is Eulerian, then the order of G is odd and every non-identity element has prime order.

Conversely, assume that $|G|$ is odd and every non-identity element of G has prime order. Thus, for any element $a \in G$, we have ${E_a}^* = G \setminus \{a\}$. Also,

$$\deg(a) = |E_a^*| = |G \setminus \{a\}|. \tag{3.1}$$

Since $|G|$ is an odd number, so $|G \setminus \{a\}|$ is an even number for every $a \in G$. Using Equation (3.1), we find that for every $a \in G$, $\deg(a)$ must be an even number. Thus, $\Theta(G)$ is connected and every vertex in $\Theta(G)$ has an even degree. Using Theorem 2.1, we conclude that $\Theta(G)$ is Eulerian. Thus, the result follows. $\square$

Theorem 3.5. *The graph $\Theta(G)$ is complete if and only if G has no elements of composite order.*

Proof. Suppose $\Theta(G)$ is complete. Let $g \in G$ be an element of composite order. Clearly $g \neq g^{-1}$. Since $\gcd(o(g), o(g^{-1})) = o(g)$, we find that g is not adjacent to g^{-1} in $\Theta(G)$. Thus, $\Theta(G)$ is not complete which is a contradiction. Hence, we conclude that G has no element whose order is composite. Conversely, if all elements of G have prime order, then for any two elements $x, y \in G$, $\gcd(o(x), o(y))$ is equal to 1 or a prime number, which in turn implies that $\Theta(G)$ is complete. $\square$

Corollary 3.6. *Let G be a finite cyclic group of order n. Then, $\Theta(G)$ is complete if and only if n is a prime number.*

Corollary 3.7. *Let G be a finite commutative group of order p^m where p is a prime and $m > 1$. Then, $\Theta(G)$ is complete if and only if $G \cong (\mathbb{Z}_p)^m$.*

Proof. We know that any finite commutative group is a direct product of cyclic groups. Hence, $G \cong \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p^{\alpha_k}}$ where $\alpha_1 + \alpha_2 + \cdots + \alpha_k = m$ and $1 \leq \alpha_i \leq m$. Assume that $\Theta(G)$ is complete. Now if $\alpha_i = 1$ for all $1 \leq i \leq k$, we are done. So, let us assume that there exists α_i such that $\alpha_i > 1$ for some i. Since $\mathbb{Z}_{p^{\alpha_i}}$ is a cyclic group of order p^{α_i}, it has $\varphi(p^{\alpha_i}) \geq 2$ generators, and hence we can find $x \in \mathbb{Z}_{p^{\alpha_i}}$ such that $o(x) = p^{\alpha_i}$. Consider the element $\mathbf{x} = (0, 0, \ldots, 0, x, 0 \ldots, 0, 0) \in G$. Clearly $o(\mathbf{x})$ is composite which contradicts Theorem 3.5. Hence, $\alpha_i = 1$ for all $1 \leq i \leq m$ which implies that $G \cong (\mathbb{Z}_p)^m$. The converse part is trivial and hence skipped. $\square$

Corollary 3.8. *The graph $\Theta(D_n)$ for $n \geq 2$ is complete if and only if n is prime.*

Proof. We know that the dihedral group of order $2n$ has the following presentation:

$$\mathrm{D}_n = \{\langle r, s\rangle : r^n = s^2 = 1, rs = sr^{-1}\}.$$

We partition the vertex set of $\Theta(\mathrm{D}_n)$ as $\mathrm{D}_n = A \cup B$ where $A = \{r^i : 0 \leq i \leq n-1\}$, and $B = \{sr^i : 0 \leq i \leq n-1\}$. The graph induced by the elements of A forms a subgraph of $\Theta(\mathrm{D}_n)$, and is isomorphic to $\Theta(\mathbb{Z}_n)$. Since every element of B has order 2, so the subgraph induced by the elements of B is isomorphic to K_n. Also since the order of each member of B is 2, every vertex of A is adjacent to every vertex of B. Using the above facts, we observe that $\Theta(\mathrm{D}_n)$ is complete if and only if $\Theta(\mathbb{Z}_n)$ is complete. Using Corollary 3.6, $\Theta(\mathbb{Z}_n)$ is complete if and only if n is prime. Thus, $\Theta(\mathrm{D}_n)$ is complete if and only if n is prime.

$\square$

Theorem 3.9. *The graph $\Theta(\mathbb{Z}_n)$ is planar if and only if $n = 3$ or $n = 2^i$ where $i \in \mathbb{N}$.*

Proof. Assume that $\Theta(\mathbb{Z}_n)$ is planar. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where p_i's are primes, and α_i's are positive integers. Assume that $\alpha_i \geq 1$ for $i = i_0$ and $i = i_1$. We choose two elements $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{i_0}^{\alpha_{i_0}-1} \cdots p_k^{\alpha_k}$ and $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{i_1}^{\alpha_{i_1}-1} \cdots p_k^{\alpha_k}$, and consider the subgroup generated by these two elements. Every element in the subgroup generated by $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{i_0}^{\alpha_{i_0}-1} \cdots p_k^{\alpha_k}$ except the identity element has order p_{i_0}, which is prime. Every element in the subgroup generated by $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{i_1}^{\alpha_{i_1}-1} \cdots p_k^{\alpha_k}$ except the identity element has order p_{i_1}, which is also prime. Thus, we obtain $p_{i_0} + p_{i_1} - 2$ elements of prime order. Note that $p_{i_0} + p_{i_1} \geq 5$ is always true, and hence we can always get at least 3 elements of prime order. If we take 3 elements of prime order, together with the zero element and a generator of $\mathbb{Z}_n$, then we can find 5 elements that are adjacent to each other. Thus, there always exists a subgraph in $\Theta(\mathbb{Z}_n)$ which is isomorphic to K_5. By Theorem 2.3, we conclude that $\Theta(\mathbb{Z}_n)$ is not planar, which is contrary to our assumption. Hence, we cannot find i_0 and i_1 such that $\alpha_{i_0}, \alpha_{i_1} \geq 1$. Thus, $\alpha_i \geq 1$ for at most one i. Hence, $n = p^i$ where p is a prime and $i \geq 1$.

