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Certain subclass of meromorphic functions associated with the Bessel function

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Abstract. In this paper, we introduce and study a new subclass of meromorphic univalent functions defined by the Bessel function. We obtain coefficient inequalities, distortion properties, closure theorems, Hadamard product. Finally we obtain integral transforms for the class $\sigma_p^*(\beta, \alpha, \lambda, \upsilon, s, t)$.

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1. Introduction

Let Σ denote the class of meromorphic functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the punctured unit disc

$$U^* := \{z : z \in \mathbb{C},\ 0 < |z| < 1\} = U \setminus \{0\}. \tag{1.2}$$

Let $g \in \Sigma$ be given by

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k \tag{1.3}$$

then the Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.4}$$

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A function f in Σ is said to be meromorphically starlike of order γ if

$$-\Re\Big\{\frac{zf'(z)}{f(z)}\Big\} > \gamma,\ z \in U,\ \text{for some } \gamma\ (0 \le \gamma < 1). \tag{1.5}$$

We denote by $\Sigma^*(\gamma)$ the class of all meromorphically starlike functions of order γ. Similarly, a function f in Σ is said to be meromorphically convex of order γ if

$$-\Re\Big\{1 + \frac{zf''(z)}{f'(z)}\Big\} > \gamma,\ z \in U,\ \text{for some } \gamma\ (0 \le \gamma < 1). \tag{1.6}$$

We denote by $\Sigma_k^*(\gamma)$ the class of meromorphically convex functions of γ. The classes $\Sigma^*(\gamma)$ and $\Sigma_k^*(\gamma)$ were introduced and studied by Pommerenke [14], Miller [11], Mogra et al. [12], Aouf et al. [1, 2, 3], EL-Ashwah et al. [9], Mostafa et al. [13] and Venkateswarlu et al. [18, 19, 17].

Many important functions in applied sciences are defined via improper integrals or series (or infinite products). The general name of these important functions is special functions. Bessel functions are important special functions which are playing an important role in studying solutions of differential equations. Especially, the linear PDE describing various chemical transfer processes, allow the exact solution expressed in terms of one special kind of Bessel functions and they are associated with a wide range of problems in important areas of mathematical physics, modelling of transfer processes in chemical engineering, as well as in the related fields like hydrodynamics, heat transfer, diffusion, bioprocesses and so on. By using the method of separation of variables, exact solution in terms of Bessel functions can be used to calculate several important parameters which are needed in design and construction of chemical engineering apparatuses and equipment like heat exchangers and their components. Typical example for the efficiency calculation is applied in a Brazilian powdered milk plant [15]. In another case when Bessel functions arise is heat transfer modelling which was considered in [8]. Here the problem of a cross-flow streaming of heated object with large value of the length to diameter ratio (like thermoanemometer) is solved for small Pe numbers using the theory of analytic functions.

Let us consider the second order linear homogenous differential equation (see, Baricz [6], p. 7]):

$$z^2w''(z) + szw'(z) + [tz^2 - \upsilon^2 + (1-s)]w(z) = 0 \quad (s, t, \upsilon \in \mathbb{C}). \tag{1.7}$$

The function $w_{\upsilon,s,t}(z)$, which is called the generalized Bessel function of the first kind of order υ where υ is an unrestricted (real or complex) number, defines a particular solution of (1.7). The function $w_{\upsilon,s,t}(z)$, has the representation

$$w_{\upsilon,s,t}(z) = \sum_{k=0}^{\infty} \frac{(-t)^k}{\Gamma(k+1)\Gamma(k+\upsilon+\frac{s+1}{2})}\Big(\frac{z}{2}\Big)^{2k+\upsilon}.$$

Let us define

$$\begin{aligned}\mathfrak{L}_{\upsilon,s,t}(z) &= \frac{2^{\upsilon}\Gamma(\upsilon+\frac{s+1}{2})}{z^{\frac{\upsilon}{2}+1}}w_{\upsilon,s,t}(z^{\frac{1}{2}})\\ &= \frac{1}{z}+\sum_{k=1}^{\infty}\frac{(-t)^k\Gamma(\upsilon+\frac{s+1}{2})}{4^k\Gamma(k+1)\Gamma(k+\upsilon+\frac{s+1}{2})}z^k.\end{aligned}$$

The operator $\mathfrak{L}_{\upsilon,s,t}$ is a modification of the of the operator introduced by Deniz [7] for analytic functions.

