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Compactness and Lindelöfness using somewhere dense and cs-dense sets

Tareq M. Al-shami[1][2] **and Takashi Noiri**[3]

Abstract. In this study, by using somewhere dense sets and cs-dense sets, we introduce and investigate the notions of almost SD-compact and almost SD-Lindelöf spaces, nearly SD-compact and nearly SD-Lindelöf spaces, and mildly SD-compact and mildly SD-Lindelöf spaces. We show the relationships between them with the help of illustrative examples. In addition, we characterize them and study their behaviours under some types of mappings.

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1. Introduction

Generalized open sets is a major area of research in general topology. By them, many topological concepts such as continuity and compactness were reformulated. Recently, they have been utilized as a vital technique to reduce the boundary region in the approximation spaces [2, 3, 8, 12] and to solve some problems in the information systems [20, 21]. Also, some authors [4, 11, 14] applied them to study of digital topology.

In 1963, Levine [16] extended the class of open sets by introducing the concept of semi-open sets. Then, Njastad [19] introduced another type of generalized open sets lying between open sets and semi-open sets, namely α-open sets. Mashhour et al. [17] in 1982, and Abd El-Monsef et al. [1] 1983, presented and studied preopen sets and β-open sets, respectively. In 1996, Andrijević [13] defined and investigated the concept of b-open sets. These types of generalized open sets were defined by using interior and closure operators.

The tendency to study these generalizations has been increasing rapidly. Al-shami [5] introduced and studied the concepts of somewhere dense sets and ST_1-spaces. Then, Al-shami and Noiri [10] have studied further properties of somewhere dense sets, and have defined the concepts of SD-continuous and SD-homeomorphism maps. The class of somewhere dense sets contains all regular open, α-open, preopen, semi-open, β-open and b-open sets with the

[1]Department of Mathematics, Sana'a University, Sana'a, Yemen,
e-mail: tareqalshami83@gmail.com
[2]Corresponding author
[3]2949-1 Shiokita-cho, Hinagu, Yatsushiro-shi, Kumamoto-ken, 869-5142 Japan,
e-mail: t.noiri@nifty.com

exception of the empty set. Al-shami [6] presented and explored the concepts of somewhere dense and *cs*-dense sets in soft topological spaces. Then, the authors of [9, 15] employed theses concepts to introduce and discuss some notions such as soft continuity and soft connectedness. Recently, Al-shami [7] has studied the sum of topological spaces in ordered setting.

We recall that any β-open cover of a topological space does not have a finite subcover. Therefore, we cannot define compactness using β-open sets. Since every β-open set is somewhere dense, then we cannot define compactness using somewhere dense sets. However, we apply somewhere dense sets to define weak types of compactness, namely almost SD-compact and almost SD-Lindelöf spaces, nearly SD-compact and nearly SD-Lindelöf spaces, and mildly SD-compact and mildly SD-Lindelöf spaces.

2. Preliminaries

In this section, we recall some definitions and results which help us to investigate and discuss our new concepts.

Definition 2.1. A subset E of a topological space (X, ζ) is said to be:

(i) semi-open [16] if $E \subseteq cl(int(E))$.

(ii) α-open [19] if $E \subseteq int(cl(int(E)))$.

(iii) preopen [17] if $E \subseteq int(cl(E))$.

(iv) β-open [1] if $E \subseteq cl(int(cl(E)))$.

(v) b-open [13] if $E \subseteq int(cl(E)) \bigcup cl(int(E))$.

(vi) somewhere dense [5] if $int(cl(E)) \neq \emptyset$. The complement of a somewhere dense set is said to be cs-dense. A somewhere dense and *cs*-dense set is called an SC-set. The collection of all somewhere dense sets of (X, ζ) is denoted by $S(\zeta)$.

The complement of a ξ-open set is said to be ξ-closed for $\xi \in \{regular, semi, \alpha, pre, \beta, b\}$.