Again if $p \geq 5$, we can consider the element p^{i-1}. We again notice that all elements in the set $\{p^{i-1}, 2p^{i-1}, 3p^{i-1}, \ldots, (p-1)p^{i-1}\}$ have prime order, and hence are adjacent to all other members of the graph $\Theta(\mathbb{Z}_n)$. Since $p \geq 5$, so we have $p - 1 \geq 4$. Hence, if we take 4 elements from the set $\{p^{i-1}, 2p^{i-1}, 3p^{i-1}, \ldots, (p-1)p^{i-1}\}$ together with the zero element of $\mathbb{Z}_n$, then the graph induced by them is isomorphic to K_5, and hence by Theorem 2.3 we find that $\Theta(\mathbb{Z}_n)$ is not planar. Thus, we are left with primes $p = 2, 3$. Hence, either $n = 2^i$ or $n = 3^i$ for some $i \in \mathbb{N}$. We claim that the graph $\Theta(\mathbb{Z}_n)$ is planar when $n = 2^i$ for all $i \geq 1$. If $n = 2^i$ for some i, then $\Theta(\mathbb{Z}_n)$ has exactly two vertices of degree $n-1$, and the remaining vertices will each have degree 2. We illustrate $\Theta(\mathbb{Z}_n)$ for $n = 8$ below. The graph $\Theta(\mathbb{Z}_8)$ can be drawn as in Figure 1, from which it is evident that $\Theta(\mathbb{Z}_8)$ is planar.

Using similar arguments as done for $\Theta(\mathbb{Z}_8)$, it can be established that $\Theta(\mathbb{Z}_n)$ is planar for $n = 2^i$ where $i \in \mathbb{N}$.

Now we show that $\Theta(\mathbb{Z}_n)$ is planar for $n = 3^i$ only for $i = 1$. Note that when $i = 1$, then $\Theta(\mathbb{Z}_3) \cong K_3$ which is planar. We illustrate $\Theta(\mathbb{Z}_9)$ in Figure 2.

Now for $i \geq 2$, if we take the vertices $0, 3^{i-1}, 2 \cdot 3^{i-1}$, and any three vertices other than these, then the graph obtained contains $K_{3,3}$ as a subgraph. Using Theorem 2.3, we can conclude that $\Theta(\mathbb{Z}_n)$ is not planar for $n = 3^i$ where $i \geq 2$. Hence, the graph is planar only when $n = 3$. Thus, $\Theta(\mathbb{Z}_n)$ is planar if and only if $n = 3$ or $n = 2^i$ where $i \in \mathbb{N}$. □

Theorem 3.10. *If p and q are distinct primes with $p < q$ then $\Theta(\mathbb{Z}_{pq})$ is Hamiltonian if and only if $p = 2$.*

Proof. Assume that $p = 2$. Let v_1 and v_2 be two non-adjacent vertices in $\Theta(\mathbb{Z}_{2q})$. Then, v_1 and v_2 are generators of $\mathbb{Z}_{2q}$. Note that v_1 is adjacent to any

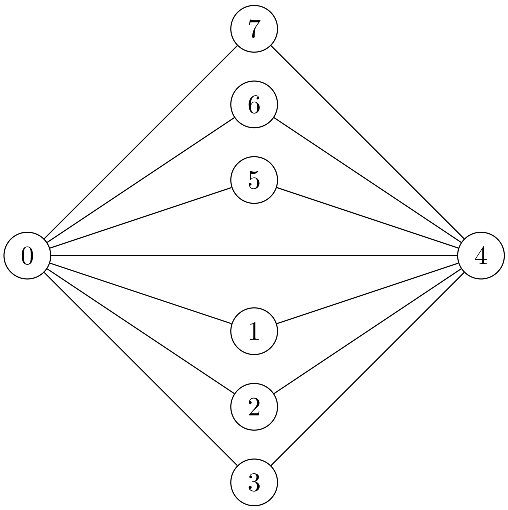

Figure 1: $\Theta(\mathbb{Z}_8)$

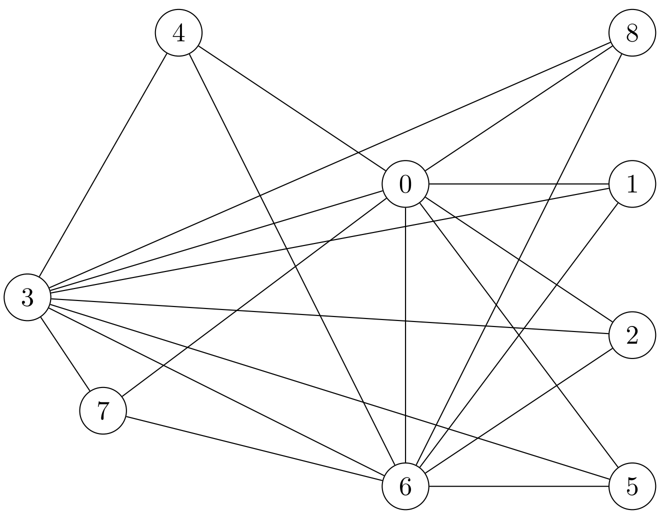

Figure 2: $\Theta(\mathbb{Z}_9)$

non-generator of $\mathbb{Z}_{2q}$ and so is v_2. Then, $\deg(v_i) = 2q - \varphi(2q) = 2q - (q-1) = q+1$, where $i \in \{1, 2\}$. Thus, $\deg(v_1) + \deg(v_2) = 2(q+1) > 2q$. Thus, the sum of degrees of two non-adjacent vertices is greater than the number of vertices in $\Theta(\mathbb{Z}_{2q})$. By Theorem 2.4, we conclude that $\Theta(\mathbb{Z}_{2q})$ is Hamiltonian.