By using the Hadamard product (or convolution), we define the operator $\mathfrak{L}_{\upsilon,s,t}$ as follows:

$$\begin{aligned}\left(\mathfrak{L}_{\upsilon,s,t}f\right)(z) &= \mathfrak{L}_{\upsilon,s,t}(z)*f(z)\\ &= \frac{1}{z}+\sum_{k=1}^{\infty}\phi_k(\upsilon,s,t)a_kz^k,\end{aligned} \tag{1.8}$$

where $\phi_k(\upsilon,s,t)=\frac{(-t)^k\Gamma(\upsilon+\frac{s+1}{2})}{4^k\Gamma(k+1)\Gamma(k+\upsilon+\frac{s+1}{2})}$.

It is easy to verify that

$$z\left(\mathfrak{L}_{\upsilon,s,t}f\right)'(z)=\left(\upsilon+\frac{s+1}{2}\right)\left(\mathfrak{L}_{\upsilon,s,t}f\right)(z)-\left(\upsilon+1+\frac{s+1}{2}\right)\left(\mathfrak{L}_{\upsilon+1,s,t}f\right)(z). \tag{1.9}$$

We note that: $\left(\mathfrak{L}_{\upsilon,1,1}f\right)(z)=\left(\mathfrak{L}_{\upsilon}f\right)(z)$ (see Aouf et al. [2]).

Motivated by Sivaprasad Kumar et al. [10], Atshan et al. [5] and Venkateswarlu et al. [17], now we define a new subclass $\sigma_p^*(\beta,\alpha,\lambda,\upsilon,s,t)$ of Σ.

Definition 1.1. For $0\le\beta<1,\ \alpha\ge0,\ 0\le\lambda<\frac{1}{2}$, we let $\sigma_p^*(\beta,\alpha,\lambda,\upsilon,s,t)$ be the subclass of Σ consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\begin{aligned}&-\Re\left(\frac{z(\mathfrak{L}_{\upsilon,s,t}f(z))'+\lambda z^2(\mathfrak{L}_{\upsilon,s,t}f(z))''}{(1-\lambda)\mathfrak{L}_{\upsilon,s,t}f(z)+\lambda z(\mathfrak{L}_{\upsilon,s,t}f(z))'}+\beta\right)\\ &\qquad>\alpha\left|\frac{z(\mathfrak{L}_{\upsilon,s,t}f(z))'+\lambda z^2(\mathfrak{L}_{\upsilon,s,t}f(z))''}{(1-\lambda)\mathfrak{L}_{\upsilon,s,t}f(z)+\lambda z(\mathfrak{L}_{\upsilon,s,t}f(z))'}+1\right|.\end{aligned} \tag{1.10}$$

In order to prove our results we need the following lemmas [4].

Lemma 1.2. *If η is a real number and ω is a complex number then*

$$\Re(\omega)\ge\eta\Leftrightarrow|\omega+(1-\eta)|-|\omega-(1+\eta)|\ge0.$$

Lemma 1.3. *If ω is a complex number and η,k are real numbers then*

$$-\Re(\omega)\ge k|\omega+1|+\eta\Leftrightarrow-\Re\left(\omega(1+ke^{i\theta})+ke^{i\theta}\right)\ge\eta,-\pi\le\ \theta\le\pi.$$

The main object of this paper is to study some usual properties of the geometric function theory such as the coefficient bounds, distortion properties, closure theorems, Hadamard product and integral transforms for the class $\sigma_p^*(\beta,\alpha,\lambda,\upsilon,s,t)$.

2. Coefficient estimates

In this section, we obtain necessary and sufficient condition for a function f to be in the class $\sigma_p^*(\beta,\alpha,\lambda,\upsilon,s,t)$.

