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Almost everywhere convergence of Riesz means of one-dimensional Fourier series on the group of 2-adic integers[1]

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Abstract. In this paper, we prove the almost everywhere convergence of Riesz means of integrable functions on the group of 2-adic integers, $\sigma_n^{\alpha,\gamma} f \to f$, where $0 < \gamma \leq 1 \leq \alpha$.

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1. Introduction

We follow the standard notation of the dyadic analysis introduced by mathematicians F. Schipp, P. Simon, W. R. Wade [8]. The set of natural numbers and the set of positive numbers are denote by $\mathbb{N} := \{0, 1, \ldots\}$ and $\mathbb{P} := \mathbb{N} \setminus \{0\}$, respectively. Let $I := [0, 1)$ be the unit interval. Denote by $\lambda(B) = |B|$ the Lebesgue measure of the set B $(B \subset I)$. $(L^p(I), \|.\|_p)$ stands for the usual Lebesgue space and the corresponding norm $(1 \leq p \leq \infty)$. Set

$$\mathcal{I} := \left\{ \left[\frac{p}{2^n}, \frac{p+1}{2^n}\right) : p, n \in \mathbb{N} \right\}$$

as the set is of dyadic intervals and for given $x \in I$ and $n \in \mathbb{N}$ $I_n(x) \in \mathcal{I}$ denotes that a dyadic interval is of length 2^{-n} and it contains x. We also use the notation $I_n := I_n(0)$. Let

$$x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)}$$

be the dyadic expansion of $x \in I$, where $x_n \in \{0, 1\}$. If x is a dyadic rational number, that is $x \in \{\frac{p}{2^n} : p, n \in \mathbb{N}\}$, then we choose an expression which terminates in 0's.

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We consider the binary form of the natural number n:

$$n = \sum_{k=0}^{\infty} n_k 2^k,$$

where $n_k \in \{0, 1\}$. We also use the following notation: $|n| := \lfloor \log_2(n) \rfloor$,

$$n_{(k)} := \sum_{i=0}^{k} n_i 2^i, \qquad \text{and} \qquad n^{(k)} := \sum_{i=k}^{\infty} n_i 2^i.$$

In the next step we define the 2-adic (or arithmetic) sum

$$a + b := \sum_{n=0}^{\infty} r_n 2^{-(n+1)} \qquad (a, b \in I)$$

where $q_n, r_n \in \{0, 1\}$ for every natural number n. The elements q_n and r_n are defined recursively as follows: $q_{-1} := 0$, $a_n + b_n + q_{n-1} = 2q_n + r_n$ for $n \in \mathbb{N}$. (These equations uniquely determine the coeffcients q_n and r_n since these elements take on only the values 0,1.) The group $(I, +)$ is called the group of 2-adic integers. Set

$$\varepsilon(t) := \exp(2\pi \iota t) \qquad (t \in \mathbb{R}),$$

where $\iota = \sqrt{-1}$. Set

$$v_{2^n}(x) := \varepsilon\left(\frac{x_n}{2} + \ldots + \frac{x_0}{2^{n+1}}\right) \qquad (x \in I, n \in \mathbb{N}),$$

and

$$v_n := \prod_{n=0}^{\infty} v_{2^j}^{n_j},$$

where $\mathbb{N} \ni n = \sum_{k=0}^{\infty} n_k 2^k$ $(n_k \in \{0, 1\}, i \in \mathbb{N})$. It is known [6] that $(v_n, n \in \mathbb{N})$ is the character system of $(I, +)$.

The next lemma highlights an important property of $v_n(x)$.

Lemma 1.1. *[3] For $k, n \in \mathbb{N}$, $k < 2^n$ we have*

$$v_{2^n-k-1}(z) = v_{2^n-1}(z)\bar{v}_k(z)$$

Consider the Dirichlet and the Fejér kernels:

$$D_n := \sum_{k=0}^{n-1} v_k, \quad K_n := \frac{1}{n+1} \sum_{k=0}^{n} D_k, \qquad D_0, K_0 := 0.$$

It is well-known that [7, 6] for $n \in \mathbb{N}$ and $x \in I$

(D1) $$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n, \end{cases}$$

and also

$$D_n(x) = v_n(x) \sum_{k=0}^{\infty} D_{2^k}(x) n_k (-1)^{x_k}$$

are statisfied.

The Fourier coefficients, the partial sum of the Fourier series, the $(C,1)$ and the Fejér means of $f \in L^1(I)$ are defined in the following way ($'-'$ is the inverse operation of $'+'$ on I)

$$\hat{f}(n) := \int_I f(x)\bar{v}_n(x)\, d\lambda(x) \qquad (n \in \mathbb{N}),$$

$$S_n f(y) := \sum_{k=0}^{n-1} \hat{f}(k)\omega_k(y) = \int_I f(x) D_n(y-x)\, d\lambda(x),$$

$$\sigma_n f(y) := \frac{1}{n+1} \sum_{k=0}^{n} S_k f(y) = \int_I f(x+y) K_n(y-x)\, d\lambda(x) \qquad (n \in \mathbb{N}, y \in I),$$

respectively.

The next lemma will play an important role in the proof of the main theorem.

