# ( $C, B$ )-resolvents of closed linear operators 

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#### Abstract

In this note, we analyze $(C, B)$-resolvents of closed linear operators in sequentially complete locally convex spaces. We provide a simple application in the qualitative analysis of solutions of abstract degenerate Volterra integro-differential equations.


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## 1. Introduction and preliminaries

The theory of abstract degenerate Volterra integro-differential equations and abstract degenerate fractional differential equations are growing fields of research. For further information about these subjects, we refer the reader to the recent monograph [6] by M. Kostić and references cited therein.

As mentioned in the abstract, the main aim of this paper is to investigate $(C, B)$-resolvents of closed linear operators in sequentially complete locally convex spaces. In such a way, we continue our recent analyses of $C$-resolvents of multivalued linear operators [6] and $C$-generalized resolvents of linear operators [8] (joint research with S. Pilipovic and D. Velinov); see also the papers [2] by R. deLaubenfels, F. Yao, S. W. Wang and 9 by Y.-C. Li and S.-Y. Shaw. We provide an illustrative example of application in the analysis of existence and uniqueness of strong solutions of abstract degenerate Volterra integro-differential equations.

We use the standard notation throughout the paper. If not stated otherwise, by $E$ we denote a complex sequentially complete locally convex space, SCLCS for short. If $\emptyset \neq \Omega \subseteq \mathbb{C}$, then by $C(\Omega: E)$ we denote the vector space consisting of all continuous functions from $\Omega$ into $E$. By $A, B$ we denote two closed linear operators with domain and range contained in $E$; the domain, kernel space and range of $A$ are denoted by $D(A), N(A)$ and $R(A)$, respectively. If $0<\tau \leq \infty$ and $a \in L_{l o c}^{1}([0, \tau))$, then we say that the function $a(t)$ is a kernel on $[0, \tau)$ iff for each $f \in C([0, \tau))$ the assumption $\int_{0}^{t} a(t-s) f(s) d s=0, t \in[0, \tau)$ implies $f(t)=0, t \in[0, \tau)$. We will use the following condition
(P1): $a(t)$ is Laplace transformable, i.e., it is locally integrable on $[0, \infty)$ and there exists $\beta \in \mathbb{R}$ such that
$\tilde{a}(\lambda):=\mathcal{L}(a)(\lambda):=\lim _{b \rightarrow \infty} \int_{0}^{b} e^{-\lambda t} a(t) d t:=\int_{0}^{\infty} e^{-\lambda t} a(t) d t$ exists for all $\lambda \in \mathbb{C}$ with $\Re \lambda>\beta$.

[^0]Consider the following abstract degenerate Volterra integral equation:

$$
\begin{equation*}
B u(t)=f(t)+\int_{0}^{t} a(t-s) A u(s) d s, t \in[0, \tau) \tag{1.1}
\end{equation*}
$$

where $0<\tau \leq \infty, t \mapsto f(t), t \in[0, \tau)$ is a continuous mapping with values in $E$ and $a \in L_{l o c}^{1}([0, \tau))$. We will use the following definition from [6]:

Definition 1.1. Let $0<\tau \leq \infty$. A function $u \in C([0, \tau): E)$ is said to be:
(i) a (mild) solution of 1.1 iff $(a * u)(t) \in D(A), t \in[0, \tau), A(a * u)(t)=$ $B u(t)-f(t), t \in[0, \tau)$ and the mapping $t \mapsto B u(t), t \in[0, \tau)$ is continuous,
(ii) a strong solution of 1.1 iff the mapping $t \mapsto A u(t), t \in[0, \tau)$ is continuous, $(a * A u)(t)=B u(t)-f(t), t \in[0, \tau)$ and the mapping $t \mapsto B u(t)$, $t \in[0, \tau)$ is continuous,

Multivalued linear operators. Now we will collect the basic definitions and properties of multivalued linear operators in SCLCSs. Let $X$ and $Y$ be two SCLCSs; by $P(Y)$ we denote the power set of $Y$. A multivalued map (multimap) $\mathcal{A}: X \rightarrow P(Y)$ is said to be a multivalued linear operator (MLO) iff the following holds:
(i) $D(\mathcal{A}):=\{x \in X: \mathcal{A} x \neq \emptyset\}$ is a linear subspace of $X$;
(ii) $\mathcal{A} x+\mathcal{A} y \subseteq \mathcal{A}(x+y), x, y \in D(\mathcal{A})$ and $\lambda \mathcal{A} x \subseteq \mathcal{A}(\lambda x), \lambda \in \mathbb{C}, x \in D(\mathcal{A})$.