Theorem 2.1. *Let $f \in \Sigma$ be given by (1.1). Then $f \in \sigma_p^*(\beta,\alpha,\lambda,\upsilon,s,t)$ if and only if*

$$\sum_{k=1}^{\infty}[(1+(k-1)\lambda)][k(\alpha+1)+(\alpha+\beta)]\phi_k(\upsilon,s,t)a_k \leq (1-\beta)(1-2\lambda). \tag{2.1}$$

Proof. Let $f \in \sigma_p^*(\beta,\alpha,\lambda,\upsilon,s,t)$. Then by Definition 1.1 and using Lemma 1.3, it is enough to show that

(2.2)

$$-\Re\left\{\frac{z(\mathfrak{L}_{\upsilon,s,t}f(z))'+\lambda z^2(\mathfrak{L}_{\upsilon,s,t}f(z))''}{(1-\lambda)\mathfrak{L}_{\upsilon,s,t}f(z)+\lambda z(\mathfrak{L}_{\upsilon,s,t}f(z))'}(1+\alpha e^{i\theta})+\alpha e^{i\theta}\right\} \geq \beta, -\pi \leq \ \theta \leq \pi.$$

For convenience

$$\begin{aligned} C(z) =& -\left[z(\mathfrak{L}_{\upsilon,s,t}f(z))'+\lambda z^2(\mathfrak{L}_{\upsilon,s,t}f(z))''\right](1+\alpha e^{i\theta}) \\ & - \alpha e^{i\theta}\left[(1-\lambda)\mathfrak{L}_{\upsilon,s,t}f(z)+\lambda z(\mathfrak{L}_{\upsilon,s,t}f(z))'\right] \\ D(z) =& (1-\lambda)\mathfrak{L}_{\upsilon,s,t}f(z)+\lambda z(\mathfrak{L}_{\upsilon,s,t}f(z))'. \end{aligned}$$

That is, the equation (2.2) is equivalent to

$$-\Re\left(\frac{C(z)}{D(z)}\right) \geq \beta.$$

In view of Lemma 1.2, we only need to prove that

$$|C(z)+(1-\beta)D(z)|-|C(z)-(1+\beta)D(z)| \geq 0.$$

Therefore

$$\begin{aligned} & |C(z)+(1-\beta)D(z)| \\ \geq\ & (2-\beta)(1-2\lambda)\frac{1}{|z|}-\sum_{k=1}^{\infty}[k-(1-\beta)][1+\lambda(k-1)]\phi_k(\upsilon,s,t)a_k|z|^k \\ & -\alpha\sum_{k=1}^{\infty}(k+1)[1+\lambda(k-1)]\phi_k(\upsilon,s,t)a_k|z|^k \end{aligned}$$

and

$$\begin{aligned} & |C(z)-(1+\beta)D(z)| \\ \leq\ & \beta(1-2\lambda)\frac{1}{|z|}+\sum_{k=1}^{\infty}[k+(1+\beta)][1+\lambda(k-1)]\phi_k(\upsilon,s,t)a_k|z|^k \\ & +\alpha\sum_{k=1}^{\infty}(k+1)[1+\lambda(k-1)]\phi_k(\upsilon,s,t)a_k|z|^k. \end{aligned}$$

It shows that

$$\begin{aligned}&|C(z)+(1-\beta)D(z)|-|C(z)-(1+\beta)D(z)|\\ \geq&2(1-\beta)(1-2\lambda)\frac{1}{|z|}-2\sum_{k=1}^{\infty}[(k+\beta)(1+(k-1)\lambda)]\phi_k(\upsilon,s,t)a_k|z|^k\\ &-2\alpha\sum_{k=1}^{\infty}(k+1)(1+(k-1)\lambda)\phi_k(\upsilon,s,t)a_k|z|^k\\ \geq&0,\ \text{by the given condition (2.1).}\end{aligned}$$

Conversely, suppose $f \in \sigma_p^*(\beta,\alpha,\lambda,\upsilon,s,t)$. Then by Lemma 1.2, we have (2.2).

Choosing the values of z on the positive real axis the inequality (2.2) reduces to

$$\Re\left\{\frac{[(1-2\lambda)(1-\beta)(1+\alpha e^{i\theta})]\frac{1}{z^2}+\sum_{k=1}^{\infty}\{k+\alpha e^{i\theta}(k+1)+\beta\}[1+\lambda(k-1)]\phi_k(\upsilon,s,t)z^{k-1}}{(1-2\lambda)\frac{1}{z^2}+\sum_{k=1}^{\infty}[1+\lambda(k-1)]\phi_k(\upsilon,s,t)a_kz^{k-1}}\right\}\geq 0.$$