Theorem 2.2. *[18] If M is an open subset of (X, ζ), then $M \bigcap cl(B) \subseteq cl(M \bigcap B)$ at each $B \subseteq X$.*

Definition 2.3. A topological space (X, ζ) with no mutually disjoint non-empty open sets is said to be hyperconnected [18]. (X, ζ) is called strongly hyperconnected [5] if a subset of X is dense if and only if it is non-empty and open.

Theorem 2.4. *[5] A subset B of (X, ζ) is cs-dense if and only if there is a proper closed subset F of X such that $int(B) \subseteq F$.*

Theorem 2.5. *[5]*

1. *Every subset of a topological space is somewhere dense or cs-dense.*

2. *Every superset of a somewhere dense set is somewhere dense.*

Theorem 2.6. *[5] The intersection of open and somewhere dense sets in a hyperconnected space is somewhere dense.*

Theorem 2.7. *[5] The intersection of cs-dense (resp. somewhere dense) subsets of a strongly hyperconnected space is cs-dense (resp. somewhere dense).*

Definition 2.8. [5] Let M be a subset of (X, ζ). Then:

(i) The S-interior of M ($Sint(M)$, in short) is the union of all somewhere dense sets contained in M.

(ii) The S-closure of M ($Scl(M)$, in short) is the intersection of all cs-dense sets containing M.

Proposition 2.9. *[5] Consider a subset M of (X, ζ). Then:*

(i) *$M \subseteq Scl(M)$; and a set $M \neq X$ is cs-dense if and only if $M = Scl(M)$.*

(ii) *$Sint(M) \subseteq M$; and a non-empty set M is somewhere dense if and only if $M = Sint(M)$.*

(iii) *$(Sint(M))^c = Scl(M^c)$ and $(Scl(M))^c = Sint(M^c)$.*

Definition 2.10. [5] (X, ζ) is called an ST_1-space if for any pair of distinct points $a, b \in X$, there are two somewhere dense sets such that one of them contains a but not b and the other contains b but not a.

Theorem 2.11. *[5] The following statements are equivalent.*

(i) *(X, ζ) is an ST_1-space;*

(ii) *For each $a \neq b \in X$, there are two disjoint somewhere dense sets such that one contains a and the other contains b.*

(iii) *For each $a \neq b \in X$, there are two disjoint sets containing a and b, respectively, such that they are both SC-sets.*

Definition 2.12. [10] A map $g : (X, \tau) \rightarrow (Y, \theta)$ is said to be:

(i) SD-continuous at $a \in X$ if for any open set U containing $g(a)$, there is a somewhere dense set E containing a such that $g(E) \subseteq U$.

(ii) SD-continuous on X if it is SD-continuous for each $a \in X$.

(iii) SD-irresolute provided that the inverse image of each somewhere dense subset of Y is empty or a somewhere dense subset of X.

Theorem 2.13. *[10] For a map $g : (X, \tau) \rightarrow (Y, \theta)$, the following statements are equivalent:*

(i) *g is SD-continuous;*

(ii) *The inverse image of each closed set is X or cs-dense;*

(iii) $Scl(g^{-1}(F)) \subseteq g^{-1}(cl(F))$ *for each* $F \subseteq Y$*;*

(iv) $g(Scl(H)) \subseteq cl(g(H))$ *for each* $H \subseteq X$*;*

(v) $g^{-1}(int(F)) \subseteq Sint(g^{-1}(F))$ *for each* $F \subseteq Y$*.*

Definition 2.14. (X, ζ) is said to be almost compact (resp. almost Lindelöf) if every open cover of X has a finite (resp. countable) subfamily such that the closures of the members cover X.

3. Almost SD-compact and SD-Lindelöf spaces

In this section, we introduce the concepts of almost SD-compact spaces and almost SD-Lindelöf spaces and explore their essential properties. To illustrate the relationship between them, we give some examples.

Definition 3.1. A family of somewhere dense subsets of (X, ζ) is called a somewhere dense cover (briefly, SD-cover) of X if the family is a cover of X.

Definition 3.2. (X, ζ) is said to be almost SD-compact (resp. almost SD-Lindelöf) if every SD-cover of X has a finite (resp. countable) subfamily such that the union of S-closures of the members cover X.