If $X=Y$, then we say that $\mathcal{A}$ is an MLO in $X$. An almost immediate consequence of the definition is that, for any $x, y \in D(\mathcal{A})$ and $\lambda, \eta \in \mathbb{C}$ with $|\lambda|+|\eta| \neq 0$, we have $\lambda \mathcal{A} x+\eta \mathcal{A} y=\mathcal{A}(\lambda x+\eta y)$. If $\mathcal{A}$ is an MLO, then $\mathcal{A} 0$ is a linear manifold in $Y$ and $\mathcal{A} x=f+\mathcal{A} 0$ for any $x \in D(\mathcal{A})$ and $f \in \mathcal{A} x$. Set $R(\mathcal{A}):=\{\mathcal{A} x: x \in D(\mathcal{A})\}$. The set $\mathcal{A}^{-1} 0=\{x \in D(\mathcal{A}): 0 \in \mathcal{A} x\}$ is called the kernel of $\mathcal{A}$ and it is denoted henceforth by $N(\mathcal{A})$ or $\operatorname{Kern}(\mathcal{A})$. The inverse $\mathcal{A}^{-1}$ of an MLO is defined by $D\left(\mathcal{A}^{-1}\right):=R(\mathcal{A})$ and $\mathcal{A}^{-1} y:=\{x \in D(\mathcal{A}): y \in \mathcal{A} x\}$. It is checked at once that $\mathcal{A}^{-1}$ is an MLO in $X$, as well as that $N\left(\mathcal{A}^{-1}\right)=\mathcal{A} 0$ and $\left(\mathcal{A}^{-1}\right)^{-1}=\mathcal{A}$.

If $\mathcal{A}, \mathcal{B}: X \rightarrow P(Y)$ are two MLOs, then we define its sum $\mathcal{A}+\mathcal{B}$ by $D(\mathcal{A}+\mathcal{B}):=D(\mathcal{A}) \cap D(\mathcal{B})$ and $(\mathcal{A}+\mathcal{B}) x:=\mathcal{A} x+\mathcal{B} x, x \in D(\mathcal{A}+\mathcal{B})$. Then $\mathcal{A}+\mathcal{B}$ is likewise an MLO.

Assume that $\mathcal{A}: X \rightarrow P(Y)$ and $\mathcal{B}: Y \rightarrow P(Z)$ are two MLOs, where $Z$ is an SCLCS. The product of $\mathcal{A}$ and $\mathcal{B}$ is defined by $D(\mathcal{B A}):=\{x \in D(\mathcal{A})$ : $D(\mathcal{B}) \cap \mathcal{A} x \neq \emptyset\}$ and $\mathcal{B} \mathcal{A} x:=\mathcal{B}(D(\mathcal{B}) \cap \mathcal{A} x)$. Then $\mathcal{B} \mathcal{A}: X \rightarrow P(Z)$ is an MLO and $(\mathcal{B A})^{-1}=\mathcal{A}^{-1} \mathcal{B}^{-1}$. The scalar multiplication of an MLO $\mathcal{A}: X \rightarrow P(Y)$ with the number $z \in \mathbb{C}, z \mathcal{A}$ for short, is defined by $D(z \mathcal{A}):=D(\mathcal{A})$ and $(z \mathcal{A})(x):=z \mathcal{A} x, x \in D(\mathcal{A})$.

Let $C \in L(X)$ be injective, let $\mathcal{A}$ is an MLO in $X$, and let $C \mathcal{A} \subseteq \mathcal{A} C$. Then the $C$-resolvent set of $\mathcal{A}, \rho_{C}(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda \in \mathbb{C}$ for which
(i) $R(C) \subseteq R(\lambda-\mathcal{A})$;
(ii) $(\lambda-\mathcal{A})^{-1} C$ is a single-valued linear continuous operator on $X$.

The operator $\lambda \mapsto(\lambda-\mathcal{A})^{-1} C$ is called the $C$-resolvent of $\mathcal{A}\left(\lambda \in \rho_{C}(\mathcal{A})\right)$; the resolvent set of $\mathcal{A}$ is defined by $\rho(\mathcal{A}):=\rho_{I}(\mathcal{A}), R(\lambda: \mathcal{A}) \equiv(\lambda-\mathcal{A})^{-1}$ $(\lambda \in \rho(\mathcal{A}))$.

In [6, Section 3.9], we have also introduced and analyzed $C$-resolvents of multivalued linear operators in the case that the regularizing operator $C$ is not injective. In this paper, the injectiveness of the operator $C$ will be a blank hypothesis.

For further information concerning multivalued linear operators, the reader may consult the monographs [3], 6] and references cited therein.

## 2. The notion and properties of $(C, B)$-resolvents of closed lienar operators

In this section, we investigate the main structural properties of $(C, B)$ resolvents of closed linear operators in sequentially complete locally convex space $E$. We assume that:

1. $A: D(A) \subseteq E \rightarrow E$ and $B: D(B) \subseteq E \rightarrow E$ are closed linear operators;
2. $C \in L(E)$ is an injective operator satisfying $C A \subseteq A C$ and $C B \subseteq B C$;
3. The closed graph theorem holds for mappings from $E$ into $E$.