Since $\Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, the above inequality reduces to

$$\Re\left\{\frac{[(1-2\lambda)(1-\beta)(1+\alpha e^{i\theta})]\frac{1}{r^2}+\sum_{k=1}^{\infty}\{k+\alpha(k+1)+\beta\}[1+\lambda(k-1)]\phi_k(\upsilon,s,t)a_kr^{k-1}}{(1-2\lambda)\frac{1}{r^2}+\sum_{k=1}^{\infty}[1+\lambda(k-1)]\phi_k(\upsilon,s,t)r^{k-1}}\right\}\geq 0.$$

Letting $r \to 1^-$ and by the mean value theorem, we have obtained the inequality (2.1). $\square$

Corollary 2.2. *If $f \in \sigma_p^*(\beta,\alpha,\lambda,\upsilon,s,t)$ then*

$$a_k \leq \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)][k(1+\alpha)+(\beta+\alpha)]\phi_k(\upsilon,s,t)}. \tag{2.3}$$

The result is sharp for the function

$$f(z)=\frac{1}{z}+\frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)][k(1+\alpha)+(\beta+\alpha)]\phi_k(\upsilon,s,t)}z^k. \tag{2.4}$$

By taking $\lambda = 0$ in Theorem 2.1, we get the following corollary.

Corollary 2.3. *If $f \in \sigma_p^*(\beta,\alpha,\upsilon,s,t)$ then*

$$a_k \leq \frac{1-\beta}{[k(1+\alpha)+(\beta+\alpha)]\phi_k(\upsilon,s,t)}. \tag{2.5}$$

3. Distortion theorem

Theorem 3.1. *If $f \in \sigma_p^*(\beta,\alpha,\lambda,\upsilon,s,t)$ then for $0 < |z| = r < 1$,*

$$\frac{1}{r}-\frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)\phi_1(\upsilon,s,t)}\ r \leq |f(z)| \leq \frac{1}{r}+\frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)\phi_1(\upsilon,s,t)}r. \tag{3.1}$$

This result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)\phi_1(\upsilon,s,t)}\ z. \tag{3.2}$$

Proof. Since $f(z) = \frac{1}{z} + \sum\limits_{k=1}^{\infty} a_k z^k$, we have

$$|f(z)| = \frac{1}{r} + \sum_{k=1}^{\infty} a_k r^k \leq \frac{1}{r} + r\sum_{k=1}^{\infty} a_k. \tag{3.3}$$

Since $k \geq 1$,

$$(2\alpha+\beta+1)\phi_1(\upsilon,s,t) \leq [1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\phi_k(\upsilon,s,t),$$

using Theorem 2.1, we have

$$\begin{aligned}(2\alpha+\beta+1)\phi_1(\upsilon,s,t)\sum_{k=1}^{\infty} a_k &\leq \sum_{k=1}^{\infty}[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\phi_k(\upsilon,s,t)\\ &\leq (1-\beta)(1-2\lambda)\\ \Rightarrow \sum_{k=1}^{\infty} a_k &\leq \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)\phi_1(\upsilon,s,t)}.\end{aligned}$$

Using the above inequality in (3.3), we have

$$\begin{aligned}|f(z)| &\leq \frac{1}{r} + \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)\phi_1(\upsilon,s,t)}r\\ \text{and } |f(z)| &\geq \frac{1}{r} - \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)\phi_1(\upsilon,s,t)}r.\end{aligned}$$

The result is sharp for the function $f(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)\phi_1(\upsilon,s,t)}z$. □

The proof of the following corollary is analogous to that of Theorem 3.1 and so we omit the proof.

Corollary 3.2. *If $f \in \sigma_p^*(\beta,\alpha,\lambda,\upsilon,s,t)$ then*

$$\frac{1}{r^2} - \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)\phi_1(\upsilon,s,t)} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{((1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)\phi_1(\upsilon,s,t)}.$$

The result is sharp for the function given by (3.2).

4. Closure theorems

Let the functions f_j is defined, for $j = 1,2,\cdots,m$, by

$$f_j(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,j} z^k, a_{k,j} \geq 0. \tag{4.1}$$

Theorem 4.1. *Let the functions $f_j, j = 1, 2, \cdots, m$ defined by (4.1) be in the class $\sigma_p^*(\beta, \alpha, \lambda, \upsilon, s, t)$. Then the function h defined by*

$$h(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{m} \sum_{j=1}^{m} a_{k,j} \right) z^k \tag{4.2}$$

also belongs to the class $\sigma_p^*(\beta, \alpha, \lambda, \upsilon, s, t)$.