Now we show that if $2 < p < q$ then the graph $\Theta(\mathbb{Z}_{2q})$ is not Hamiltonian. Consider the sets

$$A = \{i : \gcd(i, n) = 1\} \text{ and } B = \{0\} \cup \{i : \gcd(i, n) \neq 1\}.$$

If we remove all vertices of $\Theta(\mathbb{Z}_{2q})$ which are in B then $\Theta(\mathbb{Z}_{2q})$ has $|A| = \varphi(pq)$ components. Since $2 < p < q$, we obtain,

$$\begin{aligned}
(p-2)(q-2) > 2 &\implies pq - 2p - 2q + 2 > 0 \\
&\implies pq + 2 > 2(p+q) \\
&\implies pq - p - q + 1 > p + q - 1 \\
&\implies (p-1)(q-1) > p + q - 1 \\
&\implies \varphi(pq) > pq - \varphi(pq) \\
&\implies |A| > |B|.
\end{aligned} \tag{3.2}$$

Using Equation (3.2) we find that $\Theta(\mathbb{Z}_{pq})$ is not 1-tough. Using Theorem 2.5 we conclude that $\Theta(\mathbb{Z}_{pq})$ is not Hamiltonian when $2 < p < q$. Thus, $\Theta(\mathbb{Z}_{pq})$ is Hamiltonian if and only if $p = 2$.

□

We observe that if two groups G_1 and G_2 are isomorphic, then the corresponding graphs $\Theta(G_1)$ and $\Theta(G_2)$ are isomorphic to each other. However, the converse is false. To illustrate it we consider the following example:

Example 3.11. *Consider the unitriangular matrix group* $\mathfrak{F} = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{F}_3 \right\}$ *where* $\mathbb{F}_3$ *denotes the finite field of order* 3*. Clearly* $\mathfrak{F}$ *forms a group under matrix multiplication. Now consider the group* $(\mathbb{Z}_3)^3$*. Using Corollary 3.7,* $\Theta((\mathbb{Z}_3)^3)$ *is complete. We also notice that each non-identity element of the group* $\mathfrak{F}$ *has order* 3*. Thus, any two elements of* $\Theta(\mathfrak{F})$ *are adjacent to each other which in turn implies that* $\Theta(\mathfrak{F})$ *is complete. Since* $\mathfrak{F}$ *is non-commutative whereas* $(\mathbb{Z}_3)^3$ *is commutative, we find that the two groups are not isomorphic to each other. However,* $\Theta(\mathfrak{F})$ *and* $\Theta((\mathbb{Z}_3)^3)$ *are isomorphic to each other as both are complete graphs having* 27 *elements.*

4. Vertex Connectivity of $\Theta(G)$

In this section, we first investigate the vertex connectivity of $\Theta(\mathbb{Z}_n)$ for $n \geq 2$. For a given group G, we fix the following notations: Let $S^*(G)$ denote set of all those elements G which have prime order. Let $S(G) = \{e\} \cup S^*(G)$ where e denotes the identity element of G.

Proposition 4.1. *If n is prime, then $\kappa(\Theta(\mathbb{Z}_n)) = n - 1$.*

Proof. Since n is prime, $\Theta(\mathbb{Z}_n)$ is complete (Corollary 3.6). Since vertex connectivity of a complete graph on n vertices is $n - 1$, we conclude that $\kappa(\Theta(\mathbb{Z}_n)) = n - 1$. $\square$

Theorem 4.2. *If n is composite, then $\kappa(\Theta(\mathbb{Z}_n)) = |S(\mathbb{Z}_n)|$.*

Proof. Let v_0 be a generator of $\mathbb{Z}_n$. Thus, $o(v_0) = n$. Since v_0 is a generator so $\deg(v_0) \leq \deg(w)$ for all vertices $w \in \Theta(\mathbb{Z}_n)$ which implies that $\delta(\Theta(\mathbb{Z}_n)) = \deg(v_0)$. Now we notice that the vertex $v_0 \in \Theta(\mathbb{Z}_n)$ is adjacent only to all the elements of $S(\mathbb{Z}_n)$ and nothing else. Thus, $\deg(v_0) = |S(\mathbb{Z}_n)|$. By Theorem 2.2, $\kappa(\Theta(\mathbb{Z}_n)) \leq |S(\mathbb{Z}_n)|$. Now we claim that $S(\mathbb{Z}_n)$ is a minimum separating set of $\Theta(\mathbb{Z}_n)$. If not, then suppose we remove $|S(\mathbb{Z}_n)| - 1$ elements from the vertex set of $\Theta(\mathbb{Z}_n)$. Then, there exists $a \in S(\mathbb{Z}_n)$ such that a is adjacent to all other vertices of $\Theta(\mathbb{Z}_n)$, making $\Theta(\mathbb{Z}_n)$ connected. Thus, $S(\mathbb{Z}_n)$ is a minimum separating set of $\Theta(\mathbb{Z}_n)$, which proves the fact that $\kappa(\Theta(\mathbb{Z}_n)) = |S(\mathbb{Z}_n)|$. $\square$