Then the set

$$
\rho_{C}^{B}(A):=\left\{\lambda \in \mathbb{C}:(\lambda B-A)^{-1} C \in L(E)\right\}
$$

is called the $(C, B)$-resolvent set of $A$; the $(C, B)$-spectrum of $A$ is defined by $\sigma_{C}^{B}(A):=\mathbb{C} \backslash \rho_{C}^{B}(A)$. Sometimes we also write $\rho_{C}(A, B)\left(\sigma_{C}(A, B)\right)$ for $\rho_{C}^{B}(A)$ $\left(\sigma_{C}^{B}(A)\right) ; \rho^{B}(A) \equiv \rho_{I}^{B}(A)$ and $\sigma^{B}(A) \equiv \sigma_{I}^{B}(A)$. If $C \neq I$, then the $(C, B)$ resolvent set of the operator $A$ need not be open (for a counterexample of this type, with $B=I, E$ being the Hardy space $H^{2}(\{z \in \mathbb{C}:|z| \leq 1\}), A \in L(E)$ being injective and $C=A$, see [2, Example 2.5]). For any $\lambda \in \rho_{C}^{B}(A)$, we define the right $(C, B)$-resolvent of $A, R_{\lambda}^{C, B}(A)$ for short, and the left $(C, B)$-resolvent of $A, L_{\lambda}^{C, B}(A)$ for short, by

$$
R_{\lambda}^{C, B}(A):=(\lambda B-A)^{-1} C B \text { and } L_{\lambda}^{C, B}(A):=B(\lambda B-A)^{-1} C \in L(E)
$$

It is checked at once that the existence of an operator $B^{-1} \in L(E)$ implies the closedness of the operator $A B^{-1}$, with domain and range contained in $E$, as well as $\rho_{C}^{B}(A) \subseteq \rho_{C}\left(A B^{-1}\right)$ and

$$
\begin{equation*}
\left(\lambda-A B^{-1}\right)^{-1} C=B(\lambda B-A)^{-1} C, \quad \lambda \in \rho_{C}^{B}(A) \tag{2.1}
\end{equation*}
$$

Now we will analyze the case in which $\rho_{C}^{B}(A) \neq \emptyset$, the operator $B$ is injective and $B^{-1} \notin L(E)$. Fix temporarily a number $\lambda \in \rho_{C}^{B}(A)$. Suppose that $\left(x_{\tau}\right)$ is a
net in $E$ as well as that $x_{\tau} \rightarrow x$ as $\tau \rightarrow \infty$ and $A B^{-1} x_{\tau} \rightarrow y$ as $\tau \rightarrow \infty$ (see 10 for the notion). This simply implies $B(\lambda B-A)^{-1} C x_{\tau} \rightarrow B(\lambda B-A)^{-1} C x$ as $\tau \rightarrow \infty$ and $B(\lambda B-A)^{-1} C\left(\lambda-A B^{-1}\right) x_{\tau}=B(\lambda B-A)^{-1} C(\lambda B-A) B^{-1} x_{\tau}=$ $C x_{\tau} \rightarrow B(\lambda B-A)^{-1} C(\lambda x-y)$ as $\tau \rightarrow \infty$. Hence, $C x=B(\lambda B-A)^{-1} C(\lambda x-y)$, $C x \in D\left(A B^{-1}\right)$ and $A B^{-1} C x=A B^{-1} B(\lambda B-A)^{-1} C(\lambda x-y)=A(\lambda B-$ $A)^{-1} C(\lambda x-y)$. Further on, $B(\lambda B-A)^{-1} C y=\lambda B(\lambda B-A)^{-1} C x-C x=$ $A(\lambda B-A)^{-1} C x,(\lambda B-A)^{-1} C y=B^{-1} A(\lambda B-A)^{-1} C x=-B^{-1} C x+\lambda(\lambda B-$ $A)^{-1} C x$, whence it easily follows that $C y=-(\lambda B-A) B^{-1} C x+C x$ and $C y=$ $A B^{-1} C x$. Hence, the operator $A B^{-1}$ is closable and the supposition $C^{-1} \in$ $L(E)$ implies that the operator $A B^{-1}$ is closed; before proceeding further, we want to observe that the operator $A B^{-1}$ need not be closed if the above requirements hold and $C^{-1} \notin L(E)$ (let $A=B=C$, and let $R(C)$ be a proper dense subspace of $E$; then $\overline{C C^{-1}}=I \neq C C^{-1}$, see [2, Example 2.2]). It is not problematic to verify that the operator $\overline{A B^{-1}}$ commutes with the operator $B(\lambda B-A)^{-1} C\left(\lambda \in \rho_{C}^{B}(A)\right)$, and that the operator $\lambda+\overline{A B^{-1}}$ is injective $\left(\lambda \in \rho_{C}^{B}(A)\right)$. By the foregoing, we have $\rho_{C}^{B}(A) \subseteq \rho_{C}\left(\overline{A B^{-1}}\right)$ and the following modification of 2.1):

$$
\begin{equation*}
\left(\lambda-\overline{A B^{-1}}\right)^{-1} C=B(\lambda B-A)^{-1} C, \quad \lambda \in \rho_{C}^{B}(A) \tag{2.2}
\end{equation*}
$$

