Novi Sad J. Math. Vol. 52, No. 2, 2022, 177-187
https://doi.org/10.30755/NSJOM.12375

On mixed C_0-groups of bounded linear operators on non-Archimedean Banach spaces[1]

Aziz Blali[2], Abdelkhalek El amrani[3] and Jawad Ettayb[45]

Abstract

In this paper, we introduce and check some properties of mixed C_0-group of bounded linear operators on non-Archimedean Banach spaces. Our main result extends theorems for mixed C_0-groups of bounded linear operators on non-Archimedean Banach spaces. In contrast with the classical setting, the parameter of mixed C_0-groups belongs to a clopen ball Ω_r of the ground field $\mathbb{K}$. As an illustration, we will discuss the solvability of some inhomogeneous p-adic differential equations for mixed C_0-groups when $\alpha = -1$. Examples are given to support our work.

AMS Mathematics Subject Classification (2010): 47D03; 47S10

Key words and phrases: non-Archimedean Banach spaces, mixed C_0-groups of linear operators, C_0-groups of linear operators

1 Introduction and preliminaries

In the classical functional analysis, the Cauchy equations $f(x+y) = f(x)f(y)$ and $f(x+y) = f(x) + f(y)$ for $x, y \in \mathbb{R}^+$ can be generalized as the form $f(x+y) = H(f(x), f(y))$, where H is a scalar-valued function of two variables. This enabled S. Harsinder to discover and study the mixed semigroup of linear operators on Archimedean Banach spaces ([8]). The first equation has been studied on the classical Banach space by Hille-Yosida [1], [7] and [11]. Generalized semigroups and cosine functions were studied by M. Kostić, for more details, we refer to [10].

In the non-Archimedean operators theory, T. Diagana [3] introduced the concept of C_0-groups of bounded linear operators on free non-Archimedean Banach space. Also, A. El amrani, A. Blali, J. Ettayb and M. Babahmed introduced and studied the notions of $C-$groups and cosine families of bounded linear operators on non-Archimedean Banach spaces. For more details, we refer to [2] and [6].

[1]Dedicated to our Professor Rachid Ameziane Hassani on the occasion of his retirement

[2]Ecole Normale Superieure, Department of Mathematics, Sidi Mohamed Ben Abdellah University, B. P. 5206 Bensouda-Fès, Morocco,
e-mail: aziz.blali@usmba.ac.ma

[3]Department of mathematics and computer science, Sidi Mohamed Ben Abdellah University , Faculty of Sciences Dhar El Mahraz, Atlas Fès, Morocco.
e-mail: abdelkhalek.elamrani@usmba.ac.ma

[4]Department of mathematics and computer science, Sidi Mohamed Ben Abdellah University, Faculty of Sciences Dhar El Mahraz, Atlas Fès, Morocco.
e-mail: jawad.ettayb@usmba.ac.ma

[5]Corresponding author

Throughout this paper, X is a non-Archimedean (n.a) Banach space over a (n.a) non trivially complete valued field $\mathbb{K}$ with valuation $|\cdot|$, $B(X)$ denotes the set of all bounded linear operators from X into X, $\mathbb{Q}_p$ is the field of p-adic numbers ($p \geq 2$ being a prime) equipped with p-adic valuation $|.|_p$, $\mathbb{Z}_p$ denotes the ring of p-adic integers (the ring of p-adic integers $\mathbb{Z}_p$ is the unit ball of $\mathbb{Q}_p$). For more details and related issues, we refer to [9] and [13]. We denote the completion of the algebraic closure of $\mathbb{Q}_p$ under the p-adic absolute value $|\cdot|_p$ by $\mathbb{C}_p$ [9]. Let $r > 0$, Ω_r denote the clopen ball of $\mathbb{K}$ centred at 0 with radius r, that is $\Omega_r = \{k \in \mathbb{K} : |k| < r\}$. The aim of this paper is to introduce and study the notion of mixed C_0-group on non-Archimedean Banach spaces over $\mathbb{K}$. In contrast with the classical setting, the parameter of a given a mixed C_0-groups belongs to a clopen ball Ω_r of the ground field $\mathbb{K}$. As an illustration, we will discuss the solvability of some inhomogeneous p-adic differential equations for mixed C_0-groups, see Remark 2.16. We begin with some preliminaries.