We give the following two examples which we need to illustrate some results.

Example 3.3. Let the set of real numbers $\mathbb{R}$ be the universal set and let $\zeta = \{\emptyset, X, \{1\}\}$ be a topology on $\mathbb{R}$. Note that every superset of $\{1\}$ is somewhere dense, but not *cs*-dense. Therefore, every member of any SD-cover of $(\mathbb{R}, \zeta)$ is a superset of $\{1\}$. Thus, the S-closure of every member of the SD-cover is $\mathbb{R}$. Hence, $(\mathbb{R}, \zeta)$ is almost SD-compact.

Example 3.4. Let $\zeta = \{\emptyset, \mathbb{N}, \{1\}, \mathbb{N} \setminus \{1\}\}$ be a topology on the set of natural numbers $\mathbb{N}$. Note that every subset of $(\mathbb{N}, \zeta)$ (except for the empty and universal sets) is both somewhere dense and *cs*-dense. Therefore, the collection of all singleton subsets of $\mathbb{N}$ forms an SD-cover of $(\mathbb{N}, \zeta)$. Thus, the S-closure of every member of the SD-cover is itself; therefore, this cover does not have a finite subcover satisfying a condition of almost SD-compactness. Hence, $(\mathbb{N}, \zeta)$ is not almost SD-compact.

Proposition 3.5. *Every almost SD-compact space is almost SD-Lindelöf.*

Proof. It directly follows from Definition 3.2. □

Example 3.4 shows that the converse of Proposition 3.5 fails.

Proposition 3.6. *Every almost SD-compact (resp. almost SD-Lindelöf) space is almost compact (resp. almost Lindelöf).*

Proof. Let $\{G_i : i \in I\}$ be an open cover of (X, ζ). Then it is an SD-cover. Since (X, ζ) is almost SD-compact, then there is a finite subfamily G_1, G_2, ..., G_n such that $X = \bigcup_{i=1}^{n} Scl(G_i)$. Since $Scl(G_i) \subseteq cl(G_i)$, then $X = \bigcup_{i=1}^{n} cl(G_i)$, as required.

Similarly, the proof is given in the case of almost SD-Lindelöf □

By replacing the natural numbers set $\mathbb{N}$ in Example 3.4 by the real numbers set $\mathbb{R}$, we note that $(\mathbb{R}, \zeta)$ is Lindelöf, but not almost SD-compact. This clarifies that the converse of Proposition 3.6 fails.

Definition 3.7. A subset G of (X, ζ) is said to be almost SD-compact (resp. almost SD-Lindelöf) if for any cover $\{V_\alpha : \alpha \in \Delta\}$ of G by somewhere dense sets V_α of X, there exists a finite (resp. countable) subset Δ_0 of Δ such that $G \subseteq \cup\{Scl(V_\alpha) : \alpha \in \Delta_0\}$.

Proposition 3.8. *A finite (resp. countable) union of almost SD-compact (resp. almost SD-Lindelöf) subsets of (X, ζ) is almost SD-compact (resp. almost SD-Lindelöf).*

Proof. We prove the proposition in the case of almost SD-Lindelöfness. One can prove the other case similarly.

Let $\{G_n : n \in \mathbb{N}\}$ be a countable family of almost SD-Lindelöf subsets of (X, ζ) and let $\Lambda = \{H_i : i \in I\}$ be an SD-cover of $\bigcup_{n\in\mathbb{N}} G_n$. By the hypothesis, for each $n \in \mathbb{N}$ there exists a countable set S_n of Λ such that $G_n \subseteq \cup_{i\in S_n} Scl(H_i)$. It is clear that $\cup_{n\in\mathbb{N}} S_n$ is a countable set. Therefore, $\bigcup_{n\in\mathbb{N}} G_n \subseteq \bigcup_{i\in\bigcup_{n\in\mathbb{N}} S_n} Scl(H_i)$. Hence, the desired result is proved. □

Proposition 3.9. *If A is almost SD-compact (resp. almost SD-Lindelöf) and B is an SC-set, then $A \cap B$ is almost SD-compact (resp. almost SD-Lindelöf).*