Corollary 4.3. *If $n = pq$ where p, q are distinct primes with $p < q$, then $\kappa(\Theta(\mathbb{Z}_n)) = p + q - 1$.*

Proof. If $n = pq$, then $S(\mathbb{Z}_{pq}) = \{0, p, 2p, 3p, \ldots, (q-1)p, q, 2q, 3q, \ldots, (p-1)q\}$. Since $|S(\mathbb{Z}_{pq})| = p + q - 1$, the result follows. $\square$

Corollary 4.4. *If $n = p^m$ where p is a prime and $m \in \mathbb{N}$ then $\kappa(\Theta(\mathbb{Z}_n)) = p$.*

Proof. If $n = p^m$, then $S(\mathbb{Z}_{p^m}) = \{0, p^{m-1}, 2p^{m-1}, 3p^{m-1}, \ldots, (p-1)p^{m-1}\}$. Since $|S(\mathbb{Z}_{p^m})| = p$, the result follows. $\square$

Now it is quite natural to ask that if $\Theta(G)$ is not complete, is it true that $\kappa(\Theta(G))$ equals $|S(G)|$? We show that it is false. Consider the Dicyclic group Dic_n of order $4n$ given by :

$$\mathrm{Dic}_n = \{\langle a, x\rangle : a^{2n} = 1, x^2 = a^n, ax = xa^{-1}\}.$$

We illustrate $\Theta(\mathrm{Dic}_3)$ in Figure 3.

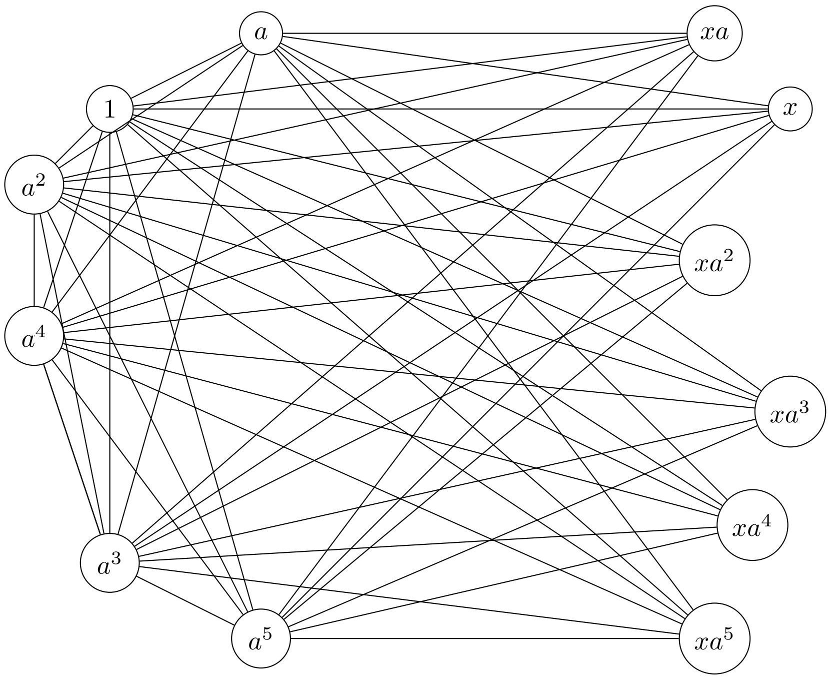

Figure 3: $\Theta(\mathrm{Dic}_3)$

From Figure 3, we observe that $\{1, a, a^2, a^3, a^4, a^5\}$ is a minimum separating set of $\Theta(\mathrm{Dic}_3)$. So, $\kappa(\Theta(\mathrm{Dic}_3)) = 6$. Also, $S(\mathrm{Dic}_n) = \{1, a^2, a^3, a^4\}$. Hence, $\kappa(\Theta(\mathrm{Dic}_3)) > |S(\mathrm{Dic}_3)|$.

We thus end this section by proposing the following open problem which can be considered for further research.

Problem 4.5. *Characterize all finite groups G, such that $\Theta(G)$ is not complete, but $\kappa(\Theta(G)) = |S(G)|$.*

5. Signless Laplacian Spectrum of $\Theta(G)$

In this section, we shall find the signless Laplacian spectra of $\Theta(\mathbb{Z}_n)$ and $\Theta(\mathrm{D}_n)$ for $n \in \{pq, p^m\}$ where p, q are distinct primes with $p < q$ and $m \in \mathbb{N}$. We denote the signless Laplacian matrix of $\Theta(G)$ by $Q = Q(\Theta(G))$.