Definition 1.1 ([4], Definition 2.1)**.** Let X be a vector space over $\mathbb{K}$. A non-negative real valued function $\|\cdot\| : X \to \mathbb{R}_+$ is called a non-Archimedean norm if:

(1) For all $x \in X$, $\|x\| = 0$ if and only if $x = 0$,

(2) For any $x \in X$ and $\lambda \in \mathbb{K}$, $\|\lambda x\| = |\lambda|\|x\|$,

(3) For any $x\ , y \in X$, $\|x + y\| \leq \max(\|x\|, \|y\|)$.

Property (3) of Definition 1.1 is referred to as the ultrametric or strong triangle inequality.

Definition 1.2 ([4], Definition 2.2)**.** A non-Archimedean normed space is a pair $(X; \|\cdot\|)$ where X is a vector space over $\mathbb{K}$ and $\|\cdot\|$ is a non-Archimedean norm on X.

Definition 1.3 ([3], Definition 2.2)**.** A non-Archimedean Banach space is a vector space endowed with non-Archimedean norm, which is complete.

For more details on non-Archimedean Banach spaces and related issues, see for example [4].

Proposition 1.4 ([4], Proposition 2.16)**.** *(1) A closed subspace of a non-Archimedean Banach space is a non-Archimedean Banach space;*

(2) The direct sum of two non-Archimedean Banach spaces is a non-Archimedean Banach space.

Examples 1.5 ([4], Example 2.20)**.** Let $c_0(\mathbb{K})$ denote the set of all sequences $(x_i)_{i\in\mathbb{N}}$ in $\mathbb{K}$ such that $\lim_{i\to\infty} x_i = 0$. Then, $c_0(\mathbb{K})$ is a vector space over $\mathbb{K}$ and

$$\| (x_i)_{i\in\mathbb{N}} \| = \sup_{i\in\mathbb{N}} |x_i|$$

is a non-Archimedean norm for which $(c_0(\mathbb{K}), \|\cdot\|)$ a non-Archimedean Banach space.

In this section, we define and discuss properties of non-Archimedean Banach spaces which have bases.

Definition 1.6 ([3], Definition 2.5)**.** A non-Archimedean Banach space $(X, \|\cdot\|)$ over a non-Archimedean valued field (complete) $(K, |\ |)$ is said to be a free non-Archimedean Banach space if there exists a family $(x_i)_{i\in I}$ of elements of X indexed by a set I such that each element $x \in X$ can be written uniquely like a pointwise convergent series defined by $x = \sum_{i\in I} \lambda_i x_i$, and $\|x\| = \sup_{i\in I} |\lambda_i|\|x_i\|$.

The family $(x_i)_{i\in I}$ is then called a t-orthogonal basis for X. If, for all $i \in I$, $\|x_i\| = 1$, then $(x_i)_{i\in I}$ is called an orthonormal basis of X. For more details of orthogonality and the concepts of bases in non-Archimedean case, we refer to [12] and [14].

However, the treatment of those non-Archimedean Banach spaces in the general case can be found in [3] and [5]. Moreover, X is a free non-Archimedean Banach space over $\mathbb{K}$ if and only if X is isometrically isomorphic to $c_0(I, u)$ for certain index set I and an application $u : I \to \mathbb{R}_+^*$. By [12, Theorem 2.58] $c_0(I)$ is of countable type if and only if I is countable. For more details we refer to [12] and [14]. In this work the basis of free $n.a$ Banach spaces considered is countable, and we assume $I = \mathbb{N}$.

Definition 1.7. [4] Let $(X, \|\cdot\|)$ be a non-Archimedean Banach space. The non-Archimedean Banach space $(B(X), \|\cdot\|)$ is the collection of all bounded linear operators from X into itself equipped with the operator-norm defined by

$$(\forall A \in B(X)) \ \ \|A\| = \sup_{x\in X\setminus\{0\}} \frac{\|A(x)\|}{\|x\|}.$$

For more details on non-Archimedean linear operators theory, we refer to [4], [5], [12] and [14].