If the operator $B$ is not injective, then $A B^{-1}$ is an MLO in $E$ and, in this case, we can simply prove that $\rho_{C}^{B}(A) \subseteq \rho_{C}\left(A B^{-1}\right)$ and 2.1 continues to hold. Therefore, we have arrived at the following propositions.
Proposition 2.1. Suppose that $\rho_{C}^{B}(A) \neq \emptyset$ and the operator $B$ is injective.
(i) If $B^{-1} \in L(E)$ or $C^{-1} \in L(E)$, then the operator $A B^{-1}$ is closed, $\rho_{C}^{B}(A) \subseteq \rho_{C}\left(A B^{-1}\right)$ and 2.1 holds.
(ii) Suppose $B^{-1} \notin L(E)$ and $C^{-1} \notin L(E)$. Then the operator $A B^{-1}$ is closable, $\rho_{C}^{B}(A) \subseteq \rho_{C}\left(\overline{A B^{-1}}\right)$ and 2.2 holds.
Proposition 2.2. Suppose that the operator $B$ is not injective. Then $A B^{-1}$ is an MLO in $E, \rho_{C}^{B}(A) \subseteq \rho_{C}\left(A B^{-1}\right)$ and 2.1 holds in the sense of multivalued linear operators.

The inclusion $\rho_{C}\left(A B^{-1}\right) \subseteq \rho_{C}^{B}(A)$ also holds in some cases (for example, if $B \in L(E)$ ), but we will not go into further details concerning this question here. Using the trivial identities

$$
\begin{aligned}
& (\lambda B-A)(\mu B-A)^{-1} C=C+(\lambda-\mu) B(\mu B-A)^{-1} C, \mu \in \rho_{C}^{B}(A), \lambda \in \mathbb{C} \\
& \quad(\mu B-A)^{-1} C(\lambda B-A) x=C x \\
& \quad+(\lambda-\mu) B(\mu B-A)^{-1} C x, \mu \in \rho_{C}^{B}(A), \lambda \in \mathbb{C}, x \in D(A) \cap D(B)
\end{aligned}
$$

and observing that for each $\lambda \in \rho_{C}^{B}(A)$ we have $B(\lambda B-A)^{-1} C^{2}=C B(\lambda B-$ $A)^{-1} C$, the following version of Hilbert resolvent equation readily follows:

$$
\begin{equation*}
(\lambda B-A)^{-1} C^{2}-(\mu B-A)^{-1} C^{2}=(\mu-\lambda)(\mu B-A)^{-1} C B(\lambda B-A)^{-1} C \tag{2.3}
\end{equation*}
$$

for any $\lambda, \mu \in \rho_{C}^{B}(A)$. From this, we may conclude the following:
(RE1) Suppose $\lambda, \mu \in \rho_{C}^{B}(A)$. Then

$$
\begin{equation*}
L_{\lambda}^{C, B}(A) C-L_{\mu}^{C, B}(A) C=(\mu-\lambda) L_{\mu}^{C, B}(A) L_{\lambda}^{C, B}(A) \tag{2.4}
\end{equation*}
$$

and

$$
L_{\mu}^{C, B}(A) L_{\lambda}^{C, B}(A)=L_{\lambda}^{C, B}(A) L_{\mu}^{C, B}(A)
$$

Hence, the nonemptiness of the set $\rho_{C}^{B}(A)$ implies that the function $\lambda \mapsto$ $L_{\lambda}^{C, B}(A) \in L(E), \lambda \in \rho_{C}^{B}(A)$ is a $C$-pseudoresolvent in the sense of 9, Definition 3.1] and the following holds ( 9 ):
(RE1)' The spaces $N\left(L_{\lambda}^{C, B}(A)\right), C^{-1}\left(R\left(L_{\lambda}^{C, B}(A)\right)\right), N\left(C-\lambda L_{\lambda}^{C, B}(A)\right)$ and $C^{-1}\left(R\left(C-\lambda L_{\lambda}^{C, B}(A)\right)\right)$ are independent of $\lambda \in \rho_{C}^{B}(A)$.
(RE1)" Suppose, additionally, that $N\left(L_{\lambda}^{C, B}(A)\right)=\{0\}$ for some $\lambda \in \rho_{C}^{B}(A)$. Then we can define the closed linear operator $W$ on $E$ by

$$
\begin{aligned}
D(W) & :=C^{-1}\left(R\left(L_{\lambda}^{C, B}(A)\right)\right) \\
& \text { and } W x:=\left(\lambda-\left(L_{\lambda}^{C, B}(A)\right)^{-1} C\right) x \text { for } x \in D(W)
\end{aligned}
$$

observe that (RE1)' implies that the definition of $W$ is independent of $\lambda \in \rho_{C}^{B}(A)$. Then $C^{-1} W C=W, \rho_{C}^{B}(A) \subseteq \rho_{C}(W)$ and $L_{\lambda}^{C, B}(A)=(\lambda-$ $W)^{-1} C, \lambda \in \rho_{C}^{B}(A)$.