5.1. Signless Laplacian Spectrum of $\Theta(\mathbb{Z}_n)$

Theorem 5.1. *If $n = pq$, then the eigenvalues of $Q(\Theta(\mathbb{Z}_n))$ are $p+q-1$ with multiplicity $pq-p-q$, $pq-2$ with multiplicity $p+q-2$ and the other two are solutions of the equation $x^2 - x(pq+2p+2q-4) + 2(p+q-1)(p+q-2) = 0$.*

Proof. The rows and columns of $Q = Q(\Theta(\mathbb{Z}_n))$ have been indexed in the following way:

We start with the zero element 0 of $\mathbb{Z}_n$. We then list those elements $m \in \mathbb{Z}_n$ such that $\gcd(m, n) \neq 1$. Finally, we list those elements $m \in \mathbb{Z}_n$ such that $\gcd(m, n) = 1$. Using the above indexing, Q takes the following form,

$$Q = \begin{pmatrix} ((n-2)I + J)_{(n-\varphi(n))\times(n-\varphi(n))} & J_{(n-\varphi(n))\times\varphi(n)} \\ J^T_{\varphi(n)\times(n-\varphi(n))} & (n-\varphi(n))I_{\varphi(n)\times\varphi(n)} \end{pmatrix}. \tag{5.1}$$

Here, $J_{m\times n}$ is a matrix of order $m \times n$ all of whose entries are 1.
If we consider the matrix $Q - (n-2)I$, we obtain

$$Q - (n-2)I = \begin{pmatrix} J_{n-\varphi(n)} & J_{(n-\varphi(n))\times\varphi(n)} \\ J^T_{\varphi(n)\times(n-\varphi(n))} & (2-\varphi(n))I_{\varphi(n)} \end{pmatrix}. \tag{5.2}$$

Since the matrix in eq. (5.2) has $n - \varphi(n)$ identical rows, we conclude that $n-2$ is an eigenvalue of Q with multiplicity at least $n - \varphi(n) - 1$. Similarly, if we consider the matrix $Q - (n - \varphi(n))I$ we find that it has $\varphi(n)$ identical rows, which makes us conclude that $n-\varphi(n)$ is an eigenvalue of Q with multiplicity at least $\varphi(n) - 1$. We shall use the concept of *equitable partitions* (see [7, Section 5]) to find the remaining two eigenvalues of Q. In short, given a graph $\mathcal{G}$, a partition π of $V(\mathcal{G}) = V_1 \cup V_2 \cup \cdots \cup V_k$ is an equitable partition of $\mathcal{G}$, if every vertex in V_i has the same number of neighbors in V_j for all $i, j \in \{1, 2, \ldots, k\}$. Also, given an equitable partition π of $\mathcal{G}$, and its signless Laplacian matrix Q, we can form the matrix $Q_\pi = (q_{ij})$ in the following way

$$q_{ij} = \begin{cases} b_{ij} & \text{if } i \neq j \\ b_{ii} + \sum_{j=1}^{k} b_{ij} & \text{if } i = j \end{cases} \tag{5.3}$$

where b_{ij} is the number of neighbors a vertex $v \in V_i$ has in V_j, and b_{ii} is the number of neighbors a vertex $v \in V_i$ has in V_i. We refer to the matrix Q_π as the matrix corresponding to the partition π of $\mathcal{G}$. It is further known that the multiset of eigenvalues of Q_π is contained in the multiset of eigenvalues of Q [7, Lemma 5.1].

We partition the vertex set V of $\Theta(\mathbb{Z}_n)$ as $V_1 \cup V_2$ where $V_1 = \{0\} \cup \{m : \gcd(m, n) \neq 1\}$ and $V_2 = V \setminus V_1$. We notice that each vertex v in V_1 has $n - \varphi(n) - 1$ neighbors in V_1, and $\varphi(n)$ neighbors in V_2. Also, each vertex v in V_2 has $n - \varphi(n)$ neighbors in V_1, and 0 neighbors in V_2. We call this partition π. Using eq. (5.3), we can construct the *equitable quotient* matrix Q_π corresponding to this partition π of $\Theta(\mathbb{Z}_n)$,

$$Q_\pi = \begin{pmatrix} 2n - \varphi(n) - 2 & \varphi(n) \\ n - \varphi(n) & n - \varphi(n) \end{pmatrix}.$$

The characteristic polynomial of Q_π is $\Lambda(x) = x^2 + x(2 - 3n + 2\varphi(n)) + (2n - 2\varphi(n))(n - 1 - \varphi(n))$. The solutions of $\Lambda(x) = 0$ are $\frac{1}{2}\{3n - 2 - 2\varphi(n) \pm$

$\sqrt{4n\varphi(n) - 4\varphi(n)^2 + n^2 - 4n + 4}\}$. Since $n > 2$, so $\varphi(n) \geq 2$. Hence we have

$$\Lambda(n - \phi(n)) = \phi(n)(\phi(n) - n) \neq 0$$
$$\text{and } \Lambda(n - 2) = 2(\phi(n) - 1)(\phi(n) - n) \neq 0.$$

Since the eigenvalues of Q_π are different from $n - \varphi(n)$ and $n - 2$, using Lemma 5.1 of [7] we find that the remaining eigenvalues of Q are $\frac{1}{2}\{3n - 2 - 2\varphi(n) \pm \sqrt{4n\varphi(n) - 4\varphi(n)^2 + n^2 - 4n + 4}\}$. Thus, the eigenvalues of Q are $n - \varphi(n)$ with multiplicity $\varphi(n) - 1$, $n - 2$ with multiplicity $n - \varphi(n) - 1$, and the other two are solutions of the equation $x^2 + x(2 - 3n + 2\varphi(n)) + (2n - 2\varphi(n))(n - 1 - \varphi(n)) = 0$. On substituting $n = pq$, we find that the eigenvalues of Q are $p + q - 1$ with multiplicity $pq - p - q$, $pq - 2$ with multiplicity $p + q - 2$, and the other two are solutions of the equation $x^2 - x(pq + 2p + 2q - 4) + 2(p + q - 1)(p + q - 2) = 0$, and hence the result follows.