Proof. Since $f_j, j = 1, 2, \cdots, m$ are in the class $\sigma_p^*(\beta, \alpha, \lambda, \upsilon, s, t)$, it follows from Theorem 2.1 that

$$\sum_{k=1}^{\infty}[1 + \lambda(k-1)][k(1+\alpha) + (\alpha+\beta)]\phi_k(\upsilon, s, t)a_{k,j} \leq (1-\beta)(1-2\lambda),$$

for every $j = 1, 2, \cdots, m$. Hence

$$\begin{aligned} &\sum_{k=1}^{\infty}[1 + \lambda(k-1)][k(1+\alpha) + (\alpha+\beta)]\phi_k(\upsilon, s, t) \left(\frac{1}{m} \sum_{j=1}^{m} a_{k,j} \right) \\ = \ & \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{k=1}^{\infty}[1 + \lambda(k-1)][k(1+\alpha) + (\alpha+\beta)]\phi_k(\upsilon, s, t)a_{k,j} \right) \\ \leq \ & (1-\beta)(1-2\lambda). \end{aligned}$$

From Theorem (2.1), it follows that $h \in \sigma_p^*(\beta, \alpha, \lambda, \upsilon, s, t)$.

This completes the proof of Theorem 4.1. □

Theorem 4.2. *The class $\sigma_p^*(\beta, \alpha, \lambda, \upsilon, s, t)$ is closed under convex linear combinations.*

Proof. Let the functions $f_j, j = 1, 2, \cdots, m$ defined by (4.1) is in the class $\sigma_p^*(\beta, \alpha, \lambda, \upsilon, s, t)$. Then it is sufficient to show that the function

$$h(z) = \varsigma f_1(z) + (1-\varsigma)f_2(z), \ 0 \leq \varsigma \leq 1 \tag{4.3}$$

is in the class $\sigma_p^*(\beta, \alpha, \lambda, \upsilon, s, t)$. Since for $0 \leq \varsigma \leq 1$,

$$h(z) = \frac{1}{z} + \sum_{k=1}^{\infty}[\varsigma a_{k,1} + (1-\varsigma)a_{k,1}]z^k, \tag{4.4}$$

with the aid of Theorem 2.1, we have

$$\begin{aligned} &\sum_{k=1}^{\infty}[1 + \lambda(k-1)][k(1+\alpha) + (\alpha+\beta)]\phi_k(\upsilon, s, t)[\varsigma a_{k,1} + (1-\varsigma)a_{k,1}] \\ \leq \ & \varsigma(1-\beta)(1-2\lambda) + (1-\varsigma)(1-\beta)(1-2\lambda) \\ = \ & (1-\beta)(1-2\lambda), \end{aligned}$$

which implies that $h \in \sigma_p^*(\beta, \alpha, \lambda, \upsilon, s, t)$. □

Theorem 4.3. *Let $\xi \geq 0$. Then* $\sigma_p^{*\xi}(\beta,\alpha,\lambda,\upsilon,s,t) \subseteq N(\alpha,\xi)$, *where*

$$\xi = 1 - \frac{2(1-\beta)(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)+(1-\beta)(1-2\lambda)}. \tag{4.5}$$

Proof. If $f \in \sigma_p^{*\xi}(\beta,\alpha,\lambda,\upsilon,s,t)$ then

$$\sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\phi_k(\upsilon,s,t)}{(1-\beta)(1-2\lambda)} a_k \leq 1.$$

We need to find the value of ξ such that

$$\sum_{k=1}^{\infty} \frac{[k(1+\alpha)+(\alpha+\xi)]\phi_k(\upsilon,s,t)}{1-\xi} a_k \leq 1.$$

Thus it is sufficient to show that

$$\frac{[k(1+\alpha)+(\alpha+\xi)]\phi_k(\upsilon,s,t)}{1-\xi} \leq \frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\phi_k(\upsilon,s,t)}{(1-\beta)(1-2\lambda)}.$$

Then

$$\xi \leq 1 - \frac{(k+1)(1-\beta)(1-2\lambda)(1+\alpha)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]+(1-\beta)(1-2\lambda)}.$$