Throughout this paper, X is a $(n.a)$ Banach space over a $(n.a)$ non trivially complete valued field $\mathbb{K}$ of characteristic zero with valuation $|.|$, $B(X)$ is equipped with the norm of Definition 1.7 and for all $r > 0$, $\Omega_r^* = \Omega_r\setminus\{0\}$, denotes the clopen ball of center 0 with radius r deprived of zero.

Definition 1.8 ([3], Definition 3.1)**.** Let $r > 0$ be a chosen real number such that $(T(t))_{t\in\Omega_r}$ are well defined. A one-parameter family $(T(t))_{t\in\Omega_r}$ of bounded linear operators from X into X is a group of bounded linear operators on X if

(i) $T(0) = I$, where I is the unit operator of X;

(ii) For all $t, s \in \Omega_r$ $T(t + s) = T(t)T(s)$.

The group $(T(t))_{t\in\Omega_r}$ will be called of class C_0 or strongly continuous if the following condition holds:

- For each $x \in X$, $\lim_{t\to 0} \|T(t)x - x\| = 0$.

A group of bounded linear operators $(T(t))_{t\in\Omega_r}$ is uniformly continuous if and only if $\lim_{t\to 0}\|T(t)-I\|=0$.
The linear operator A defined by

$$D(A)=\{x\in X:\lim_{t\to 0}\frac{T(t)x-x}{t}\ exists\},$$

and

$$Ax=\lim_{t\to 0}\frac{T(t)x-x}{t}\ for\ each\ x\in D(A),$$

is called the infinitesimal generator of the group $(T(t))_{t\in\Omega_r}$.

As an application of C_0-groups of linear operators, consider the p-adic abstract Cauchy problem for differential equations on non-Archimedean Banach space X given by:

$$ACP(A;x)\begin{cases}\frac{du(t)}{dt}=Au(t), & t\in\Omega_r,\\ u(0)=x,\end{cases}$$

where $A:D(A)\to X$ is a linear operator and $x\in D(A)$.

2 Main results

Recall that k is the residue class field of $\mathbb{K}$. Througout this paper, we assume that $\mathbb{K}$ is a complete non-Archimedean valued field of characteristic zero with $char(k)=p$ $\Big(p$ is a prime number$\Big)$. We begin with some definitions.

Definition 2.1. Let $r>0$ be a real number. A family $(T(t))_{t\in\Omega_r}$ of bounded linear operators is said to satisfy a p-adic $H-$generalized Cauchy equation of bounded linear operators on X if

$$for\ all\ t,\ s\in\Omega_r,\ T(t+s)=H\Big(T(s),\ T(t)\Big),$$

where $H:\ B(X)\times B(X)\to B(X)$ is a function.

Remark 2.2. If $H\Big(T(s),\ T(t)\Big)=T(s)T(t)$, with $T(0)=I$, $(T(t))_{t\in\Omega_r}$ is a group of bounded linear operators on X.

Definition 2.3. Let $r>0$ be a real number. A family $(S(t))_{t\in\Omega_r}$ of bounded linear operators is said to be a $H-C_0-$group or a generalized C_0-group of bounded linear operators on X if

(i) $S(0)=I$, where I is the identity operator of X.

(ii) there is a C_0-group $(T(t))_{t\in\Omega_r}$ of bounded linear operators and $D\in B(X)$ such that $for\ all\ t,\ s\in\Omega_r$,

$$\begin{aligned}S(s+t)&=H\Big(S(s),\ S(t)\Big)\\&=S(s)S(t)+D\Big(S(s)-T(s)\Big)\Big(S(t)-T(t)\Big);\end{aligned}$$

(iii) for each $x \in X, S(\cdot)x : \Omega_r \longrightarrow S(t)x$ is continuous on Ω_r.

The linear operator A defined by

$$D(A) = \{x \in X : \lim_{t\to 0} \frac{S(t)x - x}{t} \ exists\}$$

and

$$for\ each\ x \in D(A),\ Ax = \lim_{t\to 0} \frac{S(t)x - x}{t},$$

is called the infinitesimal generator of the $H - C_0-$group $(S(t))_{t\in\Omega_r}$.