Lemma 1.2. *[3] Let B and t be fixed natural numbers. Then*

$$\int_{\bar{I}_t} \sup_{|n| \geq B} |K_n(z)|\, d\lambda(z) \leq \begin{cases} C(t-B), & \text{if } B < t, \\ C\dfrac{(B-t)^2}{2^{B-t}}, & \text{if } B \geq t. \end{cases}$$

Corollary 1.3. *[2]*

$$\|K_n\| \leq C \qquad \text{for all } n \in \mathbb{N}.$$

Denote by the kernel function $K_n^{\alpha,\gamma}$ of the Riesz summability method

$$K_n^{\alpha,\gamma} := \frac{1}{n^{\alpha\gamma}} \sum_{k=0}^{n-1} (n^\gamma - k^\gamma)^\alpha v_k,$$

where $n \in \mathbb{N}$ and $0 < \gamma \leq 1 \leq \alpha < \infty$. The Riesz means of the integrable functions f are

$$\sigma_n^{\alpha,\gamma} f(y) := \frac{1}{n^{\alpha\gamma}} \sum_{j=0}^{n-1} (n^\gamma - k^\gamma)^\alpha \hat{f}(j).$$

The Riesz means are called Fejér means if $\alpha = \gamma = 1$.

In 1955, N.J. Fine proved the Fejér-Lebesgue theorem for the Walsh-system [1], stating that the almost everywhere convergence $\sigma_n^1 f \to f$ holds for $f \in L^1$.

For the character system of the group of 2-adic integers, Fejér means related to Taibleson [9] question, which was open for a long time. In 1997, Gát [4] answered the Taibleson's question in the affirmative.

For the trigonometric system, the almost everywhere convergence of Riesz-means was proved by Riesz ([5, 11]). The case $0 < \gamma \leq 1 \leq \alpha$ it was proved in the Walsh case by Weisz [10].

2. Main results

The main aim of this paper is to prove the same result in the 2-adic case.

Theorem 2.1. *Let $0 < \gamma \le 1 \le \alpha$. Then we have $\sigma_n^{\alpha,\gamma} f \to f$ for every $f \in L^1(I)$, where I is the group of 2-adic integers.*

During the proof we use the following notations. Let $a_{n,k} = (n^\gamma - k^\gamma)^\alpha = n^{\alpha\gamma}\Theta\left(\frac{k}{n}\right)$, where

$$\Theta(x) = (1 - x^\gamma)^\alpha.$$

Moreover,

$$\Delta_1 a_{n,k} := a_{n,k} - a_{n,k+1}, \qquad \text{and} \qquad \Delta_2 a_{n,k} := \Delta_1 a_{n,k} - \Delta_1 a_{n,k+1}.$$

It is easy to prove the following: if $0 < \gamma \le 1 \le \alpha < \infty$, then Θ' is increasing and it has negative values. The derivative, $\Theta'(x)$, is bounded. Since $\alpha - 1 \ge 0$, then $0 \le (1 - x^\gamma)^{\alpha-1} \le 1$ and consequently $|\Theta'(x)| \le \alpha\gamma x^{\gamma-1}$.

Furthermore, Θ'' and $\Delta_2\Theta$ have only positive values.

Now we investigate $\Delta_2\Theta(x)$. By the theorem of Lagrange we get

$$\begin{aligned}\Delta_2\Theta(x) &= \Theta(x) - \Theta(x+h) - (\Theta(x+h) - \Theta(x+2h))\\ &= \Theta'(\xi)\cdot(-h) - \Theta'(\eta)(-h)\end{aligned}$$

for some $x < \xi < x + h$ and $x + h < \eta < x + 2h$. We apply the Lagrange theorem again. Then we have

$$\Theta'(\xi)\cdot(-h) - \Theta'(\eta)(-h) = h(\Theta'(\eta) - \Theta'(\xi)) = h \cdot \bar{h} \cdot \Theta''(\zeta)$$

with some $0 < \bar{h} < 2h$ and $\xi < \zeta < \eta$. The difference h will mainly be something like $1/n$.

Therefore, the second difference of $\Theta(x)$ is positive for $0 < \gamma \le 1 \le \alpha < \infty$.

Lemma 2.2. *Let n and t be natural numbers. An upper bound for the Riesz kernel can be written in the following form:*

$$|K_n^{\alpha,\gamma}| \le \sum_{j=1}^{7} T_j + \left|\tilde{K}_n^{\alpha,\gamma}\right|,$$

where

$$T_1 := n^{-\alpha\gamma} \sum_{k=0}^{t-1} n_k \sum_{l=0}^{2^k-1} (l+2)\left|\Delta_2 a_{n,n^{(k)}-l-1}\right| |K_l|,$$

$$T_2 := n^{-\alpha\gamma} \sum_{k=t}^{|n|-1} n_k \sum_{l=0}^{2^t-1} (l+2)\left|\Delta_2 a_{n,n^{(k)}-l-1}\right| |K_l|,$$

$$T_3 := n^{-\alpha\gamma} \sum_{k=t}^{|n|-1} n_k \sum_{l=2^t}^{2^k-1} (l+2)\left|\Delta_2 a_{n,n^{(k)}-l-1}\right| |K_l|,$$