Proof. Let us prove the proposition in the case of almost SD-compactness. The case between parentheses can be achieved similarly.
Let $\{H_\alpha : \alpha \in \Delta\}$ be any cover of $A \cap B$ by SD-open sets of X. Then $\{H_\alpha : \alpha \in \Delta\} \cup (X \setminus B)$ is an SD-cover of A. Since A is almost SD-compact, there exists a finite subset Δ_0 of Δ such that $A \subset [\cup\{Scl(H_\alpha) : \alpha \in \Delta_0\}] \cup Scl(X \setminus B)$. Since B is an SC-set, we have $A \cap B \subseteq \cup\{Scl(H_\alpha) : \alpha \in \Delta_0\}$. Therefore, $A \cap B$ is almost SD-compact. □

Corollary 3.10. *Every SC-subset of an almost SD-compact (resp. almost SD-Lindelöf) space is almost SD-compact (resp. almost SD-Lindelöf).*

In Example 3.3, a subset $\{1, 2\}$ of $(\mathbb{R}, \zeta)$ is almost SD-compact; however, it is not cs-dense. This shows that the converse of the above corollary is false.

Definition 3.11. A family $\Lambda = \{F_i : i \in I\}$ of sets is said to have the first type of finite (resp. countable) SD-intersection property if $\bigcap_{i\in S} Sint(F_i) \neq \emptyset$ for any finite (resp. countable) subset S of I.

Theorem 3.12. *(X, ζ) is almost SD-compact (resp. almost SD-Lindelöf) if and only if every family of cs-dense sets satisfying the first type of finite (resp. countable) SD-intersection property, has, itself, a nonempty intersection.*

Proof. We will start with the proof for almost SD-compactness, because the proof for almost SD-Lindelöfness is analogous.
Suppose that $\Lambda = \{F_i : i \in I\}$ is a family of cs-dense subsets of X such that $\bigcap_{i\in I} F_i = \emptyset$. Then $X = \bigcup_{i\in I} F_i^c$. Since (X,ζ) is almost SD-compact, then there exist finite subsets $F_1, F_2, ..., F_n$ of Λ such that $X = \bigcup_{i=1}^{n} Scl(F_i^c)$. Therefore $\emptyset = (\bigcup_{i=1}^{n} Scl(F_i^c))^c = \bigcap_{i=1}^{n} Sint(F_i)$, as required.
Conversely, let $\Lambda = \{G_i : i \in I\}$ be an SD-cover of X. Then $\emptyset = \bigcap_{i\in I} G_i^c$. By the hypothesis of the first type of finite SD-intersection property, we have $\emptyset = \bigcap_{i=1}^{n} Sint(G_i^c)$ for some finite subsets $G_1, G_2, ..., G_n$ of Λ. Therefore, $X = \bigcup_{i=1}^{n} Scl(G_i)$. Hence, (X,ζ) is almost SD-compact. □

Proposition 3.13. *The SD-continuous image of an almost SD-compact (resp. almost SD-Lindelöf) set is almost compact (resp. almost Lindelöf).*

Proof. Let $g : X \to Y$ be an SD-continuous map and F be an almost SD-Lindelöf subset of X. Suppose that $\{H_i : i \in I\}$ is an open cover of $g(F)$. Then $g(F) \subseteq \bigcup_{i\in I} H_i$. Now, $F \subseteq \bigcup_{i\in I} g^{-1}(H_i)$ and $g^{-1}(H_i)$ is the empty set or somewhere dense for each $i \in I$. By the hypotheses, since F is almost SD-Lindelöf, then $F \subseteq \bigcup_{i\in S} Scl(g^{-1}(H_i))$, where S is a countable subset of I, therefore $g(F) \subseteq \bigcup_{i\in S} g(Scl(g^{-1}(H_i)))$. It follows from Theorem 2.13 that $g(Scl(g^{-1}(H_i))) \subseteq cl(g(g^{-1}(H_i))) \subseteq cl(H_i)$. Thus $g(F) \subseteq \bigcup_{i\in S} cl(H_i)$. Hence, $g(F)$ is almost Lindelöf.
A similar proof is given in the case of almost SD-compactness. □

In a similar way, one can prove the following result.