2.1 Question

Can we characterize the infinitesimal generator of an $H - C-$group of linear operators on infinite dimensional non-Archimedean Banach space?

Remark 2.4. Let $(S(t))_{t\in\Omega_r}$ be a generalized C_0-group on X, if $D = 0$, then $(S(t))_{t\in\Omega_r}$ is a C_0-group of linear operators on X.

From Definition 2.3, when $D = \alpha I$ for $\alpha \in \mathbb{K}$, we have the following definition.

Definition 2.5. Let $r > 0$ be a real number. A family $(S(t))_{t\in\Omega_r}$ is said to be a mixed C_0-group of bounded linear operators on X if

(i) $S(0) = I$;

(ii) there is a C_0-group $(T(t))_{t\in\Omega_r}$ of bounded linear operators and $\alpha \in \mathbb{K}$ such that for all $s, t \in \Omega_r$

$$\begin{aligned} S(s+t) &= H\Big(S(s),\ S(t)\Big) \\ &= S(s)S(t) + \alpha\Big(S(s) - T(s)\Big)\Big(S(t) - T(t)\Big); \end{aligned}$$

(iii) for each $x \in X,\ S(\cdot)x : \Omega_r \longrightarrow S(t)x$ is continuous on Ω_r.

The linear operator A defined by

$$D(A) = \{x \in X : \lim_{t\to 0} \frac{S(t)x - x}{t} \ exists\}$$

and

$$for\ each\ x \in D(A),\ Ax = \lim_{t\to 0} \frac{S(t)x - x}{t},$$

is called the infinitesimal generator of the mixed C_0-group $(S(t))_{t\in\Omega_r}$.

Remark 2.6. Let $(S(t))_{t\in\Omega_r}$ be a mixed C_0-group on X. If $\alpha = 0$, then $(S(t))_{t\in\Omega_r}$ is a C_0-group of linear operators on X.

Example 2.7. Let $r = p^{\frac{-1}{p-1}}$, suppose that X is a non-Archimedean Banach space over $\mathbb{Q}_p$, $A \in B(X)$ such that $\|A\| < r$. Set

$$for\ all\ t \in \Omega_r,\ S(t) = e^{tA} + tAe^{tA}.$$

Then one can see that with $D = -I$, $\{S(t)\}_{t\in\Omega_r}$ is a $H - C-$group where for all $t \in \Omega_r$, $T(t) = e^{tA}$. In this case, for all $t,\ s \in \Omega_r,\ S(s)S(t) = S(t)S(s)$.
In fact, $D = -I$, if $D = -I$, then for all $t, s \in \Omega_r$ we have $S(t + s) = e^{(t+s)A} + (t + s)Ae^{(t+s)A}$ and

$$\begin{aligned} S(t)S(s) &= \Big(e^{tA} + tAe^{tA}\Big)\Big(e^{sA} + sAe^{sA}\Big) \\ &= e^{tA+sA} + sAe^{tA+sA} + tAe^{tA+sA} + tsAAe^{tA+sA} \\ &= e^{(t+s)A} + (t + s)Ae^{(t+s)A} + tsAAe^{(t+s)A}, \end{aligned}$$

and

$$\Big(S(t) - T(t)\Big)\Big(S(s) - T(s)\Big) = tsAAe^{(t+s)A}.$$

Hence,

$$\begin{aligned} S(t)S(s) - \Big(S(t) - T(t)\Big)\Big(S(s) - T(s)\Big) &= e^{(t+s)A} + (t + s)Ae^{(t+s)A} \\ &= S(s + t). \end{aligned}$$

(i) and (iii) of Definition 2.3 are easy to verify so $\{S(t)\}_{t\in\Omega_r}$ is a $H-C_0-$group.

The following proposition gives a condition for which all operators in an $H - C_0-$group family commute.