It is well known that the existence of the operator $W$ from (RE1)" cannot be proved in the case when there exists $\lambda \in \rho_{C}^{B}(A)$ such that the kernel space of the operator $L_{\lambda}^{C, B}(A)$ is non-trivial (cf. also Example 2.6 below; then (RE1)" holds).

Further on, it is not difficult to prove that

$$
A(\lambda B-A)^{-1} C B x=B(\lambda B-A)^{-1} C A x, \quad x \in D(A) \cap D(B), \lambda \in \rho_{C}^{B}(A)
$$

and (see the second equality in [11, Lemma 2.1.2] with $C=I$ ):

$$
N\left(L_{\lambda}^{C, B}(A)\right)=C^{-1}[\{A x: x \in D(A) \cap N(B)\}], \quad \lambda \in \rho_{C}^{B}(A) .
$$

The proof of the next resolvent equation follows from (2.4) and the fact that, for every $x \in D(B)$, one has $B(\lambda B-A)^{-1} C B C x=C B(\lambda B-A)^{-1} C B x$ :
(RE2) Suppose $\lambda, \mu \in \rho_{C}^{B}(A)$ and $x \in D(B)$. Then

$$
R_{\lambda}^{C, B}(A) C x-R_{\mu}^{C, B}(A) C x=(\mu-\lambda) R_{\mu}^{C, B}(A) R_{\lambda}^{C, B}(A) x
$$

and

$$
R_{\mu}^{C, B}(A) R_{\lambda}^{C, B}(A) x=R_{\lambda}^{C, B}(A) R_{\mu}^{C, B}(A) x .
$$

Taking into account (RE2) and proceeding as in the proofs of 9, Lemma 3.2, Lemma 3.3], we can deduce the following:
(RE2)' The spaces $N\left(R_{\lambda}^{C, B}(A)\right), C^{-1}\left(R\left(R_{\lambda}^{C, B}(A)\right)\right) \cap D(B), N\left(C-\lambda R_{\lambda}^{C, B}(A)\right)$ and $C^{-1}\left(R\left(C-\lambda R_{\lambda}^{C, B}(A)\right)\right) \cap D(B)$ are independent of $\lambda \in \rho_{C}^{B}(A)$.
Furthermore, if $B \in L(E)$ is injective, then it is not difficult to show that the operator $B^{-1} A$ is closed, as well as that $\rho_{C}\left(B^{-1} A\right)=\rho_{C}^{B}(A)$ and

$$
\left(\lambda-B^{-1} A\right)^{-1} C x=(\lambda B-A)^{-1} C B x, \quad x \in E
$$

cf. also (RE1)", [9, Theorem 3.4] and [3, Theorem 1.15]. Making use of [5, Lemma 3.3], 2.3) and the argumentation from [2, Section 2] (cf. [2, Proposition 2.6, Remark 2.7]), we can prove the following:

Proposition 2.3. Let $\emptyset \neq \Omega \subseteq \rho_{C}^{B}(A)$ be open, and let $x \in E$.
(i) The local boundedness of the mapping $\lambda \mapsto B(\lambda B-A)^{-1} C x, \lambda \in \Omega$, resp. the assumption that $E$ is barreled and local boundedness of the mapping $\lambda \mapsto B(\lambda B-A)^{-1} C \in L(E), \lambda \in \Omega$, implies the analyticity of the mappings $\lambda \mapsto(\lambda B-A)^{-1} C^{3} x, \lambda \in \Omega$ and $\lambda \mapsto B(\lambda B-A)^{-1} C^{3} x, \lambda \in \Omega$, resp. $\lambda \mapsto(\lambda B-A)^{-1} C^{3} \in L(E), \lambda \in \Omega$ and $\lambda \mapsto B(\lambda B-A)^{-1} C^{3} \in$ $L(E), \lambda \in \Omega$. Furthermore, if $R(C)$ is dense in $E$, resp. if $R(C)$ is dense in $E$ and $E$ is barreled, then the mappings $\lambda \mapsto(\lambda B-A)^{-1} C x, \lambda \in \Omega$ and $\lambda \mapsto B(\lambda B-A)^{-1} C x, \lambda \in \Omega$ are analytic, resp. the mappings $\lambda \mapsto$ $(\lambda B-A)^{-1} C \in L(E), \lambda \in \Omega$ and $\lambda \mapsto B(\lambda B-A)^{-1} C \in L(E), \lambda \in \Omega$ are analytic.
(ii) The continuity of the mapping $\lambda \mapsto B(\lambda B-A)^{-1} C x, \lambda \in \Omega$ implies its analyticity. The continuity of mappings $\lambda \mapsto B(\lambda B-A)^{-1} C x, \lambda \in \Omega$ and $\lambda \mapsto(\lambda B-A)^{-1} C x, \lambda \in \Omega$ implies the analyticity of the mapping $\lambda \mapsto$ $(\lambda B-A)^{-1} C x, \lambda \in \Omega$; the strong continuity of the mapping $\lambda \mapsto(\lambda B-$ $A)^{-1} C \in L(E), \lambda \in \Omega\left(\lambda \mapsto(\lambda B-A)^{-1} C B, \lambda \in \Omega\right.$; with the meaning clear) implies the analyticity of the mapping $\lambda \mapsto(\lambda B-A)^{-1} C x, \lambda \in$ $\Omega\left(\lambda \mapsto(\lambda B-A)^{-1} C B x, \lambda \in \Omega\right.$, provided that $\left.x \in D(B)\right)$, as well. Furthermore, if $E$ is barreled, then the continuity of the mapping $\lambda \mapsto$ $(\lambda B-A)^{-1} C \in L(E), \lambda \in \Omega\left(\lambda \mapsto B(\lambda B-A)^{-1} C \in L(E), \lambda \in \Omega\right)$ implies its analyticity; the same conclusion holds for the mapping $\lambda \mapsto$ $(\lambda B-A)^{-1} C B \in L(E), \lambda \in \Omega$, provided that $E$ is barreled and $B \in L(E)$.
For clarity's sake, we will prove parts (i) and (ii) of the following extension of [2, Corollary 2.8].