□

Proposition 5.2. *If $n = p$, then the eigenvalues of $Q(\Theta(\mathbb{Z}_n))$ are $2(n-1)$ with multiplicity 1 and $n - 2$ with multiplicity $n - 1$.*

Proof. If $n = p$, then using Corollary 3.6, $\Theta(\mathbb{Z}_n)$ is complete. Thus, $Q = (n-2)I + J$. Using Theorem 2.6, the eigenvalues of Q are $2(n-1)$ with multiplicity 1 and $n - 2$ with multiplicity $n - 1$. □

Theorem 5.3. *If $n = p^m$, where $m \geq 2$, then the eigenvalues of $Q(\Theta(\mathbb{Z}_n))$ are p with multiplicity $p^m - p - 1$, $p^m - 2$ with multiplicity $p - 1$, and the other two are given by the solutions of the equation $x^2 - x(p^m + 2p - 2) + 2p(p-1) = 0$.*

Proof. The rows and columns of the matrix Q have been indexed in the following way,
We start with the zero element 0 of $\mathbb{Z}_n$. We then list the following elements of $\mathbb{Z}_n$:

$$\{p^{m-1}, 2p^{m-1}, 3p^{m-1}, \ldots, (p-2)p^{m-1}, (p-1)p^{m-1}\}.$$

We then list the remaining non-generators of $\mathbb{Z}_n$, and finally we list the generators of $\mathbb{Z}_n$. Since each element of the set $\{p^{m-1}, 2p^{m-1}, 3p^{m-1}, \ldots, (p-2)p^{m-1}, (p-1)p^{m-1}\}$ has order p we find that they are adjacent to all other vertices of $\Theta(\mathbb{Z}_n)$. Using the above indexing, Q takes the following form,

$$Q = \begin{pmatrix} ((n-2)I + J)_{p \times p} & J_{p \times (n-p)} \\ \\ J^T_{(n-p) \times p} & pI_{(n-p) \times (n-p)} \end{pmatrix}.$$

If we consider the matrix $Q - (n-2)I$, we find that it has p identical rows, and hence $n - 2$ is an eigenvalue of Q with multiplicity at least $p - 1$. Similarly, if we consider the matrix $Q - pI$, we find that it has $p^m - p$ identical rows, and hence p is an eigenvalue of Q with multiplicity at least $p^m - p - 1$.

We again partition the vertex set V of $\Theta(\mathbb{Z}_n)$ as $V = V_1 \cup V_2$ where $V_1 = \{0, p^{m-1}, 2p^{m-1}, \ldots, (p-2)p^{m-1}, (p-1)p^{m-1}\}$ and $V_2 = V \setminus V_1$. We observe that each vertex v in V_1 has $p-1$ neighbors in V_1 and $p^m - p$ neighbors in V_2. Similarly, each vertex v in V_2 has p neighbors in V_1 and 0 neighbors in V_2. Thus, the partition is an equitable partition, and we call the partition π.

According to eq. (5.3), the equitable quotient matrix corresponding to the partition π becomes

$$Q_\pi = \begin{pmatrix} p^m + p - 2 & p^m - p \\ p & p \end{pmatrix}$$

The characteristic polynomial of Q_π is

$$\Lambda(x) = x^2 - x(p^m + 2p - 2) + 2p(p-1).$$

Note that for a given p and $m \geq 2$,

$$\begin{aligned} \Lambda(p) &= p(p - p^m) \neq 0 \\ \text{and } \Lambda(p^m - 2) &= -2p(p^m - p - 1) \neq 0 \end{aligned}$$

The solutions of $\Lambda(x) = 0$ are $\frac{1}{2}\left(p^m + 2p - 2 \pm \sqrt{(p^m + 2p - 2)^2 - 8p(p-1)}\right)$. Since the eigenvalues of Q_π are different from p and $p^m - 2$, using Lemma 5.1 of [7] we find that the eigenvalues of Q are p with multiplicity $p^m - p - 1$, $p^m - 2$ with multiplicity $p - 1$, and the other two are solutions of the equation $x^2 - x(p^m + 2p - 2) + 2p(p-1) = 0$.

□

5.2. Signless Laplacian Spectrum of $\Theta(\mathrm{D}_n)$

In this section, we shall find the signless Laplacian spectrum of $\Theta(\mathrm{D}_n)$. We know that

$$\mathrm{D}_n = \{\langle r, s\rangle : r^n = s^2 = 1, rs = sr^{-1}\}.$$

We first index the elements r^i, and then index the elements sr^i where $0 \leq i \leq n-1$. We also note that $o(sr^i) = 2$ for all $0 \leq i \leq n-1$, and hence $\gcd(o(sr^i), o(sr^j)) = 2$ for all $0 \leq i, j \leq n-1$. Also, r^i is adjacent to sr^j for all $0 \leq i, j \leq n-1$. The signless Laplacian matrix of $\Theta(\mathrm{D}_n)$ is given by:

$$Q(\Theta(\mathrm{D}_n)) = \begin{pmatrix} (Q(\Theta(\mathbb{Z}_n)) + nI)_{n\times n} & J_{n\times n} \\ J^T_{n\times n} & ((2n-2)I + J)_{n\times n} \end{pmatrix}. \tag{5.4}$$

Using eq. (5.4), we find that the signless Laplacian matrix of $\Theta(D_n)$ depends on the signless Laplacian matrix of $\Theta(\mathbb{Z}_n)$. In the previous section we had determined the eigenvalues of $Q(\Theta(\mathbb{Z}_n))$ for $n \in \{p^m, pq\}$. We will use those in this section to find the eigenvalues of $Q(\Theta(D_n))$ for $n \in \{p^m, pq\}$.