Proposition 3.14. *The SD-irresolute image of an almost SD-compact (resp. almost SD-Lindelöf) set is almost SD-compact (resp. almost SD-Lindelöf).*

Proposition 3.15. *Let (X,ζ) be strongly hyperconnected and ST_1-space. Then every almost SD-compact subset of (X,ζ) is cs-dense.*

Proof. Let F be an almost SD-compact subset of (X,ζ) and let $x \in F^c$. Since (X,ζ) is ST_1, then for each $y_i \in F$ there are two disjoint sets U_i and V_i containing x and y_i, respectively, such that U_i and V_i are both somewhere dense and cs-dense. Now, $\{V_i : i \in I\}$ is an SD-cover of F. By the hypothesis, there exists a finite subset $\{1, 2, ..., n\}$ of I such that $F \subseteq \bigcup_{i=1}^{n} Scl(V_i) = \bigcup_{i=1}^{n} V_i$. Since (X,ζ) is strongly hyperconnected, then $\bigcap_{i=1}^{n} U_i$ is a somewhere dense set. It is clear that $(\bigcap_{i=1}^{n} U_i) \bigcap (\bigcup_{i=1}^{n} V_i) = \emptyset$. Therefore, $(\bigcap_{i=1}^{n} U_i) \subseteq F^c$ which means that F^c is a somewhere dense set. Hence, F is cs-dense, as required. □

4. Nearly SD-compact and nearly SD-Lindelöf spaces

In this section, we formulate the concepts of nearly SD-compact spaces and nearly SD-Lindelöf spaces. We demonstrate their main properties and explain the relationship between them with the help of examples.

Definition 4.1. (X, ζ) is said to be nearly SD-compact (resp. nearly SD-Lindelöf) if every SD-cover of X has a finite (resp. countable) subfamily whose S-closure covers X.

Proposition 4.2. (i) *Every nearly SD-compact space is nearly SD-Lindelöf.*

(ii) *Every almost SD-compact (resp. almost SD-Lindelöf) space is nearly SD-compact (resp. nearly SD-Lindelöf).*

Proof. Straightforward. □

The following examples show that the converse of the above proposition is not true.

Example 4.3. Assume that $(\mathcal{N}, \zeta)$ is the same as in Example 3.4. One can easily check that $(\mathcal{N}, \zeta)$ is nearly SD-Lindelöf, but not nearly SD-compact.

Example 4.4. Let the set of real numbers $\mathbb{R}$ be the universal set and let $\zeta = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}\}$ be a topology on $\mathbb{R}$. Note that every superset of $\{1, 2\}$ is somewhere dense, but not *cs*-dense. The collection $\{\{2\}, \{1, x\} : x \neq 2\}$ forms an SD-cover of $(\mathbb{R}, \zeta)$. It is clear that the S-closure of every member of this SD-cover is itself; therefore, $(\mathbb{R}, \zeta)$ is not almost SD-compact. On the other hand, every SD-cover of $(\mathbb{R}, \zeta)$ contains at least two subsets such that $\{1, 2\}$ is contained in them. It is clear that $Scl(\{1, 2\}) = \mathbb{R}$. Hence, $(\mathbb{R}, \zeta)$ is nearly SD-compact.

Definition 4.5. A family $\Lambda = \{F_i : i \in I\}$ of sets is said to have the second type of finite (resp. countable) SD-intersection property if $Sint[\bigcap_{i\in S} F_i] \neq \emptyset$ for any finite (resp. countable) subset S of I.

It is clear that if a collection satisfies the second type of finite (resp. countable) SD-intersection property, then it satisfies the first type of finite (resp. countable) SD-intersection property.

Theorem 4.6. (X, ζ) *is nearly SD-compact (resp. nearly SD-Lindelöf) if and only if every family of cs-dense subsets of* (X, ζ)*, satisfying the second type of finite (resp. countable) SD-intersection property, has, itself, a nonempty intersection.*

Proof. We only prove the theorem when (X, ζ) is nearly SD-compact. The case between parentheses can be made similarly.