Proposition 2.8. *Let $\{S(t)\}_{t\in\Omega_r}$ be an $H - C_0-$group family on X. If $I + D$ is injective and for all $t,\ s \in \Omega_r$, $T(s)S(t) = S(t)T(s)$. Then for all $t,\ s \in \Omega_r,\ S(s)S(t) = S(t)S(s)$.*

Proof. Assume that $I + D$ is injective and for all $t,\ s \in \Omega_r,\ T(s)S(t) = S(t)T(s)$, then for all $t,\ s \in \Omega_r$,

$$\begin{aligned} S(t)S(s) + D\Big(S(t) - T(t)\Big)\Big(S(s) - T(s)\Big) &= S(t + s) \\ &= S(s + t) \\ &= S(s)S(t) + D\Big(S(s) - T(s)\Big) \\ &\quad \times\Big(S(t) - T(t)\Big). \end{aligned}$$

Thus, $(I + D)\Big(S(t)S(s) - S(s)S(t)\Big) = 0$, then for all $t,\ s \in \Omega_r,\ S(s)S(t) = S(t)S(s)$. ☐

We have the following theorem.

Theorem 2.9. *Let $\{S(t)\}_{t\in\Omega_r}$ be an $H-C_0-$group family of infinitesimal generator A on X with $\{T(t)\}_{t\in\Omega_r}$ is C_0-group of infinitesimal generator A_0 such that there is a $M_1,\ M_2>0$ such that for all $t,\ s\in\Omega_r$, $\|S(t)\|\le M_1$, $\|T(t)\|\le M_2$, $T(s)S(t)=S(t)T(s)$, and $S(t)S(s)=S(s)S(t)$. If $x\in D(A)$, then for all $t\in\Omega_r$, $S(t)x$, $T(t)x\in D(A)$, and $AS(t)x=S(t)Ax$. Furthermore, $S(t)x$, $T(t)x\in D(A_0)$ and $A_0S(t)x=S(t)A_0x$, $A_0T(t)x=T(t)A_0x$ for any $x\in D(A_0)$.*

Proof. Let $x\in D(A)$ and let $s\in\Omega_r^*$ *and* $t\in\Omega_r$. From the boundedness of $\{S(t)\}_{t\in\Omega_r}$, it is easy to see that

$$\left(\frac{S(s)S(t)x-S(t)x}{s}\right)=S(t)\left(\frac{S(s)x-x}{s}\right)\to S(t)Ax \text{ as } s\to 0. \tag{2.1}$$

Consequently, for all $t\in\Omega_r$, $S(t)Ax\in D(A)$ and $AS(t)x=S(t)Ax$.
Let $x\in D(A)$ and let $s\in\Omega_r^*$ *and* $t\in\Omega_r$. From the boundedness of $\{T(t)\}_{t\in\Omega_r}$

$$\left(\frac{S(s)T(t)x-T(t)x}{s}\right)=T(t)\left(\frac{S(s)x-x}{s}\right)\to T(t)Ax \text{ as } s\to 0. \tag{2.2}$$

Consequently, for all $t\in\Omega_r$, $T(t)x\in D(A)$ and $AT(t)x=T(t)Ax$.
The last part can be proved similarly. □

Set $A_1=(1+\alpha)A-\alpha A_0$, where $\alpha\in\mathbb{K}\backslash\{-1\}$ and A_0 is the infinitesimal generator of the C_0-group $\{T(t)\}_{t\in\Omega_r}$ and A is the infinitesimal generator of a mixed C_0-group $\{S(t)\}_{t\in\Omega_r}$. We have the following theorem.

Theorem 2.10. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $\{S(t)\}_{t\in\Omega_r}$ be a mixed C_0-group family of bounded linear operators on X with $\alpha\in\mathbb{K}\backslash\{-1\}$. Let for all $t\in\Omega_r$, $T_1(t)=(1+\alpha)S(t)-\alpha T(t)$, then $\{T_1(t)\}_{t\in\Omega_r}$ is a C_0-group of bounded linear operators whose infinitesimal generator is an extension of A_1. Furthermore, for all $x\in X$, and $t\in\Omega_r$,*

$$S(t)x=\frac{1}{1+\alpha}T_1(t)x+\frac{\alpha}{1+\alpha}T(t)x.$$

Proof.