Theorem 5.4. *If $n = pq$, then the eigenvalues of $Q(\Theta(D_n))$ are $2(n-1)$ with multiplicity $2n - \varphi(n) - 1$, $2n - \varphi(n)$ with multiplicity $\varphi(n) - 1$, and $3n - \varphi(n) - 1 \pm \sqrt{n^2 + 2\varphi(n)n - \varphi(n)^2 - 2n + 1}$, each with multiplicity 1.*

Proof. If $n = pq$, using eqs. (5.1) and (5.4), we find that the signless Laplacian matrix of $\Theta(\mathrm{D}_n)$ is of the following form:

$$Q = \begin{pmatrix} ((2n-2)I+J)_{(n-\varphi(n))^2} & J_{(n-\varphi(n)\times\varphi(n))} & J_{(n-\varphi(n)\times n)} \\ J_{\varphi(n)\times(n-\varphi(n))} & (2n-\varphi(n))I_{(\varphi(n))^2} & J_{\varphi(n)\times n} \\ J_{n\times(n-\varphi(n))} & J_{n\times\varphi(n)} & ((2n-2)I+J)_{n^2} \end{pmatrix}.$$

If we consider the matrix $Q-(2n-2)I$, we find that it has $2n-\varphi(n)$ identical rows. Thus, $2n-2$ is an eigenvalue of Q with multiplicity at least $2n-\varphi(n)-1$. We also note that $Q-(2n-\varphi(n))I$ has $\varphi(n)$ identical rows, which makes us conclude that $2n-\varphi(n)$ is an eigenvalue of Q with multiplicity at least $\varphi(n)-1$.

We partition the vertex set V of $\Theta(\mathrm{D}_n)$ in the following way: $V_1 = \{1\}\cup\{r^i : \gcd(i,n)\neq 1\}$, $V_2 = \{r^i : \gcd(i,n) = 1\}$ and $V_3 = \{sr^i : 0 \le i \le n-1\}$. Each vertex $v \in V_1$ has $n-\varphi(n)-1$ neighbors in V_1, $\varphi(n)$ neighbors in V_2 and n neighbors in V_3. Each vertex $v \in V_2$ has $n-\varphi(n)$ neighbors in V_1, 0 neighbors in V_2 and n neighbors in V_3, and each vertex $v \in V_3$ has $n-\varphi(n)$ neighbors in V_1, $\varphi(n)$ neighbors in V_2 and $n-1$ neighbors in V_3. Hence, the partition is an equitable partition, and we call it π. Using eq. (5.3), the equitable quotient matrix corresponding to π is given by:

$$Q_\pi = \begin{pmatrix} 3n-2-\varphi(n) & \varphi(n) & n \\ n-\varphi(n) & 2n-\varphi(n) & n \\ n-\varphi(n) & \varphi(n) & 3n-2 \end{pmatrix}.$$

The characteristic polynomial of Q_π is

$$\begin{aligned}\Lambda(x) = x^3 &+ (2\varphi(n)-8n+4)x^2 + (2\varphi(n)^2 - 12\varphi(n)n \\ &+ 20n^2 + 6\varphi(n) - 20n + 4)x - 4\varphi(n)^2 n + 16\varphi(n)n^2 - 16n^3 \\ &+ 4\varphi(n)^2 - 20\varphi(n)n + 24n^2 + 4\varphi(n) - 8n.\end{aligned}$$

On solving, we find that solutions of $\Lambda(x) = 0$ are $2(n-1)$ with multiplicity 1 and $3n-\varphi(n)-1 \pm \sqrt{n^2+2\varphi(n)n-\varphi(n)^2-2n+1}$, each with multiplicity 1. We further note that $\Lambda(2n-\varphi(n)) \neq 0$, as otherwise

$$\begin{aligned} 2n-\varphi(n) &= 3n-\varphi(n)-1 \pm \sqrt{n^2+2\varphi(n)n-\varphi(n)^2-2n+1} \\ &\text{which implies } (n-1)^2 = n^2+2n\varphi(n)-\varphi(n)^2-2n+1 \\ &\text{which implies } 2n\varphi(n)-\varphi(n)^2 = 0 \\ &\text{which implies } (2n-\varphi(n))\varphi(n) = 0 \text{ which is false for } n = pq. \end{aligned}$$

Again, we further note that

$2n-2 \neq 3n-\varphi(n)-1 \pm \sqrt{n^2+2\varphi(n)n-\varphi(n)^2-2n+1}$, as otherwise

$$2n-2 = 3n-\varphi(n)-1 \pm \sqrt{n^2+2\varphi(n)n-\varphi(n)^2-2n+1}$$

$$\text{which implies } \Big(n+1-\varphi(n)\Big)^2 = n^2+2\varphi(n)n-\varphi(n)^2-2n+1$$

$$\text{which implies } 2\varphi(n)^2+4n-2\varphi(n)-4n\varphi(n)=0$$

$$\text{which implies } \Big(\varphi(n)-2n\Big)\Big(\varphi(n)-1\Big)=0$$

which implies either $\varphi(n)=2n$ or $\varphi(n)=1,$ which are both false.

We thus conclude that the eigenvalues $3n-\varphi(n)-1 \pm \sqrt{n^2+2\varphi(n)n-\varphi(n)^2-2n+1}$ of Q_π are distinct from both $2n-\varphi(n)$ and $2(n-1)$ for $n=pq$. Using Lemma 5.1 of [7], we find that $3n-\varphi(n)-1 \pm \sqrt{n^2+2\varphi(n)n-\varphi(n)^2-2n+1}$ are eigenvalues of Q, each with multiplicity 1. Thus, the eigenvalues of Q are $2(n-1)$ with multiplicity $2n-\varphi(n)-1$, $2n-\varphi(n)$ with multiplicity $\varphi(n)-1$ and $3n-\varphi(n)-1 \pm \sqrt{n^2+2\varphi(n)n-\varphi(n)^2-2n+1}$, each with multiplicity 1.