Let $\Lambda = \{F_i : i \in I\}$ be a family of *cs*-dense subsets of X. Suppose that $\bigcap_{i\in I} F_i = \emptyset$. Then $X = \bigcup_{i\in I} F_i^c$. Since (X, ζ) is nearly SD-compact, then there exist finite subsets $F_1, F_2, ..., F_n$ of Λ such that $X = Scl(\bigcup_{i=1}^n F_i^c)$. Therefore $\emptyset = (Scl(\bigcup_{i=1}^n F_i^c))^c = Sint(\bigcap_{i=1}^n F_i)$. Hence, the necessary condition holds.

Conversely, Let Λ be a family of *cs*-dense subsets of X which satisfies the second type of finite SD-intersection property. Then it also satisfies the first type of finite SD-intersection property. Since Λ has a nonempty intersection, then (X, ζ) is an almost SD-compact space. It follows from Proposition 4.2 that (X, ζ) is nearly SD-compact. □

Definition 4.7. A subset G of (X, ζ) is said to be nearly SD-compact (resp. nearly SD-Lindelöf) if for every cover $\{V_\alpha : \alpha \in \Delta\}$ of G by somewhere dense sets V_α of X, there exists a finite (resp. countable) subset Δ_0 of Δ such that $G \subset Scl(\cup\{V_\alpha : \alpha \in \Delta_0\})$.

Proposition 4.8. *A finite (resp. countable) union of nearly SD-compact (resp. nearly SD-Lindelöf) subsets of (X, ζ) is nearly SD-compact (resp. nearly SD-Lindelöf).*

Proof. We prove the proposition in the case of nearly SD-Lindelöfness. One can prove the other case similarly.

Let $\{G_n : n \in \mathbb{N}\}$ be a countable family of nearly SD-Lindelöf subsets of (X, ζ) and let $\Lambda = \{H_i : i \in I\}$ be an SD-cover of $\bigcup_{n\in\mathbb{N}} G_n$. By the hypothesis, for each $n \in \mathbb{N}$ we obtain a countable subset S_n of I such that $G_n \subseteq Scl[\bigcup_{i\in S_n}(H_i)]$. It is clear that $\bigcup_{n\in\mathbb{N}} S_n$ is a countable set. Therefore, $G_1 \subseteq Scl(\bigcup_{i\in S_1} H_i)$, $\ldots$, $G_n \subseteq Scl(\bigcup_{i\in S_n} H_i)$, $\ldots$. Thus, $\bigcup_{n\in\mathbb{N}} G_n \subseteq Scl[\bigcup_{i\in\bigcup_{n\in\mathbb{N}} S_n}(H_i)]$. Hence, the desired result is proved. □

Proposition 4.9. *The SD-continuous image of a nearly SD-compact (resp. nearly SD-Lindelöf) set is nearly compact (resp. nearly Lindelöf).*

Proof. Let $g : X \to Y$ be an SD-continuous map and F be a nearly SD-Lindelöf subset of X. Suppose that $\{H_i : i \in I\}$ is an open cover of $g(F)$. Then $g(F) \subseteq \bigcup_{i\in I} H_i$. Now, $F \subseteq \bigcup_{i\in I} g^{-1}(H_i)$ and $g^{-1}(H_i)$ is the empty set or somewhere dense for each $i \in I$. By the hypotheses, since F is nearly SD-Lindelöf, then $F \subseteq Scl(\bigcup_{i\in S} g^{-1}(H_i))$, where S is a countable subset of I; hence, $g(F) \subseteq g(Scl(\bigcup_{i\in S} g^{-1}(H_i)))$. It follows from Theorem 2.13 that $g(Scl(\bigcup_{i\in S} g^{-1}(H_i))) \subseteq g(g^{-1}(cl(\bigcup_{i\in S} H_i))) \subseteq cl(\bigcup_{i\in S} H_i)$. Thus $g(F) \subseteq cl(\bigcup_{i\in S} H_i)$. Hence, $g(F)$ is nearly Lindelöf.
A similar proof is given in the case of nearly SD-compactness. □

In a similar way, one can prove the following result.