$$T_4 := n^{-\alpha\gamma} \sum_{k=0}^{t-1} n_k (2^k + 1) \left|\Delta_1 a_{n,n^{(k)}} - 2^k\right| \cdot |K_{2^k}|,$$

$$T_5 := n^{-\alpha\gamma} \sum_{k=t}^{|n|-1} n_k (2^k + 1) \left|\Delta_1 a_{n,n^{(k)}-2^k}\right| \cdot |K_{2^k}|,$$

$$T_6 := n^{-\alpha\gamma} \sum_{k=0}^{t-1} n_k \left|a_{n,n^{(k)}-2^k}\right| D_{2^k},$$

$$T_7 := n^{-\alpha\gamma} \sum_{k=t}^{|n|} n_k \left|a_{n,n^{(k)}-2^k}\right| D_{2^k},$$

$$\tilde{K}_n^{\alpha,\gamma} := n^{-\alpha\gamma} \sum_{i=0}^{2^{|n|}-1} a_{n,i} v_i.$$

Proof. We use the binary form of n and Lemma 1.1 and then we have

$$\begin{aligned}
K_n^{\alpha,\gamma} &= n^{-\alpha\gamma} \sum_{i=0}^{n-1} a_{n,i} v_i \\
&= n^{-\alpha\gamma} \sum_{k=0}^{|n|-1} n_k \sum_{i=n^{(k+1)}}^{n^{(k+1)}+2^k-1} a_{n,i} v_i + n^{-\alpha\gamma} n_{|n|} \sum_{i=0}^{2^{|n|}-1} a_{n,i} v_i \\
&= n^{-\alpha\gamma} \sum_{k=0}^{|n|-1} n_k \sum_{l=0}^{2^k-1} a_{n,n^{(k)}-l-1} v_{n^{(k)}-l-1} + n^{-\alpha\gamma} \sum_{i=0}^{2^{|n|}-1} a_{n,i} v_i \\
&= n^{-\alpha\gamma} \sum_{k=0}^{|n|-1} n_k v_{n^{(k)}-1} \sum_{l=0}^{2^k-1} a_{n,n^{(k)}-l-1} \bar{v}_l + \tilde{K}_n^{\alpha,\gamma}.
\end{aligned}$$

We apply Abel's transformation twice. This implies

$$\begin{aligned}
\sum_{l=0}^{2^k-1} a_{n,n^{(k)}-l-1} \bar{v}_l &= \sum_{l=0}^{2^k-1} \Delta_1 a_{n,n^{(k)}-l-1} \sum_{j=0}^{l} \bar{v}_j + a_{n,n^{(k)}-2^k} \sum_{l=0}^{2^k-1} \bar{v}_l \\
&= \sum_{l=0}^{2^k-1} \Delta_1 a_{n,n^{(k)}-l-1} \bar{D}_{l+1} + a_{n,n^{(k)}-2^k} \bar{D}_{2^k} \\
&= \sum_{l=0}^{2^k-1} \Delta_2 a_{n,n^{(k)}-l-1} \sum_{j=0}^{l} \bar{D}_{j+1} + \Delta_1 a_{n,n^{(k)}-2^k} \sum_{l=0}^{2^k-1} \bar{D}_{l+1} \\
&\quad + a_{n,n^{(k)}-2^k} \bar{D}_{2^k} \\
&= \sum_{l=0}^{2^k-1} \Delta_2 a_{n,n^{(k)}-l-1} (l+2) \bar{K}_{l+1} + \Delta_1 a_{n,n^{(k)}-2^k} (2^k + 1) \bar{K}_{2^k-1} \\
&\quad + a_{n,n^{(k)}-2^k} \bar{D}_{2^k},
\end{aligned}$$

where $k < |n|$. Subtituting this formula we get the following estimation:

$$\begin{aligned}|K_n^{\alpha,\gamma}| &\le n^{-\alpha\gamma} \sum_{k=0}^{|n|-1} n_k \sum_{l=0}^{2^k-1} \left|\Delta_2 a_{n,n^{(k)}-l-1}\right| (l+2) \left|K_{l+1}\right| \\ &+ n^{-\alpha\gamma} \sum_{k=0}^{|n|-1} n_k (2^k+1) \left|\Delta_1 a_{n,n^{(k)}-2^k}\right| \cdot \left|K_{2^k-1}\right| \\ &+ n^{-\alpha\gamma} \sum_{k=0}^{|n|-1} n_k \left|a_{n,n^{(k)}-2^k}\right| D_{2^k} + \left|\tilde{K}_n^{\alpha,\gamma}\right|.\end{aligned}$$

The statement of the lemma comes from this estimation. □

2.1. The case $\tilde{K}_n^{\alpha,\gamma}$

The aim of this section is to give an upper estimation for the function $\tilde{K}_n^{\alpha,\gamma}$, where the definition of $\tilde{K}_n^{\alpha,\gamma}$ is

$$\tilde{K}_n^{\alpha,\gamma} = n^{-\alpha\gamma} \sum_{i=0}^{2^{|n|}-1} a_{n,i} v_i.$$