(i) Trivially, $T_1(0)=(1+\alpha)S(0)-\alpha T(0)=I$,

(ii) For all $t,\ s\in\Omega_r,\ x\in X$, we have

$$T_1(s+t)x=(1+\alpha)S(s+t)-\alpha T(s+t)x,$$

and

$$\begin{aligned}
T_1(s+t)x &= (1+\alpha)\Big(S(s)S(t)+\alpha(S(s)-T(s))(S(t)-T(t))\Big)x \\
&- \alpha T(s)T(t)x \\
&= (1+\alpha)S(s)S(t)x+\alpha(1+\alpha)S(s)S(t)x-\alpha(1+\alpha)S(s)T(t)x \\
&- \alpha(1+\alpha)T(s)S(t)x+\alpha(1+\alpha)T(s)T(t)x-\alpha T(s)T(t)x \\
&= (1+\alpha)^2S(s)S(t)x-\alpha(1+\alpha)S(s)T(t)x \\
&- \alpha(1+\alpha)T(s)S(t)x+\alpha^2T(s)T(t)x \\
&= \Big((1+\alpha)S(s)-\alpha T(s)\Big)\Big((1+\alpha)S(t)-\alpha T(t)\Big)x \\
&= T_1(s)T_1(t)x.
\end{aligned}$$

(iii) Since $(T(t))_{t\in\Omega_r}$ and $(S(t))_{t\in\Omega_r}$ are continuous on Ω_r, thus $(T_1(t))_{t\in\Omega_r}$ is continuous on Ω_r. So, $(T_1(t))_{t\in\Omega_r}$ is a C_0-group of bounded linear operators on X.

Now we show that an extension of $A_1=(1+\alpha)A-\alpha A_0$ where $\alpha\in\mathbb{K}\backslash\{-1\}$ is the infinitesimal generator of $\{T_1(t)\}_{t\in\Omega_r}$. Let B be the infinitesimal generator of $\{T_1(t)\}_{t\in\Omega_r}$. For $x\in D(A_1)\Big(=D(A)\cap D(A_0)\Big)$. By definitions of $D(A)$ and $D(A_0)$, $\lim\limits_{t\to 0}\Big(\dfrac{S(t)x-x}{t}\Big)=Ax$ and $\lim\limits_{t\to 0}\Big(\dfrac{T(t)x-x}{t}\Big)=A_0x$. Then,

$$\begin{aligned}
\lim_{t\to 0}\Big(\frac{T_1(t)x-x}{t}\Big) &= \lim_{t\to 0}\Big(\frac{(1+\alpha)S(t)x-\alpha T(t)x-x}{t}\Big) \\
&= (1+\alpha)\lim_{t\to 0}\Big(\frac{S(t)x-x}{t}\Big)-\alpha\lim_{t\to 0}\Big(\frac{T(t)x-x}{t}\Big)
\end{aligned}$$

exists in X. It follows that $x\in D(B)$ and $A_1x=Bx$, hence the infinitesimal generator of $(T_1(t))_{t\in\Omega_r}$ is an extension of A_1. □

For $\alpha\in\mathbb{K}\backslash\{-1\}$ and $D=\alpha I$, from Proposition 2.8 and Theorem 2.9, we conclude:

Proposition 2.11. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $\{S(t)\}_{t\in\Omega_r}$ be a mixed C_0-group family of bounded linear operators on X with $\alpha\in\mathbb{K}\backslash\{-1\}$ such that for all $t,\ s\in\Omega_r,\ T(s)S(t)=S(t)T(s)$. Then for all $t,\ s\in\Omega_r,\ S(s)S(t)=S(t)S(s)$.*

Theorem 2.12. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $\{S(t)\}_{t\in\Omega_r}$ be a mixed C_0-group family of infinitesimal generator A on X, while $\{T(t)\}_{t\in\Omega_r}$ is C_0-group of infinitesimal generator A_0 and $\alpha\in\mathbb{K}\backslash\{-1\}$ such that there is a $M_1,\ M_2>0$ such that for all $t,\ s\in\Omega_r,\ \|S(t)\|\leq M_1,\ \|T(t)\|\leq M_2,\ T(s)S(t)=S(t)T(s)$, and $S(s)S(t)=S(t)S(s)$. If $x\in D(A)$, then for all $t\in\Omega_r,\ S(t)x,\ T(t)x\in D(A)$, and $AS(t)x=S(t)Ax$. Furthermore, $S(t)x,\ T(t)x\in D(A_0)$ and $A_0S(t)x=S(t)A_0x,\ A_0T(t)x=T(t)A_0x$ for any $x\in D(A_0)$.*

For $\alpha=-1$, we have the following theorem.