Since

$$D(k) = 1 - \frac{(k+1)(1-\beta)(1-2\lambda)(1+\alpha)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]+(1-\beta)(1-2\lambda)}$$

is an increasing function of k, $k \geq 1$, we obtain

$$\xi \leq D(1) = 1 - \frac{2(1-\beta)(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)+(1-\beta)(1-2\lambda)}.$$

□

Theorem 4.4. *Let $f_0(z) = \frac{1}{z}$ and*

$$f_k(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\phi_k(\upsilon,s,t)} z^k, \quad k \geq 1. \tag{4.6}$$

Then f is in the class $\sigma_p^*(\beta,\alpha,\lambda,\upsilon,s,t)$ if and only if can be expressed in the form

$$f(z) = \sum_{k=0}^{\infty} \omega_k f_k(z), \tag{4.7}$$

where $\omega_k \geq 0$ and $\sum_{k=0}^{\infty} \omega_k = 1$.

Proof. Assume that

$$\begin{aligned}
f(z) &= \sum_{k=0}^{\infty} \omega_k f_k(z) \\
&= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\phi_k(\upsilon,s,t)} z^k.
\end{aligned}$$

Then it follows that

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\phi_k(\upsilon,s,t)}{(1-\beta)(1-2\lambda)} \\
&\qquad \times \frac{(1-\beta)(1-2\lambda) \quad z^k}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\phi_k(\upsilon,s,t)} \\
&= \sum_{k=1}^{\infty} \omega_k = 1 - \omega_0 \le 1
\end{aligned}$$

which implies that $f \in \sigma_p^*(\beta, \alpha, \lambda, \upsilon, s, t)$.

Conversely, assume that the function f defined by (1.1) be in the class $f \in \sigma_p^*(\beta, \alpha, \lambda, \upsilon, s, t)$. Then

$$a_k \le \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\phi_k(\upsilon,s,t)}.$$

Setting

$$\omega_k = \frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\phi_k(\upsilon,s,t)}{(1-\beta)(1-2\lambda)} a_k,$$

where

$$\omega_0 = 1 - \sum_{k=0}^{\infty} \omega_k,$$

we can see that f can be expressed in the form (4.7).

□

Corollary 4.5. *The extreme points of the class $\sigma_p^*(\beta, \alpha, \lambda, \upsilon, s, t)$ are the functions $f_0(z) = \frac{1}{z}$ and*

$$f_k(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\phi_k(\upsilon,s,t)} z^k. \tag{4.8}$$

5. Modified Hadamard products

Let the functions $f_j (j = 1, 2)$ defined by (4.1). The modified Hadamard product of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z). \tag{5.1}$$

Theorem 5.1. *Let the functions $f_j (j = 1, 2)$ defined by (4.1) be in the class $\sigma_p^*(\beta, \alpha, \lambda, \upsilon, s, t)$. Then $f_1 * f_2 \in \sigma_p^*(\beta, \alpha, \lambda, \upsilon, s, t)$, where*

$$\varphi = 1 - \frac{2(1-\beta)^2(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)^2\phi_1(\upsilon, s, t) + (1-\beta)^2(1-2\lambda)}. \tag{5.2}$$

The result is sharp for the functions $f_j (j = 1, 2)$ given by

$$f_j(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)\phi_1(\upsilon, s, t)} z, \ (j = 1, 2). \tag{5.3}$$

Proof. Employing the technique used earlier by Schild and Silverman [16], we need to find the largest real parameter φ such that

$$\sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\varphi)]\phi_k(\upsilon, s, t)}{(1-\varphi)(1-2\lambda)} a_{k,1}a_{k,2} \leq 1. \tag{5.4}$$

Since $f_j \in \sigma_p^*(\beta, \alpha, \lambda, \upsilon, s, t), \ j = 1, 2$, we readily see that

$$\sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta]\phi_k(\upsilon, s, t)}{(1-\beta)(1-2\lambda)} a_{k,1} \leq 1$$

and

$$\sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta]\phi_k(\upsilon, s, t)}{(1-\beta)(1-2\lambda)} a_{k,2} \leq 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta]\phi_k(\upsilon, s, t)}{(1-\beta)(1-2\lambda)} \sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{5.5}$$