Lemma 2.3. *An upper bound for the $\tilde{K}_n^{\alpha,\gamma}$ kernel can be written in the following form:*

$$\left|\tilde{K}_n^{\alpha,\gamma}\right| \le \sum_{i=8}^{11} T_i,$$

where

$$T_8 := n^{-\alpha\gamma} \sum_{i=0}^{2^t-1} (i+2) \left|\Delta_2 a_{n,i}\right| \cdot \left|K_{i+1}\right|,$$

$$T_{10} := n^{-\alpha\gamma} (2^{|n|}+1) \left|\Delta_1 a_{n,2^{|n|}}\right| \cdot \left|K_{2^{|n|}}\right|,$$

$$T_9 := n^{-\alpha\gamma} \sum_{i=2^t}^{2^{|n|}-1} (i+2) \left|\Delta_2 a_{n,i}\right| \cdot \left|K_{i+1}\right|,$$

$$T_{11} := n^{-\alpha\gamma} \left|a_{n,2^{|n|}}\right| D_{2^{|n|}}.$$

Proof. Applying the Abel transformation, we have

$$\begin{aligned}\left|\tilde{K}_n^{\alpha,\gamma}\right| &\le n^{-\alpha\gamma} \left|\sum_{i=0}^{2^{|n|}-1} a_{n,i} v_i\right| \\ &= n^{-\alpha\gamma} \left|\sum_{i=0}^{2^{|n|}-1} \Delta_1 a_{n,i} D_{i+1} + a_{n,2^{|n|}} D_{2^{|n|}}\right|\end{aligned}$$

$$= n^{-\alpha\gamma} \left| \sum_{i=0}^{2^{|n|}-1} \Delta_2 a_{n,i} \sum_{j=0}^{i} D_{i+1} + \Delta_1 a_{n,2^{|n|}} \sum_{i=0}^{2^{|n|}-1} D_{i+1} + a_{n,2^{|n|}} D_{2^{|n|}} \right|$$
$$\leq n^{-\alpha\gamma} \left[\sum_{i=0}^{2^{|n|}-1} (i+2) \left|\Delta_2 a_{n,i}\right| \cdot |K_{i+1}| + (2^{|n|}+1) \left|\Delta_1 a_{n,2^{|n|}}\right| \cdot \left|K_{2^{|n|}}\right| \right]$$
$$+ n^{-\alpha\gamma} \left|a_{n,2^{|n|}}\right| D_{2^{|n|}},$$

which proves the lemma.

□

Lemma 2.4. *Let n be a natural number. Then*

$$n^{-\alpha\gamma}(2^{|n|}+1) \left|\Delta_1 a_{n,2^{|n|}}\right| \leq C_{\alpha,\gamma}.$$

Proof. By the Lagrange theorem, we obtain

$$n^{-\alpha\gamma}(2^{|n|}+1) \left|\Delta_1 a_{n,2^{|n|}}\right| = n^{-\alpha\gamma}(2^{|n|}+1) \left|a_{n,2^{|n|}} - a_{n,2^{|n|}+1}\right|$$
$$= (2^{|n|}+1) \left| \left(1 - \left(\frac{2^{|n|}}{n}\right)^{\gamma}\right)^{\alpha} - \left(1 - \left(\frac{2^{|n|}+1}{n}\right)^{\gamma}\right)^{\alpha} \right|$$
$$= (2^{|n|}+1) \left| \Theta\left(\frac{2^{|n|}}{n}\right) - \Theta\left(\frac{2^{|n|}+1}{n}\right) \right|$$
$$\leq \frac{2^{|n|}+1}{n} \left|\Theta'(\xi_1)\right| =: (A),$$

for some $\frac{2^{|n|}}{n} < \xi_1 < \frac{2^{|n|}+1}{n}$.

Since $0 < \gamma \leq 1 \leq \alpha < \infty$, then $(A) \leq \frac{2^{|n|}+1}{n} \left|\Theta'\left(\frac{2^{|n|}}{n}\right)\right|$ is bounded by a constant $C_{\alpha,\gamma}$, because $\Theta'(x)$ is bounded for $1/4 \leq x \leq 1$. □

Lemma 2.5. *We suppose that $0 < \gamma \leq 1 \leq \alpha < \infty$ and $|n| \geq B \geq t$ where B is fixed. Then*

$$\int_{\bar{I}_t} \sup_{|n|\geq B} T_{10} \leq C_{\alpha,\gamma}.$$

Proof. By Lemma 2.4 we have

$$\int_{\bar{I}_t} \sup_{|n|\geq B} T_{10} = \int_{\bar{I}_t} \sup_{|n|\geq B} n^{-\alpha\gamma}(2^{|n|}+1) \left|\Delta_1 a_{n,2^{|n|}}\right| \cdot \left|K_{2^{|n|}}\right|$$
$$\int_{\bar{I}_t} \sup_{j>t} |K_{2^j}| \cdot \sup_{|n|\geq B} n^{-\alpha\gamma}(2^{|n|}+1) \left|\Delta_1 a_{n,2^{|n|}}\right| \leq C_{\alpha,\gamma} \int_{\bar{I}_t} \sup_{j>t} |K_{2^j}| \leq C_{\alpha,\gamma}.$$

The last inequality is implied by Lemma 1.2. □

Lemma 2.6. *Let n be a natural number. Then we get*

$$n^{-\alpha\gamma}\sum_{i=2^t}^{2^{|n|}-1}(i+2)\left|\Delta_2 a_{n,i}\right|\le C_{\alpha,\gamma}.$$