Proposition 2.4. Let $\emptyset \neq \Omega \subseteq \rho_{C}^{B}(A)$ be open, and let $x \in E$.
(i) Suppose that the mapping $\lambda \mapsto(\lambda B-A)^{-1} C x, \lambda \in \Omega$ is analytic. Then, for every $n \in \mathbb{N}$ and $\lambda \in \Omega$, we have

$$
(\lambda B-A) \frac{d^{n}}{d \lambda^{n}}(\lambda B-A)^{-1} C x=(-n) B \frac{d^{n-1}}{d \lambda^{n-1}}(\lambda B-A)^{-1} C x, \quad n \in
$$

$C x \in D\left(\left((\lambda B-A)^{-1} B\right)^{n-1}(\lambda B-A)^{-1}\right)$, and

$$
\frac{d^{n-1}}{d \lambda^{n-1}}(\lambda B-A)^{-1} C x=(-1)^{n-1}(n-1)!\left((\lambda B-A)^{-1} B\right)^{n-1}(\lambda B-A)^{-1} C x
$$

If, in addition, the mapping $\lambda \mapsto B(\lambda B-A)^{-1} C x, \lambda \in \Omega$ is analytic, then for each $n \in \mathbb{N}$ and $\lambda \in \Omega, C x \in D\left(B\left((\lambda B-A)^{-1} B\right)^{n-1}(\lambda B-A)^{-1}\right)$, and

$$
\begin{aligned}
& \frac{d^{n-1}}{d \lambda^{n-1}} B(\lambda B-A)^{-1} C x \\
& \quad=(-1)^{n-1}(n-1)!B\left((\lambda B-A)^{-1} B\right)^{n-1}(\lambda B-A)^{-1} C x
\end{aligned}
$$

(ii) Suppose that the mapping $\lambda \mapsto B(\lambda B-A)^{-1} C x, \lambda \in \Omega$ is analytic. Then for each $n \in \mathbb{N}$ and $\lambda \in \Omega,\left((\lambda B-A)^{-1} B\right)^{n-1}(\lambda B-A)^{-1} C^{2} x \in R(C)$, $n \in \mathbb{N}, \lambda \in \Omega$ and

$$
\begin{aligned}
\frac{d^{n-1}}{d \lambda^{n-1}} & B(\lambda B-A)^{-1} C x \\
& =C^{-1}(-1)^{n-1}(n-1)!B\left((\lambda B-A)^{-1} B\right)^{n-1}(\lambda B-A)^{-1} C^{2} x
\end{aligned}
$$

(iii) Suppose that $E$ is barreled, and the mapping $\lambda \mapsto(\lambda B-A)^{-1} C \in$ $L(E), \lambda \in \Omega$ is analytic, resp., $B \in L(E)$ and the mapping $\lambda \mapsto(\lambda B-$ $A)^{-1} C B \in L(E), \lambda \in \Omega$ is analytic. Then for each $n \in \mathbb{N}$ and $\lambda \in \Omega$, $R(C) \subseteq D\left(\left((\lambda B-A)^{-1} B\right)^{n-1}(\lambda B-A)^{-1}\right)$, resp., $R(C B) \subseteq D\left(\left((\lambda B-A)^{-1} B\right)^{n-1}(\lambda B-A)^{-1}\right)$, and

$$
\begin{aligned}
\frac{d^{n-1}}{d \lambda^{n-1}} & (\lambda B-A)^{-1} C \\
& =(-1)^{n-1}(n-1)!\left((\lambda B-A)^{-1} B\right)^{n-1}(\lambda B-A)^{-1} C \in L(E)
\end{aligned}
$$

resp.,

$$
\begin{aligned}
& \frac{d^{n-1}}{d \lambda^{n-1}}(\lambda B-A)^{-1} C B \\
& \quad=(-1)^{n-1}(n-1)!\left((\lambda B-A)^{-1} B\right)^{n-1}(\lambda B-A)^{-1} C B \in L(E)
\end{aligned}
$$