Theorem 2.13. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $\{S(t)\}_{t\in\Omega_r}$ be a mixed C_0-group family of infinitesimal generator A on X, such that there is a $M_1,\ M_2>0$ such that for all $t,\ s\in\Omega_r,\ \|S(t)\|\leq M_1,\ \|T(t)\|\leq M_2,\ T(s)S(t)=S(t)T(s)$ and $S(s)S(t)=S(t)S(s)$ where $(T(t))_{t\in\Omega_r}$ is a C_0-group of infinitesimal generator A_0. Then, for all $x\in D(A)\cap D(A_0)$,*

$$\frac{dS(t)}{dt}x=\Big(A_0S(t)x+(A-A_0)T(t)x\Big).$$

Proof. Let $x\in D(A)\cap D(A_0)$, using Theorem 2.9, we have

$$\begin{aligned}
\frac{d}{dt}S(t)x &= \lim_{h\to 0}\frac{S(h+t)x-S(t)x}{h}\\
&= \lim_{h\to 0}\frac{S(h)S(t)x+(T(h)-S(h)(S(t)-T(t))x-S(t)x}{h}\\
&= \lim_{h\to 0}\frac{T(h)S(t)x-S(t)x}{h}+\lim_{h\to 0}\frac{S(h)T(t)x-T(t)x}{h}\\
&\quad -\lim_{h\to 0}\frac{T(h)T(t)x-T(t)x}{h}\\
&= \Bigg(A_0S(t)x+AT(t)x-A_0T(t)x\Bigg).
\end{aligned}$$

□

We have the following definition.

Proposition 2.14. *Let X be a finite dimensional Banach space over $\mathbb{Q}_p$, let $\{T(t)\}_{t\in\Omega_r}$ be a C_0-group of infinitesimal generator A_0 on X, and let $A\in B(X)$ such that for all $t\in\Omega_r$, $T(t)A=AT(t)$ and $A_0T(t)=T(t)A_0$. If, for all $t\in\Omega_r$, $S(t)=T(t)+t(A-A_0)T(t)$, then $\{S(t)\}_{t\in\Omega_r}$ is a mixed C_0-group of infinitesimal generator A with $\alpha=-1$.*

Proof. Since $(T(t))_{t\in\Omega_r}$ is C_0-group, then $T(0)=I$, hence $S(0)=T(0)=I$. Let $s,\ t\in\Omega_r$, then

$$\begin{aligned}
S(t+s) &= T(t+s)+(t+s)(A-A_0)T(t+s),\\
&= T(t)T(s)+tAT(t)T(s)-tA_0T(t)T(s)+sAT(t)T(s)\\
&\quad -sA_0T(t)T(s),
\end{aligned}$$

and

$$\begin{aligned} & S(s)S(t) + \Big(T(s) - S(s)\Big)\Big(S(t) - T(t)\Big) \\ = \quad & \Big(T(s) + s(A - A_0)T(s)\Big) \times \\ & \Big(T(t) + t(A - A_0)T(t)\Big) \\ & -st(A - A_0)T(s)(A - A_0)T(t) \\ = \quad & T(s)T(t) + tT(s)AT(t) \\ & -tT(s)A_0T(t) \\ & +sAT(s)T(t) - sA_0T(s)T(t) \\ & +st(A - A_0)(A - A_0)T(s)T(t) \\ & -st(A - A_0)(A - A_0)T(s)T(t) \\ = \quad & S(t + s). \end{aligned}$$