Proof. In this case, the second difference of $a_{n,k}$ is positive. We get

$$\begin{aligned}
&n^{-\alpha\gamma}\sum_{i=2^t}^{2^{|n|}-1}(i+2)\left|\Delta_2 a_{n,i}\right|\\
&=n^{-\alpha\gamma}\sum_{m=t}^{|n|-1}\sum_{i=2^{m-1}}^{2^m-1}(i+2)\left|\Delta_1 a_{n,i}-\Delta_1 a_{n,i+1}\right|\\
&\le n^{-\alpha\gamma}\sum_{m=t}^{|n|-1}(2^m+1)\sum_{i=2^{m-1}}^{2^m-1}\left|\Delta_1 a_{n,i}-\Delta_1 a_{n,i+1}\right|\\
&\le n^{-\alpha\gamma}\sum_{m=t}^{|n|-1}(2^m+1)\sum_{i=2^{m-1}}^{2^m-1}\left(\Delta_1 a_{n,i}-\Delta_1 a_{n,i+1}\right)\\
&=n^{-\alpha\gamma}\sum_{m=t}^{|n|-1}(2^m+1)\left(\Delta_1 a_{n,2^{m-1}}-\Delta_1 a_{n,2^m}\right)\\
&\le n^{-\alpha\gamma}\sum_{m=t}^{|n|-1}(2^m+1)\left|\Delta_1 a_{n,2^{m-1}}\right|+n^{-\alpha\gamma}\sum_{m=t}^{|n|-1}(2^m+1)\left|\Delta_1 a_{n,2^m}\right|\\
&=:(A)+(B)
\end{aligned}$$

The estimation of the function (A) consists of the following steps:

$$\begin{aligned}
&n^{-\alpha\gamma}\sum_{m=t}^{|n|-1}(2^m+1)\left|\Delta_1 a_{n,2^{m-1}}\right|=n^{-\alpha\gamma}\sum_{m=t}^{|n|-1}(2^m+1)\left|a_{n,2^{m-1}}-a_{n,2^{m-1}+1}\right|\\
&=\sum_{m=t}^{|n|-1}(2^m+1)\left|\left(1-\left(\frac{2^{m-1}}{n}\right)^{\gamma}\right)^{\alpha}-\left(1-\left(\frac{2^{m-1}+1}{n}\right)^{\gamma}\right)^{\alpha}\right|\\
&=\sum_{m=t}^{|n|-1}(2^m+1)\left|\Theta\left(\frac{2^{m-1}}{n}\right)-\Theta\left(\frac{2^{m-1}+1}{n}\right)\right|\\
&=C_{\alpha,\gamma}\sum_{m=t}^{|n|-1}\frac{2^m+1}{n}\left|\Theta'(\xi)\right|\le C_{\alpha,\gamma}\sum_{m=t}^{|n|-1}\alpha\gamma\cdot\frac{2^m+1}{n}\left(\frac{2^m}{n}\right)^{\gamma-1}\le C_{\alpha,\gamma}
\end{aligned}$$

by the Lagrange theorem for some $\frac{2^{m-1}}{n}<\xi<\frac{2^{m-1}+1}{n}$. We also used the fact that $|\Theta'(x)|$ is bounded.

The upper estimation of the kernel (B) is similar to the previous case. We have

$$\begin{aligned}
(B) &= n^{-\alpha\gamma}\sum_{m=t}^{|n|-1}(2^m+1)\left|\Delta_1 a_{n,2^m}\right| = n^{-\alpha\gamma}\sum_{m=t}^{|n|-1}(2^m+1)\left|a_{n,2^m}-a_{n,2^m+1}\right| \\
&= \sum_{m=t}^{|n|-1}(2^m+1)\left|\left(1-\left(\frac{2^m}{n}\right)^{\gamma}\right)^{\alpha}-\left(1-\left(\frac{2^m+1}{n}\right)^{\gamma}\right)^{\alpha}\right| \\
&= \sum_{m=t}^{|n|-1}(2^m+1)\left|\Theta\left(\frac{2^m}{n}\right)-\Theta\left(\frac{2^m+1}{n}\right)\right| \\
&= C_{\alpha,\gamma}\sum_{m=t}^{|n|-1}\frac{2^m+1}{n}\left|\Theta'(\xi)\right| \le C_{\alpha,\gamma}\sum_{m=t}^{|n|-1}\alpha\gamma\cdot\frac{2^m+1}{n}\left(\frac{2^{m+1}}{n}\right)^{\gamma-1} \le C_{\alpha,\gamma}
\end{aligned}$$

by the Lagrange theorem again with some $\dfrac{2^m}{n} < \xi < \dfrac{2^m+1}{n}$. We also used the fact that $|\Theta'(x)|$ is bounded. □