Proposition 4.10. *The SD-irresolute image of a nearly SD-compact (resp. nearly SD-Lindelöf) set is nearly SD-compact (resp. nearly SD-Lindelöf).*

Proposition 4.11. *Let (X, ζ) be strongly hyperconnected and ST_1-space. Then every nearly SD-compact subset of (X, ζ) is cs-dense.*

Proof. The proof is similar to that of Proposition 3.15. □

5. Mildly SD-compact and mildly SD-Lindelöf spaces

In this section, we define the concepts of mildly SD-compact spaces and mildly SD-Lindelöf spaces. We reveal main properties and relationships with the help of examples.

Definition 5.1. A family of subsets of (X, ζ) is called an SC-cover of X if the family is a cover of X and every member of this family is an SC-subset of (X, ζ).

Definition 5.2. (X, ζ) is said to be mildly SD-compact (resp. mildly SD-Lindelöf) if every SC-cover of X has a finite (resp. countable) subcover.

Proposition 5.3. (i) *Every mildly SD-compact (resp. mildly SD-Lindelöf) space is mildly compact (resp. mildly Lindelöf).*

(ii) *Every mildly SD-compact space is mildly SD-Lindelöf.*

(iii) *Every almost SD-compact (resp. almost SD-Lindelöf) space is mildly SD-compact (resp. mildly SD-Lindelöf).*

Proof. The proofs of (i) and (ii) are obvious.

We prove (iii) in the case of almost SD-Lindelöfness.
Let $\Lambda = \{H_i : i \in I\}$ be an SC-cover of (X, ζ). Then there exists a countable subset S of I such that $X = \cup_{i \in S} Scl(H_i)$. Now, $Scl(H_i) = H_i$ for each $i \in I$. Therefore, (X, ζ) is mildly SD-Lindelöf.
A similar proof is given when (X, ζ) is almost SD-compact. □

Definition 5.4. A subset G of (X, ζ) is said to be mildly SD-compact (resp. mildly SD-Lindelöf) if every cover of G by SC-subsets of X has a finite (resp. countable) subcover.

Proposition 5.5. *A finite (resp. countable) union of mildly SD-compact (resp. mildly SD-Lindelöf) subsets of (X, ζ) is mildly SD-compact (resp. mildly SD-Lindelöf).*

Proof. We prove the proposition in the case of mildly SD-Lindelöfness. One can prove the other case similarly.

Let $\{G_n : n \in \mathbb{N}\}$ be a countable family of mildly SD-Lindelöf subsets of (X, ζ) and let $\Lambda = \{H_i : i \in I\}$ be a cover of $\cup_{n \in N} G_n$ by SC-subsets of X. By the hypothesis, for each $n \in \mathbb{N}$ we obtain a countable subset S_n of I such that $G_n \subseteq \bigcup_{i \in S_n} H_i$. It is clear that $\bigcup_{n \in \mathbb{N}} S_n$ is a countable set. Therefore, $G_1 \subseteq \bigcup_{i \in S_1} H_i, \ldots, G_n \subseteq \bigcup_{i \in S_n} H_i, \ldots$. Thus, $\bigcup_{n \in \mathbb{N}} G_n \subseteq \bigcup_{i \in \bigcup_{n \in \mathbb{N}} S_n} H_i$. Hence, the desired result is proved. □

Proposition 5.6. *The SD-continuous image of a mildly SD-compact (resp. mildly SD-Lindelöf) set is mildly compact (resp. mildly Lindelöf).*

Proof. Let $g : X \to Y$ be an SD-continuous map and F be a mildly SD-Lindelöf subset of X. Suppose that $\{H_i : i \in I\}$ is a clopen cover of $g(F)$. Then $g(F) \subseteq \bigcup_{i \in I} H_i$. Now, $F \subseteq \bigcup_{i \in I} g^{-1}(H_i)$ and $g^{-1}(H_i)$ is an SC-subset in X for each $i \in I$. By the hypotheses, since F is mildly SD-Lindelöf, then $F \subseteq \cup_{i \in S} g^{-1}(H_i)$, where S is a countable subfamily of I; therefore, $g(F) \subseteq g(\bigcup_{i \in S} g^{-1}(H_i)) \subseteq g(g^{-1}(\bigcup_{i \in S} H_i)) \subseteq \bigcup_{i \in S} H_i$. Thus $g(F) \subseteq \bigcup_{i \in S} H_i$. Hence, $g(F)$ is mildly Lindelöf.
A similar proof is given in the case of mildly SD-compactness. □

In a similar way, one can prove the following result.