Also, we have for all $x \in X, t \in \Omega_r \to T(t)x$ is continuous, then $x \in X, t \in \Omega_r \to S(t)x$ is continuous. Consequently, $(S(t))_{t\in\Omega_r}$ is a mixed C_0-group of bounded linear operators on X. □

Example 2.15. Assume that $\mathbb{K} = \mathbb{Q}_p$, let $A,\ A_0 \in B(X)$ such that $\|A_0\| < r = p^{\frac{1}{1-p}}$ and $AA_0 = A_0A$, we consider the family on X given by

$$for\ all\ t \in \Omega_r,\ S(t) = e^{tA_0} + t(A - A_0)e^{tA_0}.$$

It is easy to see that for $\alpha = -1$, $(S(t))_{t\in\Omega_r}$ is a mixed C_0-group of bounded linear operators of infinitesimal generator A on X.

Remark 2.16. Let X be a non-Archimedean Banach space over $\mathbb{K}$. Theorem 2.13 shows that for $\alpha = -1$, if $(S(t))_{t\in\Omega_r}$ is a mixed C_0-group of infinitesimal generator A and $(T(t))_{t\in\Omega_r}$ is a C_0-group of infinitesimal generator A_0 such that for all $t,\ s \in \Omega_r,\ T(s)S(t) = S(t)T(s),\ S(s)S(t) = S(t)S(s)$, then $u(t) = S(t)x$ is a solution of the following inhomogeneous p-adic differential equation given by

$$\frac{du(t)}{dt} = A_0u(t) + (A - A_0)f(t),\ t \in \Omega_r,$$

and $u(0) = x$, $x \in D(A_0) \cap D(A)$ with $f(t) = T(t)x$.

References

[1] Arendt, W., Batty, C. J. K., Hieber, M., and Neubrander, F. *Vector-valued Laplace transforms and Cauchy problems*, second ed., vol. 96 of *Monographs in Mathematics*. Birkhäuser/Springer Basel AG, Basel, 2011.

[2] Blali, A., amrani, A. E., Ettayb, J., and Hassani, R. A. Cosine families of bounded linear operators on non-archimedean banach spaces. *Novi Sad J. Math. 52*, 1 (2022), 173–184.

[3] Diagana, T. C_0-semigroups of linear operators on some ultrametric Banach spaces. *Int. J. Math. Math. Sci.* (2006), Art. ID 52398, 1–9.

[4] Diagana, T., and Ramaroson, F. *Non-Archimedean operator theory.* SpringerBriefs in Mathematics. Springer, Cham, 2016.

[5] Diarra, B. An operator on some ultrametric Hilbert spaces. *J. Anal. 6* (1998), 55–74.

[6] El Amrani, A., Blali, A., Ettayb, J., and Babahmed, M. A note on c 0-groups and c-groups on non-archimedean banach spaces. *Asian-European Journal of Mathematics* (2020), 2150104.

[7] Engel, K.-J., and Nagel, R. *One-parameter semigroups for linear evolution equations*, vol. 194 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 2000.

[8] Harshinder, S. The mixed semigroup relation. *Indian J. Pure Appl. Math. 9*, 4 (1978), 255–267.

[9] Koblitz, N. *p-adic analysis: a short course on recent work*, vol. 46 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge-New York, 1980.

[10] Kostić, M. *Generalized semigroups and cosine functions*, vol. 23 of *Posebna Izdanja [Special Editions].* Matematički Institut SANU, Belgrade, 2011.

[11] Pazy, A. *Semigroups of linear operators and applications to partial differential equations*, vol. 44 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1983.

[12] Perez-Garcia, C., and Schikhof, W. H. *Locally convex spaces over non-Archimedean valued fields*, vol. 119 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2010.

[13] Schikhof, W. H. *Ultrametric calculus*, vol. 4 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2006. An introduction to p-adic analysis, Reprint of the 1984 original [MR0791759].

[14] van Rooij, A. C. M. *Non-Archimedean functional analysis*, vol. 51 of *Monographs and Textbooks in Pure and Applied Math.* Marcel Dekker, Inc., New York, 1978.

Received by the editors January 14, 2021
First published online February 25, 2021