Lemma 2.7. *We suppose that* $0<\gamma\le 1\le\alpha<\infty$, $|n|\ge B\ge t$ *and* B *is fixed. Then*

$$\int_{\bar{I}_t}\sup_{|n|\ge B}T_9 \le C_{\alpha,\gamma}.$$

Proof. By Lemma 2.6, we obtain

$$\begin{aligned}
\int_{\bar{I}_t}\sup_{|n|\ge B}T_9 &= \int_{\bar{I}_t}\sup_{|n|\ge B}n^{-\alpha\gamma}\sum_{i=2^t}^{2^{|n|}-1}(i+2)\left|\Delta_2 a_{n,i}\right|\cdot\left|K_{i+1}\right| \\
&\le \int_{\bar{I}_t}\sup_{|n|\ge B}\sum_{i=2^t}^{2^{|n|}-1}n^{-\alpha\gamma}(i+2)\left|\Delta_2 a_{n,i}\right|\cdot\sup_{|j|\ge 2^t}\left|K_{j+1}\right| \\
&\le C_{\alpha,\gamma}\int_{\bar{I}_t}\sup_{|j|\ge t}\left|K_{j+1}\right| \le C_{\alpha,\gamma}.
\end{aligned}$$

This completes the proof of this lemma. □

2.2. Investigation of T_5

We are going to prove that some estimation for the function T_5 is bounded, where T_5 is defined in the following form:

$$T_5 = n^{-\alpha\gamma}\sum_{k=t}^{|n|-1}n_k(2^k+1)\left|\Delta_1 a_{n,n^{(k)}-2^k}\right|\left|K_{2^k}\right|.$$

Lemma 2.8. *We suppose that* $0<\gamma\le 1\le\alpha$. *Then*

$$n^{-\alpha\gamma}\sum_{k=t}^{|n|-1}n_k(2^k+1)\left|\Delta_1 a_{n,n^{(k)}-2^k}\right| \le C_{\alpha,\gamma}.$$

Proof. In this case, the derivative of $(1-x^\gamma)^\alpha$ $(x \in I)$ is an increasing function. By the Lagrange theorem we get

$$
\begin{aligned}
&n^{-\alpha\gamma}\sum_{k=t}^{|n|-1} n_k(2^k+1)\left|\Delta_1 a_{n,n^{(k)}-2^k}\right| \\
&= n^{-\alpha\gamma}\sum_{k=t}^{|n|-1} n_k(2^k+1)\left|a_{n,n^{(k)}-2^k}-a_{n,n^{(k)}-2^k+1}\right| \\
&= n^{-\alpha\gamma}\sum_{k=t}^{|n|-1} n_k(2^k+1)\left|\left(n^\gamma-(n^{(k)}-2^k)^\gamma\right)^\alpha - n_k\left(n^\gamma-(n^{(k)}-2^k+1)^\gamma\right)^\alpha\right| \\
&= \sum_{k=t}^{|n|-1}(2^k+1)\left|\left(1-\left(\frac{n^{(k)}-2^k}{n}\right)^\gamma\right)^\alpha - \left(1-\left(\frac{n^{(k)}-2^k+1}{n}\right)^\gamma\right)^\alpha\right| \\
&= C_{\alpha,\gamma}\sum_{k=t}^{|n|-1} n_k\frac{(2^k+1)}{n}\left|\left(1-\left(\frac{\xi}{n}\right)^\gamma\right)^{\alpha-1}\left(\frac{\xi}{n}\right)^{\gamma-1}\right| \\
&\le C_{\alpha,\gamma}\sum_{k=t}^{|n|-1} n_k\frac{(2^k+1)}{n}\left|\left(1-\left(\frac{2^k}{n}\right)^\gamma\right)^{\alpha-1}\left(\frac{2^k}{n}\right)^{\gamma-1}\right| \\
&\le C_{\alpha,\gamma}\sum_{k=t}^{|n|-1} n_k\left(\frac{2^k+1}{n}\right)^\gamma \le C_{\alpha,\gamma}
\end{aligned}
$$

for some

$$2^k+1 \le n^{(k)}-2^k < \xi < n^{(k)}-2^k+1$$

since $2^{|n|} \le n^{(k)}-2^k$ because $k<|n|$ and $n_k=1$ can be supposed. Otherwise the corresponding addend does not occur.

□

Lemma 2.9. *Suppose $|n| \ge B \ge t$ where B is fixed and $0<\gamma\le 1\le\alpha$. Then*

$$\int_{\bar{I}_t}\sup_{|n|\ge B} T_5 \le C_{\alpha,\gamma}.$$

Proof. This estimation is a result of the previous lemma and Lemma 1.2. Thus,

$$
\begin{aligned}
\int_{\bar{I}_t}\sup_{|n|\ge B} T_5 &= \int_{\bar{I}_t}\sup_{|n|\ge B} n^{-\alpha\gamma}\sum_{k=t}^{|n|-1} n_k(2^k+1)\left|\Delta_1 a_{n,n^{(k)}-2^k}\right||K_{2^k}| \\
&\le \int_{\bar{I}_t}\sup_{j\ge t}|K_{2^j}|\cdot\sup_{|n|\ge B} n^{-\alpha\gamma}\sum_{k=t}^{|n|-1}(2^k+1)n_k\left|\Delta_1 a_{n,n^{(k)}-2^k}\right| \\
&\le C_{\alpha,\gamma}\int_{\bar{I}_t}\sup_{j>t}|K_{2^j}| \le C_{\alpha,\gamma}.
\end{aligned}
$$