□

Proposition 5.5. *If $n=p$, then the eigenvalues of $Q(\Theta(D_n))$ are $2(n-1)$ with multiplicity 1 and $n-2$ with multiplicity $n-1$.*

Proof. If $n=p$, then using Corollary 3.8, $\Theta(\mathrm{D}_n)$ is complete. Thus, $Q=(n-2)I+J$. Using Theorem 2.6, the eigenvalues of Q are $2(n-1)$ with multiplicity 1 and $n-2$ with multiplicity $n-1$. □

Theorem 5.6. *If $n=p^m$ where $m \geq 2$, then the eigenvalues of $Q(\Theta(D_n))$ are $2n-2$ with multiplicity $n+p-1$, $p+n$ with multiplicity $n-p-1$, and $2n+p-1 \pm \sqrt{2n^2-2n-p^2+1}$ each with multiplicity 1.*

Proof. If $n=p^m$, using eqs. (5.1) and (5.4) we find that $Q=Q(\Theta(\mathrm{D}_n))$ is of the following form:

$$Q=\begin{pmatrix} ((2n-2)I+J)_{p\times p} & J_{p\times(n-p)} & J_{p\times n} \\ J_{(n-p)\times p} & (p+n)I_{(n-p)\times(n-p)} & J_{(n-p)\times n} \\ J_{n\times p} & J_{n\times(n-p)} & ((2n-2)I+J)_{n\times n} \end{pmatrix}.$$

We note that $Q-(2n-2)I$ has $n+p$ identical rows, and hence $2n-2$ is an eigenvalue of Q with multiplicity at least $n+p-1$. Similarly, $Q-(p+n)I$ has $n-p$ identical rows, and hence $p+n$ is an eigenvalue of Q with multiplicity at least $n-p-1$.

We now partition the vertex set V of $\Theta(\mathrm{D}_{p^m})$ as $V=V_1\cup V_2\cup V_3$ where

$$V_1=\{1, r^{p^{m-1}}, r^{2(p^{m-1})}, \ldots, r^{(p-1)(p^{m-1})}\},$$

$V_2 = \{r^i : 0 \le i \le n-1\} \setminus V_1$, and $V_3 = \{sr^i : 0 \le i \le n\}$. Each vertex $v \in V_1$ has $p-1$ neighbors in V_1, $n-p$ neighbors in V_2 and n neighbors in V_3. Each vertex $v \in V_2$ has p neighbors in V_1, 0 neighbors in V_2 and n neighbors in V_3 and each vertex $v \in V_3$ has p neighbors in V_1, $n-p$ neighbors in V_2 and $n-1$ neighbors in V_3. Hence, the partition is an equitable partition, and we call it π.

The equitable quotient matrix of Q corresponding to π is given by:

$$Q_\pi = \begin{pmatrix} 2(n-1)+p & n-p & n \\ p & n+p & n \\ p & n-p & 3n-2 \end{pmatrix}.$$

The characteristic polynomial of Q_π is

$$\begin{aligned}\Lambda(x) &= x^3 + (-6n-2p+4)x^2 + (10n^2+8np+2p^2-14n-6p+4)x \\ &\quad - 4n^3 - 8n^2p - 4np^2 + 8n^2 + 12np + 4p^2 - 4n - 4p.\end{aligned}$$

The solutions of $\Lambda(x) = 0$ are $2(n-1)$ and $2n+p-1 \pm \sqrt{2n^2-2n-p^2+1}$. We further note that, $\Lambda(n+p) \neq 0$, as otherwise it would imply,

$$\begin{aligned} n+p &= 2n+p-1 \pm \sqrt{2n^2-2n-p^2+1} \\ \text{which implies } \; -n+1 &= \pm\sqrt{2n^2-2n-p^2+1}. \\ \text{which implies } (n-1)^2 &= 2n^2-2n-p^2+1 \\ \text{which implies } n^2 &= p^2, \text{ which is false.} \end{aligned}$$

We further note that $2(n-1) \neq 2n+p-1 \pm \sqrt{2n^2-2n-p^2+1}$, as otherwise

$$\begin{aligned} 2(n-1) &= 2n+p-1 \pm \sqrt{2n^2-2n-p^2+1} \\ \text{which implies } \; -p-1 &= \pm\sqrt{2n^2-2n-p^2+1} \\ \text{which implies } (p+1)^2 &= 2n^2-2n-p^2+1 \\ \text{which implies } 2p(p+1) &= 2n(n-1) \\ \text{which implies } p+1 &= p^{m-1}(p^m-1) \\ \text{which implies } \frac{p^m-1}{p+1} &= \frac{1}{p^{m-1}} < 1, \text{ which is false.} \end{aligned}$$

Thus, the eigenvalues $2n+p-1 \pm \sqrt{2n^2-2n-p^2+1}$ of Q_π are different from both $n+p$ and $2(n-1)$ for $n = p^m$. Using Lemma 5.1 of [7], we conclude that $2n+p-1 \pm \sqrt{2n^2-2n-p^2+1}$ are eigenvalues of Q each with multiplicity 1. Thus, the eigenvalues of Q are $2n-2$ with multiplicity $n+p-1$, $p+n$ with multiplicity $n-p-1$, and $2n+p-1 \pm \sqrt{2n^2-2n-p^2+1}$ each with multiplicity 1.

□

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