Then it is sufficient to show that

$$\sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\varphi)]\phi_k(\upsilon, s, t)}{(1-\varphi)(1-2\lambda)} a_{k,1}a_{k,2}$$
$$\leq \sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta]\phi_k(\upsilon, s, t)}{(1-\beta)(1-2\lambda)} \sqrt{a_{k,1}a_{k,2}},$$

or equivalently that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{[k(1+\alpha)+(\alpha+\beta](1-\varphi)}{[k(1+\alpha)+(\alpha+\varphi](1-\beta)}.$$

Hence, it light of the inequality (5.5), it is sufficient to show that

$$\frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta]\phi_k(\upsilon, s, t)}$$
$$\leq \frac{[k(1+\alpha)+(\alpha+\beta](1-\varphi)}{[k(1+\alpha)+(\alpha+\varphi](1-\beta)}. \tag{5.6}$$

It follows from (5.6) that
$\varphi \leq 1 - \frac{(1-\beta)^2(1-2\lambda)(1+\alpha)(k+1)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta]^2\phi_k(\upsilon,s,t)+(1-\beta)^2(1-2\lambda)}$.

Now defining the function $E(k)$,
$E(k) = 1 - \frac{(1-\beta)^2(1-2\lambda)(1+\alpha)(k+1)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta]^2\phi_k(\upsilon,s,t)+(1-\beta)^2(1-2\lambda)}$.

We see that $E(k)$ is an increasing of k, $k \geq 1$. Therefore, we conclude that

$$\varphi \leq E(k) = 1 - \frac{2(1-\beta)^2(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)^2\phi_1(\upsilon,s,t)+(1-\beta)^2(1-2\lambda)},$$

which evidently completes the proof of Theorem 5.1.

□

Using arguments similar to those in the proof of Theorem 5.1, we obtain the following theorem.

Theorem 5.2. *Let the function f_1 defined by (4.1) be in the class $\sigma_p^*(\beta,\alpha,\lambda,\upsilon,s,t)$. Suppose also that the function f_2 defined by (4.1) is in the class $\sigma_p^*(\rho,\beta,\alpha,\lambda,\upsilon,s,t)$. Then $f_1 * f_2 \in \sigma_p^*(\zeta,\beta,\alpha,\lambda,\upsilon,s,t)$, where*

$$\zeta = 1 - \frac{2(1-\beta)(1-\rho)(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)(2\alpha+\rho+1)\phi_1(\upsilon,s,t)+(1-\beta)(1-\rho)(1-2\lambda)}. \tag{5.7}$$

The result is sharp for the functions $f_j (j = 1,2)$ given by

$$f_1(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)\phi_1(\upsilon,s,t)}z$$

and

$$f_2(z) = \frac{1}{z} + \frac{(1-\rho)(1-2\lambda)}{(2\alpha+\rho+1)\phi_1(\upsilon,s,t)}z.$$

Theorem 5.3. *Let the functions $f_j (j = 1,2)$ defined by (4.1) be in the class $\sigma_p^*(\beta,\alpha,\lambda,\upsilon,s,t)$. Then the function*

$$h(z) = \frac{1}{z} + \sum_{k=1}^{\infty}(a_{k,1}^2 + a_{k,2}^2)z^k \tag{5.8}$$

belongs to the class $\sigma_p^*(\varepsilon,\beta,\alpha,\lambda,\upsilon,s,t)$, where

$$\varepsilon = 1 - \frac{4(1-\beta)^2(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)^2\phi_1(\upsilon,s,t)+2(1-\beta)^2(1-2\lambda)}. \tag{5.9}$$

The result is sharp for the functions $f_j (j = 1,2)$ given by (5.3).

Proof. By using Theorem 2.1, we obtain

$$\sum_{k=1}^{\infty}\left\{\frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\phi_k(\upsilon,s,t)}{(1-\beta)(1-2\lambda)}\right\}^2 a_{k,1}^2$$
$$\leq\sum_{k=1}^{\infty}\left\{\frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\phi_k(\upsilon,s,t)}{(1-\beta)(1-2\lambda)}a_{k,1}\right\}^2\leq 1 \tag{5.10}$$

and

$$\sum_{k=1}^{\infty}\left\{\frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\phi_k(\upsilon,s,t)}{(1-\beta)(1-2\lambda)}\right\}^2 a_{k,2}^2$$
$$\leq\sum_{k=1}^{\infty}\left\{\frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\phi_k(\upsilon,s,t)}{(1-\beta)(1-2\lambda)}a_{k,2}\right\}^2\leq 1. \tag{5.11}$$