□

2.3. Investigation of T_3

In the sequel we investigate the function T_3.

$$T_3 := n^{-\alpha\gamma} \sum_{k=t}^{|n|-1} n_k \sum_{l=2^t}^{2^k-1} (l+2) \left|\Delta_2 a_{n,n^{(k)}-l-1}\right| |K_l|$$

Lemma 2.10. *We suppose that* $0 < \gamma \le 1 \le \alpha < \infty$. *Then*

$$n^{-\alpha\gamma} \sum_{k=t}^{|n|-1} n_k \sum_{l=2^t}^{2^k-1} (l+2) \left|\Delta_2 a_{n,n^{(k)}-l-1}\right| \le C_{\alpha,\gamma}.$$

Proof. In this proof we use that the second difference of $a_{n,n^{(k)}-l}$ is positive and then

$$\begin{aligned}
n^{-\alpha\gamma} &\sum_{k=t}^{|n|-1} n_k \sum_{l=2^t}^{2^k-1} (l+2) \left|\Delta_2 a_{n,n^{(k)}-l}\right| \\
&= n^{-\alpha\gamma} \sum_{k=t}^{|n|-1} n_k \sum_{m=t+1}^{k} \sum_{l=2^{m-1}}^{2^m-1} (l+2) \left|\Delta_1 a_{n,n^{(k)}-l-1} - \Delta_1 a_{n,n^{(k)}-l}\right| \\
&\le n^{-\alpha\gamma} \sum_{k=t}^{|n|-1} n_k \sum_{m=t+1}^{k} (2^m+1) \sum_{l=2^{m-1}}^{2^m-1} \left(\Delta_1 a_{n,n^{(k)}-l-1} - \Delta_1 a_{n,n^{(k)}-l}\right) \\
&= n^{-\alpha\gamma} \sum_{k=t}^{|n|-1} n_k \sum_{m=t+1}^{k} (2^m+1) \left(\Delta_1 a_{n,n^{(k)}-2^{m-1}-1} - \Delta_1 a_{n,n^{(k)}-2^m+1}\right) \\
&\le n^{-\alpha\gamma} \sum_{k=t}^{|n|-1} n_k \sum_{m=t+1}^{k} (2^m+1) \left|\Delta_1 a_{n,n^{(k)}-2^{m-1}-1}\right| \\
&\quad + n^{-\alpha\gamma} \sum_{k=t}^{|n|-1} n_k \sum_{m=t+1}^{k} (2^m+1) \left|\Delta_1 a_{n,n^{(k)}-2^m+1}\right| =: T_{31} + T_{32}.
\end{aligned}$$

In the first step, T_{32} is investigated, methods are similar to the estimation of T_5.

$$\begin{aligned}
n^{-\alpha\gamma} &\sum_{k=t}^{|n|-1} n_k \sum_{m=t+1}^{k} (2^m+1) \left|\Delta_1 a_{n,n^{(k)}-2^m+1}\right| \\
&= n^{-\alpha\gamma} \sum_{k=t}^{|n|-1} n_k \sum_{m=t+1}^{k} (2^m+1) \left|a_{n,n^{(k)}-2^m+1} - a_{n,n^{(k)}-2^m+2}\right| \\
&= \sum_{k=t}^{|n|-1} n_k \sum_{m=t+1}^{k} (2^m+1) \left| \left(1 - \left(\frac{n^{(k)}-2^m+1}{n}\right)^\gamma\right)^\alpha \right.
\end{aligned}$$

$$
\begin{aligned}
&\quad - \left(1 - \left(\frac{n^{(k)} - 2^m + 2}{n}\right)^\gamma\right)^\alpha \Bigg| \\
&= C_{\alpha,\gamma} \sum_{k=t}^{|n|-1} n_k \sum_{m=t+1}^{k} \frac{2^m+1}{n} \left| \left(1 - \left(\frac{\xi}{n}\right)^\gamma\right)^{\alpha-1} \left(\frac{\xi}{n}\right)^{\gamma-1} \right| \\
&\leq C_{\alpha,\gamma} \sum_{k=t}^{|n|-1} n_k \sum_{m=t+1}^{k} \frac{2^m+1}{n} \Bigg| \left(1 - \left(\frac{n^{(k)} - 2^m + 1}{n}\right)^\gamma\right)^{\alpha-1} \\
&\quad \cdot \left(\frac{n^{(k)} - 2^m + 2}{n}\right)^{\gamma-1} \Bigg| \\
&\leq C_{\alpha,\gamma} \sum_{k=t}^{|n|-1} n_k \sum_{m=t+1}^{k} \frac{2^m+1}{n} \left(\frac{n^{(k)} - 2^m + 2}{n}\right)^{\gamma-1} \\
&\leq C_{\alpha,\gamma} \sum_{k=t}^{|n|-1} \sum_{m=t+1}^{k} \frac{2^m+1}{n} \leq C_{\alpha,\gamma}
\end{aligned}
$$

where we applied the Lagrange theorem and where

$$
2^{k+1} \leq n^{(k+1)} = n^{(k)} - 2^k \leq n^{(k)} - 2^m < \xi < n^{(k)} - 2^m + 1
$$

and

$$
\left(\frac{n^{(k)} - 2^m + 2}{n}\right)^{\gamma-1} \leq \left(\frac{2^{|n|} - 2^k + 2}{n}\right)^{\gamma-1} \leq \left(\frac{2^{|n|} - 2^{|n|-1}}{n}\right)^{\gamma-1} \leq C_\gamma,
$$

because $k + 1 \leq |n|$.