(iv) Suppose that $E$ is barreled and the mapping $\lambda \mapsto B(\lambda B-A)^{-1} C \in L(E)$, $\lambda \in \Omega$ is analytic. Then $R\left(\left((\lambda B-A)^{-1} B\right)^{n-1}(\lambda B-A)^{-1} C^{2}\right) \subseteq R(C)$, $n \in \mathbb{N}, \lambda \in \Omega$ and

$$
\begin{aligned}
\frac{d^{n-1}}{d \lambda^{n-1}} & B(\lambda B-A)^{-1} C=C^{-1}(-1)^{n-1}(n-1)! \\
& \times B\left((\lambda B-A)^{-1} B\right)^{n-1}(\lambda B-A)^{-1} C^{2} \in L(E), \quad n \in \mathbb{N}, \lambda \in \Omega
\end{aligned}
$$

Proof. Let $n \in \mathbb{N}$, let $\lambda \in \Omega$, and let $\Gamma$ be a positively oriented circle around $\lambda$ that is contained in $\Omega$. Making use of the Cauchy integral formula, we get that

$$
\frac{d^{n}}{d \lambda^{n}}(\lambda B-A)^{-1} C x=\frac{n!}{2 \pi i} \oint_{\Gamma} \frac{(z B-A)^{-1} C x}{(z-\lambda)^{n+1}} d z
$$

Since the operators $A$ and $B$ are closed, we get from the above that $\frac{d^{n}}{d \lambda^{n}}(\lambda B-$ $A)^{-1} C x \in D(A) \cap D(B)$. Applying again the Cauchy integral formula, and taking into account that the operators $C^{-1}(\lambda B-A) C$ and $B$ are closed, we get that

$$
\begin{aligned}
(\lambda B-A) & \frac{d^{n}}{d \lambda^{n}}(\lambda B-A)^{-1} C x \\
& =C^{-1}(\lambda B-A) C \frac{n!}{2 \pi i} \oint_{\Gamma} \frac{(z B-A)^{-1} C x}{(z-\lambda)^{n+1}} d z \\
& =\frac{n!}{2 \pi i} \oint_{\Gamma} \frac{C^{-1}(\lambda B-A) C(z B-A)^{-1} C x}{(z-\lambda)^{n+1}} d z \\
& =\frac{n!}{2 \pi i} \oint_{\Gamma} \frac{(\lambda B-A)(z B-A)^{-1} C x}{(z-\lambda)^{n+1}} d z \\
& =\oint_{\Gamma} \frac{C x}{(z-\lambda)^{n+1}} d z-\oint_{\Gamma} \frac{B(z B-A)^{-1} C x}{(z-\lambda)^{n+1}} d z \\
& =-\oint_{\Gamma} \frac{B(z B-A)^{-1} C x}{(z-\lambda)^{n+1}} d z \\
& =-B \oint_{\Gamma} \frac{(z B-A)^{-1} C x}{(z-\lambda)^{n}} d z \\
& =(-n) B \frac{d^{n-1}}{d \lambda^{n-1}}(\lambda B-A)^{-1} C x
\end{aligned}
$$

which proves the first equality in (i). This implies

$$
\frac{d^{n}}{d \lambda^{n}}(\lambda B-A)^{-1} C x=(-n)\left[(\lambda B-A)^{-1} B\right] \frac{d^{n-1}}{d \lambda^{n-1}}(\lambda B-A)^{-1} C x
$$

and now the remainder of (i) simply follows by induction. To prove (ii), suppose that $\lambda_{0} \in \Omega$. Since the mapping $\lambda \mapsto B(\lambda B-A)^{-1} C x, \lambda \in \Omega$ is analytic, the Hilbert resolvent equation (2.3) shows that the mapping $\lambda \mapsto\left[\left(\lambda_{0} B-\right.\right.$ $\left.A)^{-1} C\right] B(\lambda B-A)^{-1} C x=\left(1 /\left(\lambda_{0}-\lambda\right)\right)\left[(\lambda B-A)^{-1} C^{2} x-\left(\lambda_{0} B-A\right)^{-1} C^{2} x\right]$, $\lambda \in \Omega \backslash\left\{\lambda_{0}\right\}$ is analytic, as well. From this, we may conclude that the mapping $\lambda \mapsto(\lambda B-A)^{-1} C^{2} x, \lambda \in \Omega$ is analytic. By our assumption, the mapping $\lambda \mapsto B(\lambda B-A)^{-1} C^{2} x, \lambda \in \Omega$ is likewise analytic so that part (ii) follows almost directly from (i). The proofs of (iii) and (iv) are simple and therefore omitted.