Proposition 5.7. *The SD-irresolute image of a mildly SD-compact (resp. mildly SD-Lindelöf) set is mildly SD-compact (resp. mildly SD-Lindelöf).*

Theorem 5.8. (X, ζ) *is mildly SD-compact (resp. mildly SD-Lindelöf) if and only if every collection of SC-subsets of* (X, ζ)*, satisfying the finite (resp. countable) intersection property, has, itself, a nonempty intersection.*

Proof. We only prove the theorem when (X, ζ) is mildly SD-compact. The other case can be made similarly.

Let $\Lambda = \{F_i : i \in I\}$ be a collection of SC-subsets of X. Suppose that $\bigcap_{i\in I} F_i = \emptyset$. Then $X = \cup_{i\in I} F_i^c$. Since (X, ζ) is mildly SD-compact, then there exists a finite subset I_0 of I such that $\cup_{i\in I_0} F_i^c = X$. Hence, the necessary condition holds.

Conversely, let $\{H_i : I \in I\}$ be an SC-cover of X. Then $X = \cup_{i\in I} H_i$ and $\cap_{i\in I}(X \setminus H_i) = \emptyset$. By the hypothesis, there exists a finite subset I_0 of I such that $\cap_{i\in I_0}(X \setminus H_i) = \emptyset$; hence, $\cup_{i\in I_0} H_i = X$. □

For the sake of economy, the proofs of the following two propositions will be omitted.

Proposition 5.9. *The intersection of a mildly SD-compact (resp. mildly SD-Lindelöf) set and an SC-subset of X is mildly SD-compact (resp. mildly SD-Lindelöf).*

Proposition 5.10. *Every SC-subset of a mildly SD-compact (resp. mildly SD-Lindelöf) space* (X, ζ) *is mildly SD-compact (resp. mildly SD-Lindelöf).*

Definition 5.11. (X, ζ) is called a clopen topological space if every open set is closed.

Proposition 5.12. *If* (X, ζ) *is a clopen topological space, then* $S(\zeta)$ *is the discrete topology.*

Proof. Let F be any nonemty subset of (X, ζ). Then $cl(F)$ is closed. Since (X, ζ) is a clopen topological space, then $cl(F)$ is also nonempty open; therefore, F is somewhere dense. As a result of choosing F randomly, $S(\zeta)$ is the discrete topology. □

Corollary 5.13. *If* (X, ζ) *is a clopen topological space, then* (X, ζ) *is almost SD-compact (nearly SD-compact, mildly SD-compact).*

Proposition 5.14. *Let* (X, ζ) *be strongly hyperconnected and* ST_1*-space. Then every mildly SD-compact subset of* (X, ζ) *is cs-dense.*

Proof. The proof is similar to that of Proposition 3.15. □

Conclusion

Recently, many researchers have been interested in generalizations of open subsets of a topological space. To contribute to this area, we have introduced and studied almost SD-compact and almost SD-Lindelöf spaces, nearly SD-compact and nearly SD-Lindelöf spaces, and mildly SD-compact and mildly SD-Lindelöf spaces. We have supplied several examples to elucidate the relationships between them. Also, we have characterized them using different types of finite (resp. countable) SD-intersection property.

In an upcoming paper, we will apply somewhere dense sets to initiate new types of approximation spaces. As a matter of fact, we reduce the boundary region by increasing the lower approximation (which represents the interior points from the topological viewpoint) and decreasing the upper approximation (which represents the closure points from the topological viewpoint) using somewhere dense sets.

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