□

Lemma 2.11. *Suppose* $|n| \geq B \geq t$ *and* $0 < \gamma \leq 1 \leq \alpha$. *Then*

$$
\int_{\bar{I}_t} \sup_{|n| \geq B} T_3 \leq C_{\alpha,\gamma}
$$

Proof. This estimation comes from the previous lemma. Namely,

$$
\begin{aligned}
\int_{\bar{I}_t} \sup_{|n| \geq B} T_3 &= \int_{\bar{I}_t} \sup_{|n| \geq B} n^{-\alpha\gamma} \sum_{k=t}^{|n|-1} n_k \sum_{l=2^t}^{2^k-1} (l+2) \left|\Delta_2 a_{n, n^{(k)}-l-1}\right| |K_l| \\
&\leq \int_{\bar{I}_t} \sup_{|j| > t} |K_j| \cdot \sup_{|n| \geq B} n^{-\alpha\gamma} \sum_{k=t}^{|n|-1} n_k \sum_{l=2^t}^{2^k-1} (l+2) \left|\Delta_2 a_{n, n^{(k)}-l-1}\right| \leq C_{\alpha,\gamma}.
\end{aligned}
$$

□

2.4. The proof of the main theorem

Lemma 2.12. *Let $f \in L^1(I)$ be with* $\operatorname{supp} f \subset I_k$ *and* $\int f = 0$. *Then*

$$\int_{\bar{I}_t} \sup_{n \geq 2^t} |f * K_n^{\alpha,\gamma}| \; d\lambda \leq C_{\alpha,\gamma} \cdot \|f\|_1 .$$

Proof. The upper bound for $|K_n^{\alpha,\gamma}|$ can be written in the following form by Lemmas 2.2 and 2.3:

$$|K_n^{\alpha,\gamma}| \leq T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8 + T_9 + T_{10} + T_{11}.$$

In the first step, let $m \in \{1, 2, 4, 6, 8\}$, then

$$\int_{\bar{I}_t} \sup_{n \geq 2^t} |f * T_m| \, d\lambda = 0, \tag{2.1}$$

because T_m are $\mathcal{A}_t$-measurable and $\int f = 0$.

T_7 and T_{11} are equal to 0 because of (D1).

Let j be an element of the set $\{3, 5, 9, 10\}$. We apply Lemmas 2.11, 2.9, 2.7 and 2.5. Thus,

$$\begin{aligned} \int_{\bar{I}_t} \sup_{n \geq 2^t} |f * T_j| \, d\lambda &= \int_{\bar{I}_t} \sup_{n \geq 2^t} \int_{I_t} |f(x) \cdot T_j(y - x)| \; dx \, dy \\ &= \int_{\bar{I}_t} |f(y - x)| \int_{\bar{I}_t} \sup_{n \geq 2^t} |T_j(y)| \; dy \, dx \leq C_{\alpha,\gamma} \|f\|_1 . \end{aligned}$$

□

Lemma 2.13. *Let be $f \in L^1(I)$. Then operator* $\sup_{n \in \mathbb{N}} |f * K_n^{\alpha,\gamma}|$ *is of type* (L^∞, L^∞) *and of type* (L^1, L^1).

Proof. Using Lemma 1.3, we get

$$\begin{aligned} \sup_{n \in \mathbb{N}} |f * K_n^{\alpha,\gamma}| &\leq \sup_{n \in \mathbb{N}} \left| \int_I |f(x) K_n^{\alpha,\gamma}(y - x)| \; dy \right| \\ &\leq \sum_{m=1}^{10} \|f\|_\infty \int_I \sup_{n \in \mathbb{N}} |T_m(y - x)| \; dy \leq C_{\alpha,\gamma} \|f\|_\infty . \end{aligned}$$

Furthermore,

$$\begin{aligned} \int_I \sup_{n \in \mathbb{N}} |f * K_n^{\alpha,\gamma}| &\leq \sup_{n \in \mathbb{N}} \int_I \left| \int_I |f(x) K_n^{\alpha,\gamma}(y - x)| \; dy \right| \; dx \\ &\leq \sum_{m=1}^{10} \int_I |f(x)| \; dx \cdot \int_I \sup_{n \in \mathbb{N}} |T_m(y - x)| \; dy \leq C_{\alpha,\gamma} \|f\|_1 . \end{aligned}$$

If $m \in \{1, 2, 4, 6, 8\}$, then we use (2.1). T_7 and T_{11} are equal to 0 because of (D1). Finally, if $m \in \{3, 5, 9, 10\}$, then we apply Lemmas 2.11, 2.9, 2.7 and 2.5. □

Proof of Theorem 2.1. The proof is the standard density argument (that is, the set of one-dimensional polynomials, i.e. finite linear combinations of the functions v_n is dense in $L^1(I)$) (see [8]). This fact and the previous two Lemmas complete the proof. □

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