Summa summarum, Proposition 2.3 and Proposition 2.4 taken together provide a generalization of [5, Proposition 2.16] for degenerate ( $C, B$ )-resolvents.

Remark 2.5. In the case that $C=I$ and $E$ is a Banach space, it is well known that the $(I, B)$-resolvent set of $A$ is open, as well as that the $(I, B)$-resolvent, right $(I, B)$-resolvent and left $(I, B)$-resolvent of the operator $A$ are analytic
in $\rho_{B}(A)(11)$. The corresponding statement in locally convex spaces has recently been analyzed in [7, Theorem 1].

The validity of condition (RE1)" considered in this section, and the existence of the operator $W$ obeying the properties clarified in (RE1)", enable us to formulate a great number of theoretical results about the $C$-wellposedness of abstract degenerate Volterra equation (1.1), and numerous other degenerate Cauchy problems, by using a trustworthy passing to the theory of abstract nondegenerate integro-differential equations. We will present only one illustrative example in support of this fact:

Example 2.6. Let the function $a(t)$ be a kernel on $[0, \tau)$ and let the operator $W$ generate a (local) $(a, k)$-regularized $C$-resolvent family $(R(t))_{t \in[0, \tau)}$ satisfying $W \int_{0}^{t} a(t-s) R(s) x d s=R(t) x-k(t) C x, x \in X, t \in[0, \tau)$ (the use of the operator $W$ here seems to be much better than the use of the operator $\overline{A B^{-1}}$, provided that $B$ is injective and $C \neq I$; cf. 2.2p). Then a simple computation involving the definition of the operator $W$ shows that for each element $y \in E$ such that the element $x=C^{-1} L_{\lambda}^{C, B}(A) y$ is well-defined, we have

$$
\begin{aligned}
& R(t) C^{-1} L_{\lambda}^{C, B}(A) y-k(t) L_{\lambda}^{C, B}(A) y \\
& \quad=\int_{0}^{t} a(t-s) R(s)\left[C^{-1} A(\lambda B-A)^{-1} C y\right] d s
\end{aligned}
$$

for any $t \in[0, \tau)$. In particular, if $y=(\lambda B-A) z$ for some $z \in D(A) \cap D(B)$, then the above requirements hold and we get

$$
R(t) B z-k(t) C B z=\int_{0}^{t} a(t-s) R(s) A z d s, \quad t \in[0, \tau)
$$

Assuming additionally that $R(t)$ commutes with $A$ and $B$ for all $t \in[0, \tau)$, the above implies that the function $u(t):=R(t) z, t \in[0, \tau)$ is a unique strong solution of the abstract degenerate Volterra equation 1.1), with $f(t)=k(t) C B z$, $t \in[0, \tau)$.

Remark 2.7. Assume that the functions $k(t)$ and $|a|(t)$ satisfy the condition (P1), as well as that the operator $B$ is injective. Using the definition of the operator $W$, properties stated in (RE1)" and [6, Theorem 2.1.5], we have that $W$ generates a global exponentially equicontinuous ( $a, k$ )-regularized $C$-resolvent family $(R(t))_{t \geq 0}$ (cf. [6] for the notion) provided that there exists a sufficiently large real number $\omega>0$ such that the family $\left\{e^{-\omega t} R(t): t \geq 0\right\} \subseteq L(E)$ is equicontinuous as well as that for each $\lambda \in \mathbb{C}$ with $\tilde{k}(\lambda) \tilde{a}(\lambda) \neq 0$ and $\Re \lambda>\omega$, the operator $B-\tilde{a}(\lambda) A$ is injective and

$$
\tilde{k}(\lambda) B(B-\tilde{a}(\lambda) A)^{-1} C x=\int_{0}^{\infty} e^{-\lambda t} R(t) x d t, \quad x \in E .
$$

Combined with the conclusions clarified in the above example, we are in a position to recover the assertion of [6, Theorem 2.2.8(ii)], with a much simpler proof
given provided the injectiveness of $B$ (see [6, Section 2.2] for certain applications). It is also worth noting that we can use [6, Theorem 2.1.6, Proposition 2.1.7, Theorem 2.1.29, Proposition 2.1.32] here.

We close the paper with the observation that W. Arendt has analyzed in [1] approximations of pseudoresolvents and provided certain applications in the study of approximations of degenerate strongly continuous semigroups, as well as that Q. Hualing and Z. Huaxin have analyzed in 4] approximations of $C$-pseudoresolvents and applied their results in the study of approximations of degenerate $C$-regularized semigroups. For the sake of brevity, we will not discuss related problems for $(C, B)$-resolvents of closed linear operators here.

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