It follows from (5.10) and (5.11) that

$$\sum_{k=1}^{\infty}\frac{1}{2}\left\{\frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\phi_k(\upsilon,s,t)}{(1-\beta)(1-2\lambda)}\right\}^2(a_{k,1}^2+a_{k,2}^2)\leq 1.$$

Therefore, we need to find the largest ε such that

$$\frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\varepsilon)]\phi_k(\upsilon,s,t)}{(1-\varepsilon)(1-2\lambda)}$$
$$\leq\frac{1}{2}\left\{\frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\phi_k(\upsilon,s,t)}{(1-\beta)(1-2\lambda)}\right\}^2,$$

that is

$$\varepsilon\ \leq 1-\frac{2(1-\beta)^2(1-2\lambda)(1+\alpha)(k+1)}{1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]^2\phi_k(\upsilon,s,t)+2(1-\beta)^2(1-2\lambda)}.$$

Since

$$G(k)=1-\frac{2(1-\beta)^2(1-2\lambda)(1+\alpha)(k+1)}{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]^2\phi_k(\upsilon,s,t)+2(1-\beta)^2(1-2\lambda)}$$

is an increasing function of $k,\ k\geq 1$, we obtain

$$\varepsilon\leq G(1)=\frac{4(1-\beta)^2(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)^2\phi_1(\upsilon,s,t)+2(1-\beta)^2(1-2\lambda)}$$

and hence the proof of Theorem 5.3 is completed. □

6. Integral operators

Theorem 6.1. *Let the functions f given by (1.1) be in the class $\sigma_p^*(\beta,\alpha,\lambda,\upsilon,s,t)$. Then the integral operator*

$$F(z)=c\int_0^1 u^c f(uz)du,\quad 0<u\leq 1,\ c>0 \tag{6.1}$$

is in the class $\sigma_p^*(\beta, \alpha, \lambda, \upsilon, s, t)$*, where*

$$\xi = 1 - \frac{2c(1-\beta)(1+\alpha)}{(c+2)(2\alpha+\beta+1)+c(1-\beta)}. \tag{6.2}$$

The result is sharp for the function f *given by* (3.2)*.*

Proof. Let $f \in \sigma_p^*(\beta, \alpha, \lambda, \upsilon, s, t)$. Then

$$\begin{aligned} F(z) =& c\int_0^1 u^c f(uz)du \\ =& \frac{1}{z} + \sum_{k=1}^{\infty} \frac{c}{k+c+1} a_k z^k. \end{aligned}$$

Thus it is sufficient to show that

$$\sum_{k=1}^{\infty} \frac{c[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\xi)]\phi_k(\upsilon, s, t)}{(k+c+1)(1-\xi)(1-2\lambda)} a_k \leq 1. \tag{6.3}$$

Since $f \in \sigma_p^*(\beta, \alpha, \lambda, \upsilon, s, t)$, then

$$\sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)][k(1+\alpha)+(\alpha+\beta)]\phi_k(\upsilon, s, t)}{(1-\beta)(1-2\lambda)} a_k \leq 1. \tag{6.4}$$

From (6.3) and (6.4), we have

$$\frac{[k(1+\alpha)+(\alpha+\xi)]}{(k+c+1)(1-\xi)} \leq \frac{[k(1+\alpha)+(\alpha+\beta)]}{(1-\beta)}.$$

Then

$$\xi \leq 1 - \frac{c(1-\beta)(k+1)(1+\alpha)}{(k+c+1)[k(1+\alpha)+(\alpha+\beta)]+c(1-\beta)}.$$

Since

$$Y(k) = 1 - \frac{c(1-\beta)(k+1)(1+\alpha)}{(k+c+1)[k(1+\alpha)+(\alpha+\beta)]+c(1-\beta)}$$

is an increasing function of k, $k \geq 1$, we obtain

$$\xi \leq Y(1) = 1 - \frac{2c(1-\beta)(1+\alpha)}{(c+2)(2\alpha+\beta+1)+c(1-\beta)}$$

and hence the proof of Theorem 6.1 is completed. □

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